Süleyman Demirel Üniversitesi Fen Edebiyat Fakültesi Fen Dergisi
Süleyman Demirel University Faculty of Arts and Sciences Journal of Science
2022, 17(2): 288-302
DOI: 10.29233/sdufeffd.1128768
Atıf için / For Citation: S. A. Yükçü, “The numerical evaluation methods for beta function”, *Süleyman Demirel Üniversitesi Fen Edebiyat Fakültesi Fen Dergisi*, 17(2), 288-302, 2022.

Research Article

The Numerical Evaluation Methods for Beta Function

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(Alınış / Received: 10.06.2022, Kabul / Accepted: 2.08.2022, Yayımlanma / Published: 25.11.2022)

Abstract: In this study, the beta function that is encountered in computational mathematics and physics is analyzed. The correct evaluation of this function also affects the accuracy of other mathematical functions in quantum mechanical calculations. Especially in recent years, there is an interest in studies related to the beta function for zero and negative *p* and *q* integers. This study, considering the neutrix limits of the beta function, presents new relations for the numerical computation of the beta function, especially for negative integers *p* and *q*. In addition, taking into account the definition of the beta function for positive *p* and *q* integer values, an algorithm is created to calculate the function for all integer values. Finally, numerical results obtained with the help of our new recurrence relations and algorithm are presented.

***Key words**:* Beta function, Gamma function, Quantum numbers, Mathematica.

Beta Fonksiyonu için Sayısal Değerlendirme Yöntemleri

Öz: Bu çalışmada, hesaplamalı matematik ve fizikte karşılaşılan beta fonksiyonu analiz edilmiştir. Bu fonksiyonun doğru değerlendirilmesi, kuantum mekaniksel hesaplamalardaki diğer matematiksel fonksiyonların doğruluğunu da etkilemektedir. Özellikle son yıllarda, sıfır ve negatif *p* ve *q* tam sayıları için beta fonksiyonu ile ilgili çalışmalara ilgi duyulmaktadır. Bu çalışma, beta fonksiyonunun neutrix limitlerini göz önünde bulundurarak, özellikle negatif *p* ve *q* tam sayılarında beta fonksiyonunun sayısal hesaplaması için yeni bağıntılar sunmaktadır. Ayrıca pozitif *p* ve *q* tam sayı değerleri için de beta fonksiyonunun tanımı dikkate alınarak, fonksiyonun tüm tam sayı değerlerinde hesaplanması için bir algoritma oluşturulmuştur. Son olarak, yeni yineleme bağıntılarımız ve algoritmamız yardımıyla elde edilen sayısal sonuçlar sunulmuştur.

***Anahtar kelimeler**:* Beta fonksiyonu, Gama fonksiyonu, Kuantum sayıları, Mathematica.

1. Introduction

The beta function, often referred to as the Euler beta function, is special function in mathematics. This function whose first studies were made by Euler and Legendre has a specific integral definition [1]. It is used in applied and engineering mathematics, statistics and probability, and computational physics. Recently, there is a great interest in the mathematical applications of beta function, some of these studies: [2-9]. Besides, in physics, the researchers observed that many properties of the strong nuclear force are defined by beta function, based on the data which are obtained during their research at CERN (The European Organization for Nuclear Research). This is the first time Veneziano has noticed this function in string theory [10]. Additionally, in a study in

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which the beta function is associated with string theory/M-theory, it is indicated that this function is come across in the studies of physical density integrals for flat expanding isotropic universes [11]. Furthermore, there are other studies dealing with the mathematical relationship of the beta function to string theory [12,13].

The beta function can be written and obtained from the some other mathematical functions as gamma functions, binomial coefficients and factorial functions which are frequently used in works: [14-18]. In computational physics, this situation provides great advantage in molecular integral calculations. Because, the correct calculation of each of these coefficients and functions is important for accurate numerical results, and some of these works can be found in the references: [19-27]. The evaluation of definite integrals appears repeatedly in problems of mathematical physics as well as in pure mathematics. Three moderately general techniques are useful in evaluating definite integrals: contour integration, conversion to gamma or beta functions, and numerical quadrature [28]. When the beta function is considered from this point of view, it has an important place in mathematics.

