Results in Nonlinear Analysis **5** (2022) No. 3, 372–386.
https://doi.org/10.53006/rna.1128729
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Common Attractive Point Theorems for a Finite Family of Multivalued Nonexpansive Mappings in Banach Spaces

Atit Wiriyapongsanon[a], Warunun Inthakon[b,c], Narawadee Phudolsitthiphat[b,c]

[a] *Faculty of Sport and Health Sciences, Thailand National Sports University Lampang Campus, Lampang 52100, Thailand.*

[b] *Research Group in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.*

[c] *Data Science Research Center, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.*

Abstract

Our main purpose of this paper is to introduce the modified Mann and Ishikawa iterates for finding a common attractive point of a finite family of multivalued nonexpansive mappings in the setting of uniformly convex Banach spaces. We obtain necessary and sufficient conditions to guarantee the strong convergence of the proposed algorithms without closedness of the domain of such mappings. Moreover, we derive some consequences from our main result to fixed point result of such mappings. Finally, the numerical results are provided to support our main theorem.

Keywords: Common attractive point Multivalued nonexpansive mapping Uniformly convex Banach space Demicompact
2020 MSC: 47H09; 47H10; 65J15; 54H25; 47J26

1. Introduction

Let T be a multivalued mapping on a nonempty subset K of a Banach space E. The set of fixed points of T is denoted by

$$F(T) = \{x \in K : x \in Tx\}.$$

There are many applications of fixed point theory for nonlinear mappings such as in control theory, convex optimization, differential equations, dynamic systems theory and economics, see [1, 2, 3, 4, 5] for examples and references

Email addresses: rtitkp99@gmail.com (Atit Wiriyapongsanon), warunun.i@cmu.ac.th (Warunun Inthakon), narawadee.nanan@cmu.ac.th (Narawadee Phudolsitthiphat)

Received :June 10, 2022; Accepted: August 18, 2022; Online: August 21, 2022

therein. To generalized fixed point theory for single valued mapping, studying the corresponding multivalued mappings is constanly developed. See [6, 7, 8, 9, 10, 11, 12, 13] for existence theorems and see [14, 15, 16, 17, 18, 19, 20, 21] for convergence theorems of fixed point results for both of single valued and multivalued mappings. Recently, many kinds of iterative processes have been used for approximating fixed points of multivalued nonexpansive mappings. Among these iterative processes, Sastry and Babu [22] defined Mann and Ishikawa schemes for a multivalued mapping and prove that these iterates converge to a fixed point of a multivalued nonexpansive mapping whose domain is compact and convex. In 2011, the concept of attractive points in a real Hilbert space was introduced by Takahashi and Takeuchi [23]. They established a mean convergence theorem without convexity for finding an attractive point of a nonlinear mapping which generalized the nonlinear ergodic theorem proved by Kocourek, Takahashi and Yao [24]. In 2018, Farid [25] introduced an iterative scheme to approximate common attractive points for a finite family of nonlinear mappings in a real Hilbert space using Cesàro mean approximation method. They obtained a weak convergence theorem for a sequence generated by the proposed iterative scheme. See [26, 27, 28] for more results regarding existence and convergence theorems of common attractive points for single valued mappings.

In this work, we introduce the modified Mann and Ishikawa iterates for a finite family of multivalued nonexpansive mappings in a uniformly convex Banach space. Also, we prove some strong convergence theorems for such mappings under some appropriate conditions. Finally, we give an example with numerical results to support our main theorem.

2. Preliminaries

Let E be a real Banach space. A subset K of E is called proximinal (Chebyshev) if for each $x \in E$, there exists an (unique) element $k \in K$ such that

$$\|x - k\| = \inf\{\|x - y\| : y \in K\} = d(x, K).$$

We call k as the best approximation (or nearest point) of x. It is known that weakly compact convex subsets of a Banach space and closed convex subsets of a reflexive Banach space are proximinal. Moreover, every closed convex subset of a uniformly convex Banach space is a Chebyshev set. We shall denote the family of nonempty bounded proximinal subsets of K by $P(K)$. Let $CC(K)$ be the class of all nonempty closed and covex subsets of K. Consistent with [29], let $CB(K)$ be the class of all nonempty bounded and closed subsets of K. The Hausdorff metric on $CB(E)$ is defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

for every $A, B \in CB(E)$. A multivalued mapping $T : K \to P(K)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|,$$

for all $x, y \in K$. If $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|,$$

for all $x \in K$ and $p \in F(T)$, then T is said to be quasi-nonexpansive.

Although dealing with multivalued nonexpansive mappings are more complicated than single valued nonexpansive mappings, finding fixed points of multivalued nonexpansive mappings have been approximated using various iterative approaches. In 2005, Sastry and Babu [22] suggested an iterative scheme for multivalued mappings among these iterative processes as follows:

(A) Let K be a nonempty convex subset of E, $T : K \to P(K)$ a multivalued mapping with $p \in F(T)$.

(i) The sequence of Mann iterates is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \end{cases}$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$, and $\{\alpha_n\}$ is a sequence in $(0, 1)$.

(ii) The sequence of Ishikawa iterates is defined by,

$$\begin{cases} x_1 \in K, \\ y_n = (1-\beta_n)x_n + \beta_n v_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, \end{cases}$$

where $v_n \in Tx_n,\ u_n \in Ty_n$ are such that $\|v_n - p\| = d(p, Tx_n)$ and $\|u_n - p\| = d(p, Ty_n)$, and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1)$.

Let K be a nonempty convex subset of a Banach space E, $T : K \to P(K)$ be a multivalued mapping. The set of all attractive points of T is denoted by $A(T)$, that is,

$$A(T) = \{z \in E : d(z, Tx) \leq \|z - x\|, \forall x \in K\}.$$

The set of all strongly attractive points of T is denoted by $A_S(T)$, that is,

$$A_S(T) = \{z \in E : H(z, Tx) \leq \|z - x\|, \quad \forall x \in K\}.$$

Hence, $A_S(T) \subseteq A(T)$.