The various definitions and properties for beta function are available in literature [28, 29]. The integral form of beta function is defined for positive parameters. This means that the beta function has a value for positive p and q integers. Besides, for non-positive p and q integers the integral of the beta function is divergent and its value is infinite. In these cases, for the evaluation of this integral, neutrix limits can be used. Recently, the concepts of the neutrix and neutrix limit due to van der Corput [30] have been used widely in many applications in mathematics, physics and statistics [31]. In this method, the finite value is properly obtained by neglecting the divergent parts of di vergent integral. Such an idea was devised by Hadamard and the finite integrals obtained with this method are called Hadamard type integrals [32]. Some other studies in the literature have shown that the beta function by taking the neutrix limit also has some values in cases in which that p and q are zero, negative or positive integers: [33-36]. First: one of p and q is positive integer and the other is negative integer. Second: both are negative integers. Third: one of the p and q equal to zero and the other is positive or negative integer. Fourth: both are zero.

The main purpose of this paper is to provide recurrence relations and calculation method for the beta function by taking into account the neutrix limits. Firstly, the general properties of this function and its use in quantum mechanics are given. Then, to compose the framework of the calculation method, some relations and numerical applications are presented. The mentioned expressions show that the beta function can be evaluated for all integers by knowing $B(0,0)$ and $B(0,1)$. The numerical results are obtained with the help of our algorithms. Eventually, the graphs and the tables are clearly produced to have a better understanding of the aim of this paper.

2. Material and Method

2.1 Beta function and its relation to molecular integrals

The beta function $B(p,q)$ is one of the special functions and it has an integral definition which called the Euler integral of the first kind in mathematics, it is defined by [1]:

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt \tag{1}$$

for $p>0$ and $q>0$. This function can be expressed in terms of the gamma function which is the Euler integral of second kind by the following equation [1]:

$$B\left(p,q\right)=\frac{\Gamma\left(p\right)\Gamma\left(q\right)}{\Gamma\left(p+q\right)} \tag{2}$$

where $p>0$, $q>0$ and $\Gamma\left(p\right)$ is gamma function and defined over factorial function as $\Gamma\left(p\right)=\left(p-1\right)!$. In many computational sciences, binomial coefficient is often used and its general form:

$$\binom{p}{q}=\frac{p!}{q!\left(p-q\right)!} \tag{3}$$

The combination of Equations (2) and (3), binomial coefficient can be explained in terms of beta function as:

$$\binom{p}{q}=\frac{1}{\left(p-q\right)B\left(p-q,q+1\right)} \tag{4}$$

In molecular quantum mechanical calculations, molecular integrals are appeared and their solutions are very important. At this point, a suitable basis function (orbital) must be selected. Generally, Slater type orbitals (STOs) are widely used for the solution of molecular integrals. The STOs with the *n*, *l* and *m* quantum numbers are given by

$$x_{n,l}^{m}\left(\alpha,\vec{r}\right)=\frac{\left(2\alpha\right)^{n+1/2}}{\sqrt{\left(2n\right)!}}r^{n-1}\exp\left(-\alpha r\right)Y_{l}^{m}\left(\theta,\phi\right) \tag{5}$$

where $\alpha > 0$ is the screening parameter and $Y_{l}^{m}\left(\theta,\phi\right)$ is spherical harmonics.

The overlap integrals of two STOs with screening parameters α and β are given by

$$S_{n_1 l_1 m_1}^{n_2 l_2 m_2}\left(\alpha,\beta;\vec{R}\right)=\int\left[\chi_{n_1,l_1}^{m_1}\left(\alpha,\vec{r}\right)\right]^{*}\chi_{n_2,l_2}^{m_2}\left(\beta,\vec{r}-\vec{R}\right)d^{3}\vec{r} \tag{6}$$

If we use Equation (4), the overlap integrals in Equation (22) of reference [25] have been written by beta function:

$$\begin{aligned}S_{n_1 l_1 m_1}^{n_2 l_2 m_2}\left(\alpha,\alpha;\vec{R}\right)&=\pi\frac{\left(-1\right)^{l_1}}{\left(2\alpha\right)^{3/2}}\sum_{l=l_{\min}}^{l_{\max}}{}^{(2)}\left(-1\right)^{l}\left\langle l_2 m_2\left|l_1 m_1\right|l\, m_2-m_1\right\rangle\left(n_1-l_1\right)\left(n_2-l_2\right)\\&\sqrt{n_1 n_2}\;B\left(n_1-l_1,l_1+1\right)B\left(n_2-l_2,l_2+1\right)\sqrt{B\left(n_1,n_1+1\right)B\left(n_2,n_2+1\right)}\\&\sum_{s=0}^{t}\sum_{r=0}^{L}\left(-1\right)^{s+r}2^{2(r+1)}\frac{a_s\left(l_1+1,n_1-l_1;l_2+1,n_2-l_2\right)}{\left(L-r\right)B\left(L-r,r+1\right)}\sum_{k=1}^{\mu-l}g_{\mu,k}^{l}\;\chi_{k+l,l}^{m_2-m_1}\left(\alpha,\vec{R}\right)\end{aligned} \tag{7}$$

As can be seen from the Equation (7) that overlap integrals contain beta function. Hence, the correct calculation of this function affects the accuracy of the overlap integrals. It's well known that overlap integrals are the most basic molecular integrals and all other multi-center molecular integrals can be expressed in terms of them. For example, following electric multipole moment integrals are given in terms of overlap integrals [27]:

$$M_{n_1 l_1 m_1, \nu\sigma}^{n_2 l_2 m_2}\left(\alpha,\alpha;\vec{R}_{0b},\vec{R}_{ab}\right)=\sum_{\mu=|\sigma|}^{\nu}\sum_{L=L_{\min}}^{L_{\max}} {}^{(2)}G_{\nu,\sigma,\mu,L}^{n_2,l_2,m_2}\left(\alpha,R_{0b}\right) S_{n_1,l_1,m_1}^{(\mu+n_2),\,L,\,(\sigma+m_2)}\left(\alpha,\alpha;\vec{R}_{ab}\right) \quad (8)$$

2.2 The reduction relations of beta function

A beta function can be written in terms of another and some reduction properties of it are [28]:

$$B(p,q)=B(q,p) \quad (9)$$

$$B(p,q)=B(p+1,q)+B(p,q+1) \quad (10)$$

$$B(p,q)=\frac{p+q}{q}B(p,q+1) \quad (11)$$

$$B(p,q)=\frac{q-1}{p}B(p+1,q-1) \quad (12)$$

$$B(p,q)B(p+q,t)=B(q,t)B(p,q+t) \quad (13)$$

here $t>0$. The beta function has also some properties for positive *p* and *q* integers [37]:

$$B(p,q+1)=\frac{q}{p}B(p+1,q)=\frac{q}{p+q}B(p,q) \quad (14)$$

$$B(p,1-p)=\frac{\pi}{\sin(\pi p)} \quad , \quad 0<p<1 \quad (15)$$

$$\frac{1}{B(p,q)}=q\binom{p+q-1}{p-1}=p\binom{p+q-1}{q-1} \quad (16)$$

The beta function for non-positive *p* or *q* integers is also available in the literature. Fisher, Orankitjaroe and Lin show that beta function has a finite value in case of *p* or *q* equal to zero [35]:

$$B(0,0)=0 \quad (17)$$

$$B(n,0)=-\phi(n-1) \quad (18)$$

$$B(-n,0)=-\phi(n) \quad (19)$$

for $n=1,2,\ldots$ and

$$\phi(n)=\begin{cases}0, & n=0\\ \sum_{i=1}^{n}1/i & n=1,2,\ldots\end{cases} \quad (20)$$

The another definition of beta function is [34, 36]

$$B(p,-q)=\sum_{i=0,i\neq q}^{p-1}\binom{p-1}{i}\frac{(-1)^i}{i-q} \tag{21}$$

where *q* is non-negative integer and *p* is positive integer. The another situation related to beta function is

$$B(-p,-q)=-\sum_{i=0}^{q-1}\binom{p+i}{i}\frac{1}{q-i}-\sum_{j=0}^{p-1}\binom{q+j}{j}\frac{1}{p-j} \tag{22}$$

for non-negative *p* and *q* integers [34, 36]. In addition to these, according to Fisher and Kuribayashi (1987) which cited in the work by Özçağ et al. [33], the beta function has following relations:

$$B(-p,q)=(-1)^q\frac{(q-1)!(p-q)!}{p!} \quad \text{for } q=1,2,\ldots,p, \quad p=1,2,\ldots \tag{23}$$

and

$$B(-p,q)=(-1)^p\frac{(q-1)!}{p!(q-p)!}\left[\phi(p)-\phi(q-p-1)\right] \quad \text{for } q=p+1,p+2,\ldots \tag{24}$$

and here again the symmetry property given by Equation (9) is valid for all *p* and *q* integers. These equations provide a great advantage as they allow the beta function to be calculated when *p* and *q* are zero or negative integers.