A Banach space E is called uniformly convex if for each $\varepsilon \in [0, 2]$, there is $\delta_\varepsilon > 0$ and $x, u \in E$ such that

$$\|x\| = \|u\| = 1 \ \text{ and } \ \|x - u\| \geq \varepsilon \quad \text{implies} \quad \frac{\|x + u\|}{2} < 1 - \delta_\varepsilon.$$

In order to prove our main results, we need the following Lemma.

Lemma 2.1. *[31] Let E be a uniformly convex Banach space and let $B_r(0) := \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\sum_{i=1}^{N} w^{(i)} x_i\|^2 \leq \sum_{i=1}^{N} w^{(i)} \|x_i\|^2 - w^{(j)} w^{(k)} g(\|x_j - x_k\|), \ \ \textit{for all} \ \ j, k \in \{1, 2, \ldots, N\},$$

where $\{x_i\}_{i=1}^N \subset B_r(0)$ and $\{w^{(i)}\}_{i=1}^N \subset [0, 1]$ with $\sum_{i=1}^{N} w^{(i)} = 1$.

3. Main Results

We begin this section by modifying the iteration process given by (A) in a more general setting, i.e.,

(B) Let K be a nonempty convex subset of E and $\{T_i : K \to P(K),\ i = 1, 2, \ldots, m\}$ be a finite family of a multivalued mapping.

(i) The sequence of modified Mann iterates is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_{n,0} x_n + \sum_{i=1}^{m} \alpha_{n,i} x_{n,i}, \end{cases} \tag{1}$$

where $x_{n,i} \in T_i x_n$ and $0 \leq \alpha_{n,i} < 1$ satisfying $\sum_{i=0}^{m} \alpha_{n,i} = 1$.

(ii) The sequence of modified Ishikawa iterates is defined by,

$$\begin{cases} x_1 \in K, \\ y_n = \beta_{n,0}x_n + \sum_{i=1}^{m} \beta_{n,i}v_{n,i}, \\ x_{n+1} = \alpha_{n,0}x_n + \sum_{i=1}^{m} \alpha_{n,i}u_{n,i}, \end{cases} \tag{2}$$

where $v_{n,i} \in T_i x_n$, $u_{n,i} \in T_i y_n$ and $0 \leq \alpha_{n,i}, \beta_{n,i} < 1$ satisfying $\sum_{i=0}^{m} \beta_{n,i} = \sum_{i=0}^{m} \alpha_{n,i} = 1$. We can see that (ii) can be reduce to (i) by setting $\beta_{n,0} = 1$ and $\beta_{n,i} = 0$, $\forall i = 1, 2, \dots, m$.

Before proving convergence theorems, we study some properties of attractive points for multivalued mapping as follows.

Lemma 3.1. *Let K be a nonempty closed convex subset of uniformly convex Banach space E and let $T : K \to CC(K)$ be a multivalued mapping. If $A(T) \neq \emptyset$, then $F(T) \neq \emptyset$.*

Proof. Let $z \in A(T)$. Since K is closed and convex, there is a unique $k \in K$ such that

$$\|z - k\| = d(z, K) \leq d(z, Tk). \tag{3}$$

Since Tk is closed and convex, there is $l \in Tk$ such that

$$\|z - l\| = d(z, Tk). \tag{4}$$

Since $z \in A(T)$, $d(z, Tx) \leq \|z - x\|$, $\forall x \in K$. It follows that

$$d(z, Tk) \leq \|z - k\|. \tag{5}$$

By (3), (4) and (5),

$$\|z - l\| = \|z - k\| = d(z, K).$$

By the uniqueness of the nearest point to z in K, $k = l \in Tk$, which implies that $k \in F(T)$. □

Remark 3.2. *Lemma 3.1 is still valid when K is a nonempty Chebyshev subset of Banach space E and $T : K \to P(K)$.*

Lemma 3.3. *Let K be a nonempty subset of Banach space E and let $T : K \to CB(K)$ be a multivalued mapping. Then, $A(T)$ is a closed subset of E.*

Proof. Let $\{z_n\} \subset A(T)$ be a sequence such that $z_n \to z \in E$. Let $\varepsilon > 0$ be arbitrary, choose $N \in \mathbb{N}$ such that

$$\|z - z_n\| < \varepsilon/2, \qquad \forall n \geq N.$$

For $x \in K$, we obtain

$$\|x - z_n\| \leq \|x - z\| + \|z - z_n\| < \|x - z\| + \varepsilon/2, \qquad \forall n \geq N.$$

From $z_N \in A(T)$, we have

$$d(z, Tx) \leq \|z - z_N\| + d(z_N, Tx) \leq \|z - z_N\| + \|z_N - x\| < \|z - x\| + \varepsilon.$$

Since ε is arbitrary, we have $d(z, Tx) \leq \|z - x\|$. Therefore $z \in A(T)$, it follows that $A(T)$ is closed. □

Lemma 3.4. *Let K be a nonempty subset of Banach space E and let $T : K \to CB(K)$ be a quasi-nonexpansive mapping, then $A(T) \cap K = F(T) = A_S(T) \cap K$.*

Proof. Let $z \in A(T) \cap K$. Since $z \in A(T)$,

$$d(z, Tx) \leq \|z - x\|, \quad \forall x \in K.$$

Since $z \in K$,

$$d(z, Tz) \leq \|z - z\| = 0.$$

It follows that $z \in F(T)$. Since $A_S(T) \subseteq A(T)$, then $A_S(T) \cap K \subseteq A(T) \cap K \subseteq F(T)$.

Conversely, let $z \in F(T)$. Since T is quasi-nonexpansive,

$$H(z, Tx) \leq H(Tz, Tx) \leq \|z - x\|, \quad \forall x \in K.$$

Then, $z \in A_S(T)$. Since $F(T) \subseteq K$, $z \in A_S(T) \cap K$. That is $F(T) \subseteq A_S(T) \cap K \subseteq A(T) \cap K$. Therefore, we can conclude that $A(T) \cap K = F(T) = A_S(T) \cap K$. □

Lemma 3.5. *Let K be a nonempty convex subset of a normed space E. For each $i = 1, 2, \ldots, m$, let $T_i : K \to P(K)$ be a multivalued mapping such that $\bigcap_{i=1}^{m} A_S(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined in (2). Then, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \bigcap_{i=1}^{m} A_S(T_i)$.*

Proof. Let $p \in \bigcap_{i=1}^{m} A_S(T_i)$, then

$$H(p, T_i x) \leq \|p - x\|, \quad \forall x \in K.$$

From (2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_{n,0} x_n + \sum_{i=1}^{m} \alpha_{n,i} u_{n,i} - p\| \\
&\leq \alpha_{n,0} \|x_n - p\| + \sum_{i=1}^{m} \alpha_{n,i} \|u_{n,i} - p\| \\
&\leq \alpha_{n,0} \|x_n - p\| + \sum_{i=1}^{m} \alpha_{n,i} H(T_i y_n, p) \\
&\leq \alpha_{n,0} \|x_n - p\| + \sum_{i=1}^{m} \alpha_{n,i} \|y_n - p\|. \qquad (6)
\end{aligned}$$

But

$$\begin{aligned}
\|y_n - p\| &= \|\beta_{n,0} x_n + \sum_{i=1}^{m} \beta_{n,i} v_{n,i} - p\| \\
&\leq \beta_{n,0} \|x_n - p\| + \sum_{i=1}^{m} \beta_{n,i} \|v_{n,i} - p\| \\
&\leq \beta_{n,0} \|x_n - p\| + \sum_{i=1}^{m} \beta_{n,i} H(T_i x_n, p) \\
&\leq \beta_{n,0} \|x_n - p\| + \sum_{i=1}^{m} \beta_{n,i} \|x_n - p\| \\
&= \|x_n - p\|. \qquad (7)
\end{aligned}$$

Thus, (6) becomes

$$\|x_{n+1} - p\| \leq \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{m} \alpha_{n,i}\|x_n - p\| = \sum_{i=0}^{m} \alpha_{n,i}\|x_n - p\| = \|x_n - p\|. \tag{8}$$

Therefore, the sequence $\{\|x_n - p\|\}$ is non-increasing and bounded below.

Hence, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \bigcap_{i=1}^{m} A_S(T_i)$, which implies that $\{x_n\}$ is bounded. □

Lemma 3.6. *Let E be a uniformly convex Banach space and K be a nonempty convex subset of E. For each $i = 1, 2, \ldots, m$, let $T_i : K \to P(K)$ be a multivalued mapping such that $\bigcap_{i=1}^{m} A_S(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined in* (2) *such that $\liminf_{n\to\infty} \sum_{i=1}^{m} \alpha_{n,i}(\beta_{n,0}\beta_{n,j}) > 0$, for all $j = 1, 2, \ldots, m$. Then, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, for all $i = 1, 2, \ldots, m$.*

Proof. Let $p \in \bigcap_{i=1}^{m} A_S(T_i)$. From Lemma 3.5, we suppose that $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c > 0$. By Lemma 2.1, there exists a continuous strictly increasing convex function $g_1 : [0, \infty) \to [0, \infty)$ with $g_1(0) = 0$. For all $j \in \{1, 2, \ldots, m\}$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_{n,0}(x_n - p) + \sum_{i=1}^{m} \alpha_{n,i}(u_{n,i} - p)\|^2 \\
&\leq \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_{n,i}\|u_{n,i} - p\|^2 - (\alpha_{n,0}\alpha_{n,j})g_1(\|x_n - u_{n,j}\|) \\
&\leq \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_{n,i} H(T_i y_n, p)^2 \\
&\leq \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_{n,i}\|y_n - p\|^2. \qquad (9)
\end{aligned}$$

Using Lemma 2.1 again, there exists a continuous strictly increasing convex function $g_2 : [0, \infty) \to [0, \infty)$ with $g_2(0) = 0$. For all $j \in \{1, 2, \ldots, m\}$, we obtain

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_{n,0}(x_n - p) + \sum_{i=1}^{m} \beta_{n,i}(v_{n,i} - p)\|^2 \\
&\leq \beta_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{m} \beta_{n,i}\|v_{n,i} - p\|^2 - (\beta_{n,0}\beta_{n,j})g_2(\|(x_n - p) - (v_{n,j} - p)\|) \\
&\leq \beta_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{m} \beta_{n,i} H(T_i x_n, p)^2 - (\beta_{n,0}\beta_{n,j})g_2(\|x_n - v_{n,j}\|) \\
&\leq \beta_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{m} \beta_{n,i}\|x_n - p\|^2 - (\beta_{n,0}\beta_{n,j})g_2(\|x_n - v_{n,j}\|) \\
&\leq \|x_n - p\|^2 - (\beta_{n,0}\beta_{n,j})g_2(\|x_n - v_{n,j}\|). \qquad (10)
\end{aligned}$$

By (9) and (10), we have

$$\|x_{n+1} - p\|^2 \leq \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_{n,i}\|x_n - p\|^2 - \sum_{i=1}^{m} \alpha_{n,i}(\beta_{n,0}\beta_{n,j})g_2(\|x_n - v_{n,j}\|).$$

Hence,

$$\sum_{i=1}^{m} \alpha_{n,i}(\beta_{n,0}\beta_{n,j})g_2(\|x_n - v_{n,j}\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Taking limit $n \to \infty$ on both sides, we get

$$\lim_{n\to\infty} g_2(\|x_n - v_{n,j}\|) = 0.$$

By property of the function g_2, we get

$$\lim_{n\to\infty} \|x_n - v_{n,j}\| = 0, \quad \forall j = 1, 2, \ldots, m.$$

This implies that,

$$\lim_{n\to\infty} d(x_n, T_j x_n) \le \lim_{n\to\infty} \|x_n - v_{n,j}\| = 0, \ \ \forall j = 1, 2, \ldots, m.$$

Therefore, we have $\lim_{n\to\infty} d(x_n, T_j x_n) = 0$ for all $j = 1, 2, \ldots, m$. □

Now, we present a strong convergence theorem using algorithm (2).