3. Methodology and Relations

The beta function can be shown in different mathematical forms. In this section, we consider to use these forms in the calculation processes of the beta function. When *p* and *q* are positive integers and with the consideration of $\Gamma(p)=(p-1)\Gamma(p-1)$ property of the gamma function, then the beta function in Equation (2) can be again written as below:

$$\begin{aligned}B(p,q)&=\frac{(p-1)\Gamma(p-1)\Gamma(q)}{(p+q-1)\Gamma(p+q-1)}\\&=\frac{(p-1)}{(p+q-1)}B(p-1,q)\end{aligned} \tag{25}$$

When the definition regarding the gamma function is made over *q*, the following expression is obtained similar to Equation (25):

$$B(p,q)=\frac{(q-1)}{(p+q-1)}B(p,q-1) \tag{26}$$

by using the definition given in Equation (2), the value of the beta function for a special case that $p=1$ and $q\geq 1$ is obtained as follows:

$$B(1,q)=\frac{\Gamma(1)\Gamma(q)}{\Gamma(q+1)}=\frac{1}{q} \tag{27}$$

It clearly appears that by using the Equations (9), (18) and (20) together, then the beta functions $B(1,0)=B(0,1)=0$. Furthermore, the any value of beta function can be again obtained by using a beta function (without using the factorial or gamma function). In the case of $p=0$ and $q\geq 2$, the beta function can be calculated via the knowing values of $B(0,0)$ and $B(0,1)$. When $p=1$ and $q\geq 1$, the beta function is obtained from Equation (27). The another is $p>1$ and $q>0$ situation, and then the value of beta function can be found from $B(1,q)$. All of these situations are summarized below:

$$B(p,q)=\begin{cases} 0\,; & p=0,\ q=0 \\ 0\,; & p=0,\ q=1 \\ B(p,q-1)+\dfrac{1}{1-q}\,; & p=0,\ q\geq 2 \\ \dfrac{1}{q}\,; & p=1,\ q\geq 1 \\ \dfrac{(p-1)}{(p+q-1)}B(p-1,q) & p>1,\ q>0 \end{cases} \tag{28}$$

We can give some examples right now. Let's compute the beta function for $p=3$ and $q=5$:

$$B(3,5)=\frac{2}{7}B(2,5)=\frac{2}{7}\frac{1}{6}B(1,5)=\frac{2}{7}\frac{1}{6}\frac{1}{5}=\frac{1}{105}$$

The another example for beta function is below:

$$\begin{aligned} B(0,5)&=B(0,4)+\frac{1}{1-5}=B(0,3)+\frac{1}{1-4}+\frac{1}{1-5}=B(0,2)+\frac{1}{1-3}+\frac{1}{1-4}+\frac{1}{1-5} \\ &=B(0,1)+\frac{1}{1-2}+\frac{1}{1-3}+\frac{1}{1-4}+\frac{1}{1-5}=0+\frac{1}{1-2}+\frac{1}{1-3}+\frac{1}{1-4}+\frac{1}{1-5} \\ &=-\frac{25}{12} \end{aligned}$$

At this point, if we know $B(0,0)=0$ and $B(0,1)=0$, all other values of $B(0,q)$ can be calculated for integers that less than zero of *q*. Thanks to this calculation method, the value of beta function can be found for negative *q* integers and all *p* integers as:

$$B(p,q)=\begin{cases} B(p,q+1)+\dfrac{1}{q}; & p=0,\ q<0 \\ B(p+1,q)+B(p,q+1); & p<0,\ q<0 \\ B(p-1,q)-B(p-1,q+1); & p>0, q<0 \text{ and } (\text{Cond.1or Cond.}\,2) \end{cases} \tag{29}$$

The third line of this equation requires two more conditions besides $p>0$ and $q<0$ to calculate. The summary of these conditions are below:

Condition 1: The third line of Equation (29) runs over and over again until it reaches as $B(0,a)$ or $B(b,0)$ values with zero. Here *a* and *b* are integers.