Theorem 3.7. *Let E be a real uniformly convex Banach space and K a nonempty convex subset of E. For each $i = 1, 2, \ldots, m$, let $T_i : K \to P(K)$ be a multivalued nonexpansive mapping such that $\bigcap_{i=1}^{m} A_S(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined in* (2) *such that $\liminf_{n\to\infty} \sum_{i=1}^{m} \alpha_{n,i}(\beta_{n,0}\beta_{n,j}) > 0$, for all $j = 1, 2, \ldots, m$. Then, $\{x_n\}$ converges strongly to a point of $\bigcap_{i=1}^{m} A_S(T_i)$ if and only if $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = 0$.*

Proof. Suppose that $x_n \to p \in \bigcap_{i=1}^{m} A_S(T_i)$. Then, for each $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\|x_n - p\| < \varepsilon \quad \text{for all } n \ge m_0.$$

Therefore, we obtain that

$$d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = \inf\{\|x_n - q\| : q \in \bigcap_{i=1}^{m} A_S(T_i)\} \le \|x_n - p\| \le \varepsilon, \ \ \forall n \ge m_0.$$

If follows that $\lim_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = 0$, and hence

$$\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = 0.$$

Conversely, assume that $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = 0$ and $p \in \bigcap_{i=1}^{m} A_S(T_i)$.

This means that $\{d(x_n, \bigcap_{i=1}^{m} A_S(T_i))\}$ contains a subsequence $\{d(x_{n_k}, \bigcap_{i=1}^{m} A_S(T_i))\}$ such that

$$\lim_{k\to\infty} d(x_{n_k}, \bigcap_{i=1}^{m} A_S(T_i)) = 0. \tag{11}$$

By Lemma 3.5, we assume that $\lim_{n\to\infty} \|x_n - p\| = L_p$ for some $L_p \geq 0$.
Then for each $\varepsilon > 0$, there exists $m_1 \in \mathbb{N}$ such that

$$L_p - \varepsilon \leq \|x_n - p\| \leq L_p + \varepsilon, \quad \forall n \geq m_1.$$

Taking infimum all over $p \in \bigcap_{i=1}^{m} A_S(T_i)$ on both sides, we get

$$L - \varepsilon \leq d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) \leq L + \varepsilon, \quad \forall n \geq m_1,$$

where $L = \inf\{L_p : p \in \bigcap_{i=1}^{m} A_S(T_i)\}$. This implies that

$$\lim_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = L. \tag{12}$$

By using (11) and (12), we can conclude that

$$\lim_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = 0.$$

Next, we want to show $\{x_n\}$ is a Cauchy sequence in a Banach space E. Since (8), $\|x_{n+1} - p\| \leq \|x_n - p\|$ for all $p \in \bigcap_{i=1}^{m} A_S(T_i)$. In particular, for $k > n$ we have

$$\|x_k - p\| \leq \|x_n - p\| \quad \text{for all } p \in \bigcap_{i=1}^{m} A_S(T_i).$$

Consider

$$\|x_n - x_k\| \leq \|x_n - p\| + \|p - x_k\| \leq 2\|x_n - p\|, \ \forall p \in \bigcap_{i=1}^{m} A_S(T_i).$$

Taking infimum all over $p \in \bigcap_{i=1}^{m} A_S(T_i)$ on both sides, we obtain

$$\|x_n - x_k\| \leq 2\inf\{\|x_n - p\| : p \in \bigcap_{i=1}^{m} A_S(T_i)\} = 2d(x_n, \bigcap_{i=1}^{m} A_S(T_i)).$$

Since $\lim_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = 0$, we can conclude that $\{x_n\}$ is a Cauchy sequence in a Banach space E. As a result, there exists $z \in E$ such that $\lim_{n\to\infty} \|x_n - z\| = 0$. By Lemma 3.6, we have $\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad \forall i = 1, 2, \ldots, m$.
For each i, consider

$$d(z, T_i x_n) \leq \|z - x_n\| + d(x_n, T_i x_n).$$

Therefore, $\lim_{n\to\infty} d(z, T_i x_n) = 0$.
Next, we will show that $z \in \bigcap_{i=1}^{m} A_S(T_i)$. Consider

$$\begin{aligned} H(z, T_i x) &\leq H(z, T_i x_n) + H(T_i x_n, T_i x), \\ &\leq H(z, T_i x_n) + \|x_n - x\|, \qquad \forall x \in K. \end{aligned}$$

By taking limit $n \to \infty$ on both sides, we infer that

$$H(z, T_i x) \leq \|z - x\|, \qquad \forall x \in K,\ \forall i = 1, 2, \ldots, m.$$

Therefore, $z \in \bigcap_{i=1}^{m} A_S(T_i)$. □

We also obtain the following fixed point result as a consequence of Theorem 3.7.

Corollary 3.8. *Let E be a real uniformly convex Banach space and K a nonempty convex subset of E. For each $i = 1, 2, \ldots, m$, let $T_i : K \to P(K)$ be a multivalued nonexpansive mapping such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined in* (2) *such that $\liminf_{n\to\infty} \sum_{i=1}^{m} \alpha_{n,i}(\beta_{n,0}\beta_{n,j}) > 0$, for all $j = 1, 2, \ldots, m$.*

(1) *Suppose that $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} F(T_i)) = 0$ or $\limsup_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} F(T_i)) = 0$. Then, $\{x_n\}$ converges strongly to $z \in \bigcap_{i=1}^{m} A_S(T_i)$. Additionally, if K is closed, then $\{x_n\}$ converges strongly to $z \in \bigcap_{i=1}^{m} F(T_i)$.*

(2) *Suppose that $\{x_n\}$ converges strongly to $z \in \bigcap_{i=1}^{m} A_S(T_i)$, then $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = 0$. Additionally, if K is closed, then $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} F(T_i)) = 0$.*

Proof. Since $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, we have T_i is quasi-nonexpansive mappings for all $i \in \{1, 2, \ldots, m\}$. By Lemma 3.4, we have

$$\bigcap_{i=1}^{m} A_S(T_i) \cap K = \bigcap_{i=1}^{m} F(T_i),$$

which implies that, $\bigcap_{i=1}^{m} A_S(T_i) \neq \emptyset$.

(1) Assume that

$$\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} F(T_i)) = 0.$$

Since $\bigcap_{i=1}^{m} F(T_i) \subset \bigcap_{i=1}^{m} A_S(T_i)$, we have

$$d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) \leq d(x_n, \bigcap_{i=1}^{m} F(T_i)), \quad \forall n \in \mathbb{N}.$$

Then,

$$\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) \leq \liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} F(T_i)) = 0.$$

From Theorem 3.7, we have $x_n \to z \in \bigcap_{i=1}^{m} A_S(T_i)$. Moreover, if K is closed, then $z \in K$. As a result, $x_n \to z \in \bigcap_{i=1}^{m} F(T_i)$.

(2) Assume that $x_n \to z \in \bigcap_{i=1}^{m} A_S(T_i)$. From Theorem 3.7, we have

$$\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = 0.$$

Moreover, if K is closed, then $z \in K$. Since $\bigcap_{i=1}^{m} A_S(T_i) \cap K = \bigcap_{i=1}^{m} F(T_i)$, we obtain $x_n \to z \in \bigcap_{i=1}^{m} F(T_i)$. As a result, $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} F(T_i)) = 0$. □

We now present a convergence theorem for a finite family of multivalued nonexpansive mappings satisfying condition (A) defined as:

Definition 3.9. *For each $i \in \{1, 2, \dots, m\}$, mapping $T_i : K \to P(K)$ is said to satisfy condition (A) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that*

$$f(d(x, \bigcap_{i=1}^{m} A_S(T_i))) \leq d(x, T_i x),$$

for all $x \in K$.

Theorem 3.10. *Let E be a real uniformly convex Banach space and K a nonempty convex subset of E. For each $i = 1, 2, \dots, m$, let $T_i : K \to P(K)$ be a multivalued nonexpansive mapping such that $\bigcap_{i=1}^{m} A_S(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined in (2) such that $\liminf_{n\to\infty} \sum_{i=1}^{m} \alpha_{n,i}(\beta_{n,0}\beta_{n,j}) > 0$, for all $j = 1, 2, \dots, m$. Suppose that one of T_i satisfies condition (A). Then, $\{x_n\}$ converges strongly to a point of $\bigcap_{i=1}^{m} A_S(T_i)$.*

Proof. By Lemma 3.6, we have $\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad \forall i = 1, 2, \dots, m$. Suppose that T_{i_0} satisfies condition (A) for some $i_0 \in \{1, 2, \dots, m\}$, we obtain that there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$f(d(x_n, \bigcap_{i=1}^{m} A_S(T_i))) \leq d(x_n, T_{i_0} x_n), \quad \forall n \in \mathbb{N}.$$

It follows that

$$0 \leq \lim_{n\to\infty} f(d(x_n, \bigcap_{i=1}^{m} A_S(T_i))) \leq \lim_{n\to\infty} d(x_n, T_{i_0} x_n) = 0.$$

Thus,

$$\lim_{n\to\infty} f(d(x_n, \bigcap_{i=1}^{m} A_S(T_i))) = 0.$$

Due to the non-decreasing function of f and $f(0) = 0$, we get

$$\lim_{n\to\infty} d(x_n, \bigcap_{i=1}^{m} A_S(T_i)) = 0.$$

By Theorem 3.7, we conclude that $\{x_n\}$ converges strongly to a point of $\bigcap_{i=1}^{m} A_S(T_i)$. □

Corollary 3.11. *Let E be a real uniformly convex Banach space and K a nonempty convex subset of E. For each $i = 1, 2, \ldots, m$, let $T_i : K \to P(K)$ be a multivalued nonexpansive mapping such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined in (2) such that $\liminf_{n\to\infty} \sum_{i=1}^{m} \alpha_{n,i}(\beta_{n,0}\beta_{n,j}) > 0$, for all $j = 1, 2, \ldots, m$. Suppose that one of T_i satisfies condition (A), then $\{x_n\}$ converges strongly to a point of $\bigcap_{i=1}^{m} A_S(T_i)$. Additionally, If K is closed, then $\{x_n\}$ converges strongly to $z \in \bigcap_{i=1}^{m} F(T_i)$.*

Proof. Since $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, we have T_i is quasi-nonexpansive mappings for all $i \in \{1, 2, \ldots, m\}$. By Lemma 3.4, we have $\bigcap_{i=1}^{m} A_S(T_i) \cap K = \bigcap_{i=1}^{m} F(T_i)$. It follows that $\bigcap_{i=1}^{m} A_S(T_i) \neq \emptyset$. By Theorem 3.10, we have $x_n \to z \in \bigcap_{i=1}^{m} A_S(T_i)$. If K is closed, then $z \in \bigcap_{i=1}^{m} F(T_i)$. □

Next, we will present a convergence theorem for a finite family of multivalued nonexpansive mappings satisfying hemicompact defined as:

Definition 3.12. *For each $i \in \{1, 2, \ldots, m\}$, mapping $T_i : K \to P(K)$ is called hemicompact if for any sequence $\{x_n\}$ in K such that $d(x_n, T_i x_n) \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.*

Moreover, using the hemicompactness of the mappings, we obtain the following.