Condition 2: If $p+q\geq 1$ exists, condition 1 runs and finds a numerical value. Later, the beta function that is desired to be calculated is found by dividing this numerical value by $p+q$. An example illustrating this calculation method is given below:

$$\begin{aligned} B(5,-3) &\rightarrow B(4,-3)-B(4,-2) \\ &= B(3,-3)-B(3,-2)-B(3,-2)+B(3,-1) \\ &= B(2,-3)-B(2,-2)-2B(2,-2)+2B(2,-1)+B(2,-1)-B(2,0) \\ &= B(1,-3)-B(1,-2)-3B(1,-2)+3B(1,-1)+3B(1,-1)-3B(1,0)-B(2,0) \\ &= B(0,-3)-B(0,-2)-4B(0,-2)+4B(0,-1)+6B(0,-1)-6B(0,0) \\ &\quad -3B(1,0)-B(2,0) \\ &= B(0,-3)-5B(0,-2)+10B(0,-1)-6B(0,0)-3B(1,0)-B(2,0) \\ &= -\frac{11}{6}-5\left(-\frac{3}{2}\right)+10(-1)-6(0)-3(0)-(-1) \\ &= -\frac{10}{3} \end{aligned}$$

In this numerical application, the value of $B(0,-3)$, $B(0,-2)$ and $B(0,-1)$ functions are computed from the first line of Equation (29). At the same time, $B(0,0)$, $B(1,0)$ and $B(2,0)$ are also computed from the first, second and third lines of Equation (28). After this operations, the obtained value is divided by $p+q$ and finally $B(5,-3)$ has been found as:

$$B(5,-3)=\frac{-10/3}{5+(-3)}=-\frac{5}{3}$$

The symmetry property is valid for all values of the beta function which are given here. Some examples of the cases contained in Equation (29) are presented in the diagram in Figure 1. The following diagram is given for a better understanding of the our calculation principle for beta function with negative integers. Let's take an numerical calculation for $B(-2,-3)$:

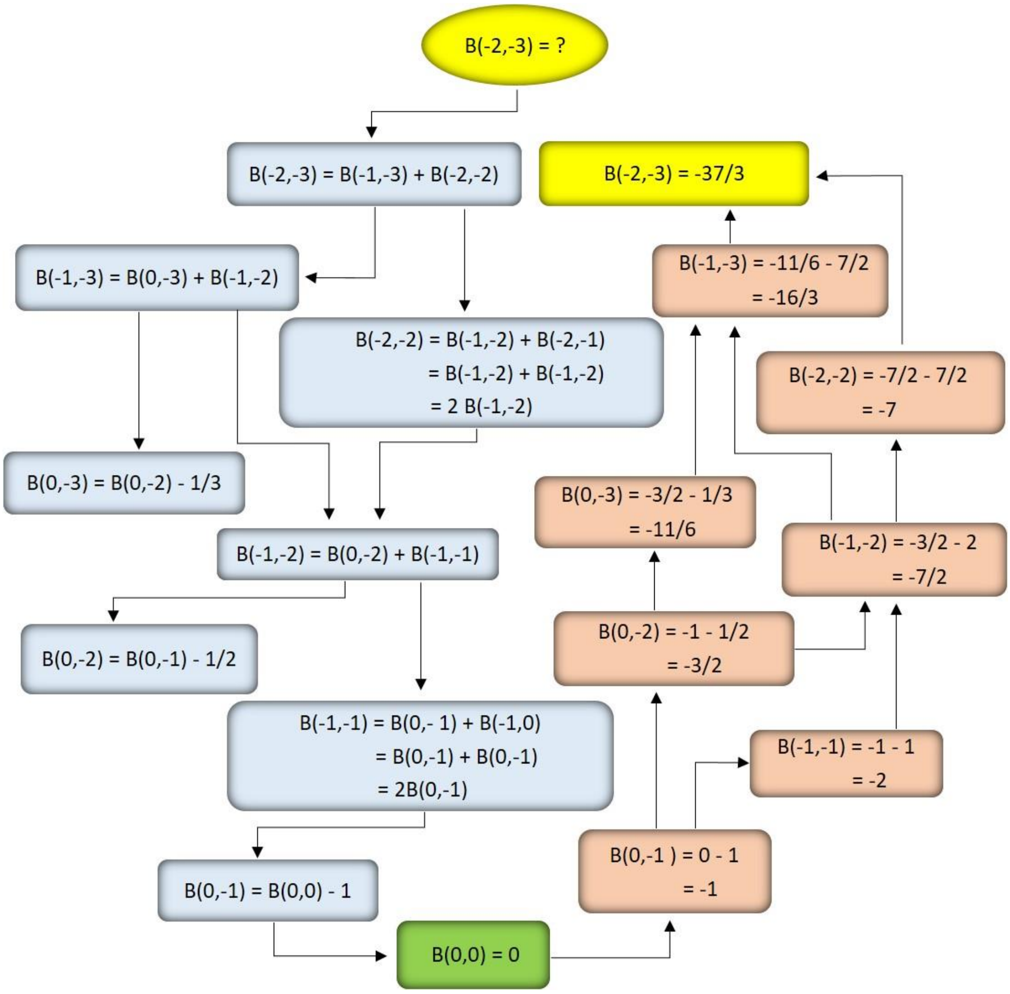

Figure 1. The diagram for the calculation of beta function for $p=-2$ and $q=-3$.

In the calculations of this diagram, first and second lines of Equation (29), and Equation (17) are used. It is clearly seen form Figure 1 that a beta function is obtained from the another beta function. The numerical calculations for many different cases are introduced in Table 1.

The beta function with $p>0$ and $q>0$ is given by the following form:

$$B(-p,q)=\frac{\Gamma(-p)\Gamma(q)}{\Gamma(-p+q)} \tag{30}$$

here, during the $-p+q<0$ or $q<p$, an indeterminate form is encountered. As *z* is a positive integer and -*z* is a simple pole function for the gamma function, the following definition exists in relation [17, 38]:

$$\Gamma(-z+\varepsilon)\approx\frac{(-1)^z}{z!}\frac{1}{\varepsilon} \tag{31}$$

When this situation is created for $-z_1$ and $-z_2$ and ε is chosen quite small, the ratio of gamma functions with negative integers is:

$$\frac{\Gamma(-z_1+\varepsilon)}{\Gamma(-z_2+\varepsilon)} = \lim_{\varepsilon\to 0} \frac{(-1)^{z_1}}{(z_1)\,!} \frac{1}{\varepsilon} \frac{(z_2)\,!}{(-1)^{z_2}} \varepsilon = (-1)^{z_1-z_2} \frac{(z_2)!}{(z_1)!} \tag{32}$$

at this point, we can use the definition given below for the ratio of gamma functions with negative integers in Equation (30):

$$\frac{\Gamma(-p)}{\Gamma(-p+q)} = (-1)^{-q} \frac{\Gamma(p-q+1)}{\Gamma(p+1)} \tag{33}$$

If this expression is used in Equation (30) and rearranged, another form of Equation (23) is obtained as:

$$B(-p,q) = (-1)^{-q} \frac{\Gamma(p-q+1)\Gamma(q)}{\Gamma(p+1)} \tag{34}$$

and then with the arrangement of this equation, following relation is achieved for the beta function:

$$B(-p,q) = (-1)^{-q} B(p-q+1,q) \tag{35}$$

Some numerical examples for this equation:

$$B(-4,1) = (-1)^{-1} B(4,1) = -\frac{1}{4}$$
$$B(-5,3) = (-1)^{-3} B(3,3) = -\frac{1}{30}$$

Even though Equation (35) is defined for $p>0$, $q>0$ and $q<p$, we confirm that this equation is also valid for $p\geq 0$, $q\geq 0$ and $q\leq p+1$. More numerical applications including these situations are presented in Table 2.