Theorem 3.13. *Let E be a real uniformly convex Banach space and K a nonempty convex subset of E. For each $i = 1, 2, \ldots, m$, let $T_i : K \to P(K)$ be a multivalued nonexpansive mapping with $\bigcap_{i=1}^{m} A_S(T_i) \neq \emptyset$. Suppose that one of T_i is hemicompact. Let $\{x_n\}$ be the sequence as defined in (2) such that $\liminf_{n\to\infty} \sum_{i=1}^{m} \alpha_{n,i}(\beta_{n,0}\beta_{n,j}) > 0$, for all $j = 1, 2, \ldots, m$. Then, $\{x_n\}$ converges strongly to a point of $\bigcap_{i=1}^{m} A_S(T_i)$.*

Proof. By Lemma 3.6, we have the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad \forall i = 1, 2, \ldots, m.$$

Suppose T_{i_0} is hemicompact, for some $i_0 \in \{1, 2, \ldots, m\}$, then there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $z \in K$ such that $\lim_{n\to\infty} \|x_{n_k} - z\| = 0$. Consider,

$$d(T_i x_{n_k}, z) \leq d(T_i x_{n_k}, x_{n_k}) + \|x_{n_k} - z\|.$$

Therefore, $\lim_{k\to\infty} d(T_i x_{n_k}, z) = 0, \quad \forall i = 1, 2, \ldots, m.$

Next, we will show that $z \in \bigcap_{i=1}^{m} A_S(T_i)$. For each $i = 1, 2, \ldots, m$, consider

$$\begin{aligned} H(z, T_i x) &\leq H(z, T_i x_{n_k}) + H(T_i x_{n_k}, T_i x), && \forall x \in K \\ &\leq H(z, T_i x_{n_k}) + \|x_{n_k} - x\|, && \forall x \in K. \end{aligned}$$

By taking limit $k \to \infty$ on both sides, we obtain

$$H(z, T_i x) \leq \|z - x\|, \qquad \forall x \in K.$$

Therefore, $z \in \bigcap_{i=1}^{m} A_S(T_i)$. Since Lemma 3.5, we get $\lim_{n\to\infty} \|x_n - z\| = 0$, we conclude that $\{x_n\}$ converges strongly to a point of $\bigcap_{i=1}^{m} A_S(T_i)$. □

Corollary 3.14. *Let E be a real uniformly convex Banach space and K a nonempty convex subset of E. For each $i = 1, 2, \dots, m$, let $T_i : K \to P(K)$ be a multivalued nonexpansive mapping with $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that one of T_i is hemicompact. Let $\{x_n\}$ be the sequence as defined in* (2) *such that $\liminf_{n\to\infty} \sum_{i=1}^{m} \alpha_{n,i}(\beta_{n,0}\beta_{n,j}) > 0$, for all $j = 1, 2, \dots, m$. Then, $\{x_n\}$ converges strongly to a point of $\bigcap_{i=1}^{m} A_S(T_i)$. Additionally, If K is closed, then $\{x_n\}$ converges strongly to $z \in \bigcap_{i=1}^{m} F(T_i)$.*

Proof. Since $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, we have T_i is quasi-nonexpansive mappings for all $i \in \{1, 2, \dots, m\}$. By Lemma 3.4, we have $\bigcap_{i=1}^{m} A_S(T_i) \cap K = \bigcap_{i=1}^{m} F(T_i)$. It follows that $\bigcap_{i=1}^{m} A_S(T_i) \neq \emptyset$. By Theorem 3.13, we obtain $x_n \to z \in \bigcap_{i=1}^{m} A_S(T_i)$. If K is closed, then $z \in \bigcap_{i=1}^{m} F(T_i)$. □

Remark 3.15. *Lemma 3.5, 3.6, Theorem 3.7, 3.10, 3.13 and Corollary 3.8, 3.11, 3.14 are valid when the sequence $\{x_n\}$ is defined in* (1) *with $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,j} > 0$ for all $j \in \{1, 2, \dots, m\}$.*

To complete this paper, we present a numerical example to support theorem 3.7 as follows.

Example 3.16. *Let $E = \mathbb{R}$ be endowed with the Euclidean norm $\|\cdot\| = |\cdot|$. Assume that $K = (0, +\infty)$ be nonempty convex subset of $\mathbb{R}$. Note that K is not closed. Define the mapping $T_i : K \to P(K)$ by $T_i x = [0, \frac{x}{4i}]$, $\forall x \in K$, $\forall i = 1, 2, 3, 4$. Then for each $i \in \{1, 2, 3, 4\}$, T_i are nonexpansive mappings such that $0 \in \bigcap_{i=1}^{4} A_S(T_i)$. We choose the parameters $v_{n,i} = \sup_{a \in T_i x_n} \{\frac{a}{2}\}$, $u_{n,i} = \sup_{b \in T_i y_n} \{\frac{b}{3}\}$, initial point $x_1 = \frac{1}{2}$. The control sequences are chosen by*

$$\beta_{n,i} = \begin{cases} \frac{n}{3n}, & i = 0 \\ \frac{n+1}{6n}, & i = 1, 3 \\ \frac{n-1}{6n}, & i = 2, 4 \end{cases} \text{ and } \alpha_{n,i} = \frac{n+i}{5n+10} \text{ for all } i = 1, ..., 4.$$

Then, we can see that $\sum_{i=0}^{4} \beta_{n,i} = 1 = \sum_{i=0}^{4} \alpha_{n,i}$ and $\liminf_{n\to\infty} \sum_{i=1}^{4} \alpha_{n,i}(\beta_{n,0}\beta_{n,j}) > 0$. Table 1 shows the values of $|x_n - 0|$, $|x_n - T_1 x_n|$, $|x_n - T_2 x_n|$, $|x_n - T_3 x_n|$ and $|x_n - T_4 x_n|$ of iteration $n = 1, 2, 3, ..., 10$.

It is evident from Table 1 that $x_n \to 0 \in \bigcap_{i=1}^{4} A_S(T_i)$, the errors $|x_n - 0| \to 0$, $|x_n - T_i x_n| \to 0$, $\forall i = 1, 2, \dots, 4$. Moreover, Figure 1 shows the convergence behavior of the modified Ishikawa iteration (2).

Table 1: Numerical results of iteration process (2)

Iteration No.	$\lvert x_n - 0\rvert$	$\lvert x_n - T_1x_n\rvert$	$\lvert x_n - T_2x_n\rvert$	$\lvert x_n - T_3x_n\rvert$	$\lvert x_n - T_4x_n\rvert$
1	5.00×10^{-1}	3.75×10^{-1}	4.38×10^{-1}	4.58×10^{-1}	4.69×10^{-1}
2	3.99×10^{-2}	2.99×10^{-2}	3.49×10^{-2}	3.66×10^{-2}	3.74×10^{-2}
3	4.51×10^{-3}	3.38×10^{-3}	3.95×10^{-3}	4.13×10^{-3}	4.23×10^{-3}
4	6.00×10^{-4}	4.50×10^{-4}	5.25×10^{-4}	5.50×10^{-4}	5.62×10^{-4}
5	8.78×10^{-5}	6.58×10^{-5}	7.68×10^{-5}	8.05×10^{-5}	8.23×10^{-5}
6	1.37×10^{-5}	1.03×10^{-5}	1.20×10^{-5}	1.25×10^{-5}	1.28×10^{-5}
7	2.23×10^{-6}	1.67×10^{-6}	1.95×10^{-6}	2.05×10^{-6}	2.09×10^{-6}
8	3.76×10^{-7}	2.82×10^{-7}	3.29×10^{-7}	3.45×10^{-7}	3.35×10^{-7}
9	6.51×10^{-8}	4.88×10^{-8}	5.69×10^{-8}	5.97×10^{-8}	6.10×10^{-8}
10	1.15×10^{-8}	8.60×10^{-9}	1.01×10^{-8}	1.05×10^{-8}	1.08×10^{-8}

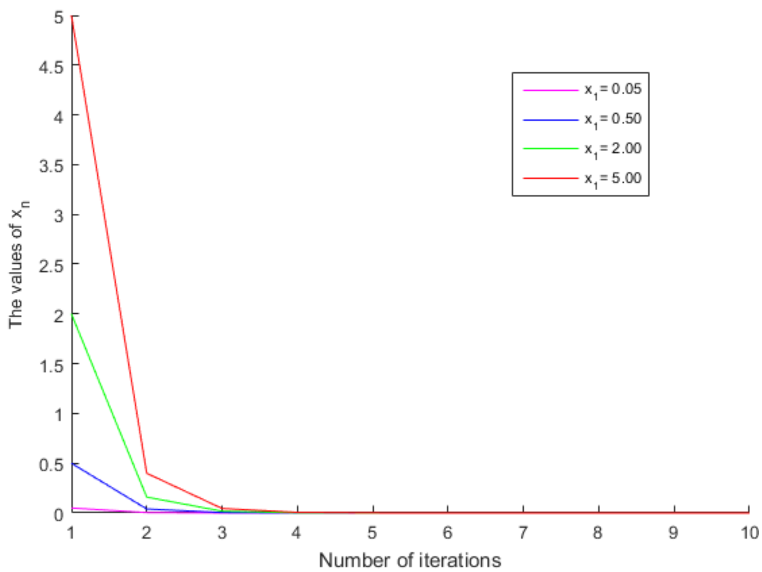

Figure 1: Convergence behavior of the iteration process (2) to an attractive point

Next, under control conditions from Theorem 3.7, we compare the rate of convergence for the sequence $\{x_n\}$ *generated by* (1) *and* (2) *which is shown in Table 2.*

From Table 2, we can see that the modified Ishikawa iteration (2) *performs with a better rate of convergence than the modified Mann iteration* (1).

4. Conclusion

In this paper, we have introduced iterative methods to approximate a solution of attractive fixed point problems for a finite family of nonlinear mappings in a uniformly convex Banach space without closedness on the domain. We have given some basic properties of attractive points and have compared them with fixed points. Further, we have shown examples and numerical results to illustrate our iteration and results.

Acknowledgment. The authors would like to thank referee(s) to complete this paper with valuable suggestions. This research was supported by Fundamental Fund 2022, Chiang Mai University.

Table 2: The values of $\{x_n\}$ for different iteration processes

Iteration No.	x_n of Iteration (1)	x_n of Iteration (2)
1	5.0000×10^{-1}	5.0000×10^{-1}
2	5.8681×10^{-2}	3.9905×10^{-2}
3	8.8632×10^{-3}	4.5103×10^{-3}
4	1.5178×10^{-3}	5.9992×10^{-4}
5	2.8038×10^{-4}	8.7794×10^{-5}
6	5.4490×10^{-5}	1.3685×10^{-5}
7	1.0983×10^{-5}	2.2309×10^{-6}
8	2.2754×10^{-6}	3.7610×10^{-7}
9	4.8160×10^{-7}	6.5100×10^{-8}
10	1.0370×10^{-7}	1.1500×10^{-8}

References

[1] W. Inthakon, Strong convergence theorems for generalized nonexpansive mappings with the system of equilibrium problems in Banach spaces, Journal of Nonlinear and Convex Analysis, **2014**, *15(4)*, 753–763.

[2] E. Karapınar, A. Fulga, M. Rashid, L. Shahid, H. Aydi, Large contractions on quasi-metric spaces with an application to nonlinear fractional differential equations, Mathematics **2019**, *7*, 444;https://doi:10.3390/math7050444.

[3] W. Inthakon, S. Suantai, P. Sarnmeta, D. Chumpungam, A new machine learning algorithm based on optimization method for regression and classification problems, Mathematics, **2020**, 8(6), 1007; https://doi.org/10.3390/math8061007.

[4] S.A.R. Hosseiniun, M. Nabiei, Some applications of fixed point theorem in economics and nonlinear functional analysis, International Mathematical Forum, **2010**, 5(49), 2407–2414.

[5] R.S. Adiguzel, ; U. Aksoy, E. Karapınar, I.M. Erhan, On the solution of a boundary value problem associated with a fractional differential equation, Mathematical Methods in the Applied Sciences, **2020**, https://doi.org/10.1002/mma.6652.

[6] N. Nanan, S. Dhompongsa, A common fixed point theorem for a commuting family of nonexpansive mappings one of which is multivalued, Fixed Point Theory and Applications, **2011**, 54, 1–10.

[7] S. Dhompongsa, N. Nanan, Fixed point theorems by ways of ultra-asymptotic centers, Abstract and Applied Analysis, **2011**, https://doi.org/10.1155/2011/826851.

[8] C. Jinakul, A. Wiwatwanich, A. Kaewkhao, Common fixed point theorem for multi-valued mappings on b-metric spaces, International Journal of Pure and Applied Mathematics, **2017**, 113(1), 167–179.

[9] N. Phudolsitthiphat, P. Charoensawan, Common fixed point results for three maps one of which is multivalued in G-metric spaces, Thai Journal of Mathematics, **2018**, 16(2), 455–465.

[10] W. Takahashi, N.C. Wong, J.C. Yao, Attractive point and mean convergence theorems for new generalized nonspreading mappings in banach spaces, Infinite Products of Operators and their Applications, **2015**, 636, 225–248.

[11] S. Dhompongsa, A. Kaewkhao, A note on properties that imply the fixed point property, Abstract and Applied Analysis, **2006**, https://doi.org/10.1155/AAA/2006/34959.

[12] S. Dhompongsa, A. Kaewcharoen, A. Kaewkhao, Fixed point property of direct sums, Nonlinear Analysis, **2005**, 63, 2177–2188.

[13] H. Aydi, M.F. Bota, E. Karapınar, S. Mitrović, A fixed point theorem for set-valued quasi-contractions in b-metric spaces, Fixed Point Theory and Applications, **2012**, 2012(1), 1–8.

[14] Panyanak, B. Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces. *Computers & Mathematics with Applications* **2007**, *54(6)*, 872–877.

[15] S.H. Khan, I. Yildirim, Fixed points of multivalued nonexpansive mappings in Banach spaces, Fixed Point Theory and Applications, **2012**, 73, 1–9.

[16] P. Sunthrayuth, P. Kumam, A new composite general iterative scheme for nonexpansive semigroups in banach spaces, International Journal of Mathematics and Mathematical Sciences, **2011**, 2011, 1–18.

[17] W. Takahashi, N.C. Wong, J.C. Yao, Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces, Journal of Nonlinear and Convex Analysis, **2012**, 13, 745–757.

[18] P. Sunthrayuth, P. Kumam, Fixed point solutions of variational inequalities for a semigroup of asymptotically nonexpansive mappings in Banach spaces, Fixed Point Theory and Applications, **2012**, 177, 1–18.

[19] Y. Song, H. Wang, Convergence of iterative algorithms for multivalued mappings in Banach spaces, Nonlinear Analysis, **2009**, 70(4), 1547–1556.

[20] Chen, L.; Yang, N.; Zhou, J. Common attractive points of generalized hybrid multi-valued mappings and applications. *Mathematics* **2020**, *8(8)*, 1307; https://doi.org/10.3390/math8081307.

[21] N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Analysis, **2009**, 71, 838–844.

[22] K.P.R. Sastry, G.V.R. Babu, Convergence of Ishikawa iterates for a multivalued mapping with a fixed point, Czechoslovak Mathematical Journal, **2005**, 55, 817–826.

[23] W. Takahashi, Y. Takeuchi, Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space, Journal of Nonlinear and Convex Analysis, **2011**, 12(2), 399–406.

[24] P. Kocourek, W. Takahashi, J.C. Yao, Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces, Taiwanese Journal of Mathematics, **2010**,14(6), 2497–2511.

[25] M. Farid, Weak convergence to common attractive points of finite families of nonexpansive mappings, JMI International Journal of Mathematical Sciences, **2018**, 9, 41–50.

[26] S.H. Khan, Iterative approximation of common attractive points of further generalized hybrid mappings, Fixed Point Theory and Applications, **2018**, 8, 1–10.

[27] P. Thongphaen, W. Inthakon, Common attractive point theorems of widely more generalized hybrid mappings in Hilbert spaces, Thai Journal of Mathematics, **2020**,18(3), 861–869.

[28] Thongphaen, C.; Inthakon, W.; Suantai, S.; Phudolsitthiphat, N. Common attractive point results for two generalized nonexpansive mappings in uniformly convex Banach spaces. *Mathematics* **2022**, *10*, 1275;https://doi.org/10.3390/math10081275.

[29] S.B. Nadler, Multivalued contraction mappings, Pacific Journal of Math, **1969** 30, 475–488.

[30] J, Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bulletin of the Australian Mathematical Society, **1991**, 43(1), 153–159.

[31] S. Plubtieng, K. Ungchittrakool, Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach spaces, Fixed Point Theory and Applications, **2008** 82, 1–19.

[32] S.S. Zhang, Generalized mixed equilibrium problem in Banach spaces, Applied Mathematics and Mechanics, **2009** 30, 1105–1112.