4. Results

In this section, the details of numerical calculations for beta function are given over the tables and figures. The algorithms in this work are written in Mathematica programming language by using the Intel(R) Core(TM) i5 CPU M465 @ 2.53 Ghz computer. Numerical values in Table 1 are compared by using Mathematica 10 [39], Maple 15 [40] and Matlab R2015a [41] programming languages. The values of the beta function for some *p* and *q* integers are given in Table 1. Here, Algorithm 1 includes Equations (2), (9), (17)-(20) and (22)-(24) which are based on previous works. On the other hand, Algorithm 2 also includes Equation (9), and our new Equations (28) and (29).

Table 1. Numerical results for beta function

B(p, q)	*Algorithm 1*	*Algorithm 2*	*Mathematica 10 Beta[a, b]*	*Maple15 beta(a,b)*	*Matlab R2015a beta(a,b)*
B (0, 0)	0	0	ComplexInfinity	Error	NaN
B (0, 1)	0	0	ComplexInfinity	Error	Inf
B (0, -1)	-1	-1	ComplexInfinity	Error	Error
B (-3, 0)	-11/6	-11/6	ComplexInfinity	Error	Error
B (3, 0)	-3/2	-3/2	ComplexInfinity	Error	Inf
B (-3, -2)	-37/3	-37/3	ComplexInfinity	Error	Error
B (-3, -1)	-16/3	-16/3	ComplexInfinity	Error	Error
B (-3, 1)	-1/3	-1/3	-1/3	-1/3	Error
B (-4, 3)	-1/12	-1/12	-1/12	-1/12	Error
B (2, -1)	-1	-1	ComplexInfinity	Error	Error
B (3, -1)	0	0	ComplexInfinity	Error	Error
B (2, -2)	1/2	1/2	1/2	1/2	Error
B (3, 1)	1/3	1/3	1/3	1/3	1/3
B (3, 5)	1/105	1/105	1/105	1/105	1/105
B (-5, 8)	-329/60	-329/60	ComplexInfinity	Error	Error

It is seen that the beta function which calculated using Algorithm 1 and Algorithm 2 has a finite value especially for $p \leq 0$ and $q \leq 0$. So, these two algorithms in Table 1 generate completely the same numerical results. However, Mathematica 10, Maple 15 and Matlab R2015a programming languages produce Complexinfinity, error, infinity (Inf), and not a number (NaN) results for the beta function in the same *p* and *q* integers.

Table 2 contains the results obtained from Equation (35) using Algorithm 1, Algorithm 2 and Mathematica 10. If we taking account Equation (35), beta function on the right side of this equation consists of zero and positive integers while the function on the left side includes zero and negative integers. This provides a great advantage to calculate beta function with zero and negative integers. In Table 2, computational results have been presented via Algorithm 1, Algorithm 2 and Mathematica 10 for the two sides of this equation. The table proves that Algorithm 1 and Algorithm 2 are successful to demonstrate the validity of this equation since they produce the same results.

Table 2. Some numerical results for Equation (35)

B(-p,q)	$(-1)^{-q}$ *B(p-q+1,q)*	*Algorithm 1*	*Algorithm 2*	*Mathematica 10*
B (0, 0)	*B* (1, 0)	0	0	ComplexInfinity
B (0, 1)	-*B* (0, 0)	0	0	ComplexInfinity
B (-4, 0)	*B* (5, 0)	-25/12	-25/12	ComplexInfinity
B (-4, 1)	-*B* (4, 1)	-1/4	-1/4	-1/4
B (-2, 3)	-*B* (0, 3)	3/2	3/2	ComplexInfinity
B (-4, 5)	-*B* (0, 5)	25/12	25/12	ComplexInfinity
B (-5, 3)	-*B* (3, 3)	-1/30	-1/30	-1/30
B (-7, 7)	-*B* (1, 7)	-1/7	-1/7	-1/7
B (-10, 2)	*B* (9, 2)	1/90	1/90	1/90

Figure 2 contains graphs about computational results of beta function by using Algorithm 1 and Mathematica 10. In these graphs, it is well seen that Algorithm 1 calculates more numerical values than Mathematica 10 for beta function with $q = -5$, $q = 0$, $q = 5$ and

$-25 \le p \le 25$. If we consider first graph in Figure 2 that Algorithm 1 calculates all values of beta function, however, Mathematica 10 finds only a few results on the right side of the origin. Furthermore, in the second graph only Algorithm 1 generates numerical results. In the third graph, Algorithm 1 gives more numerical results on the left side of the origin.

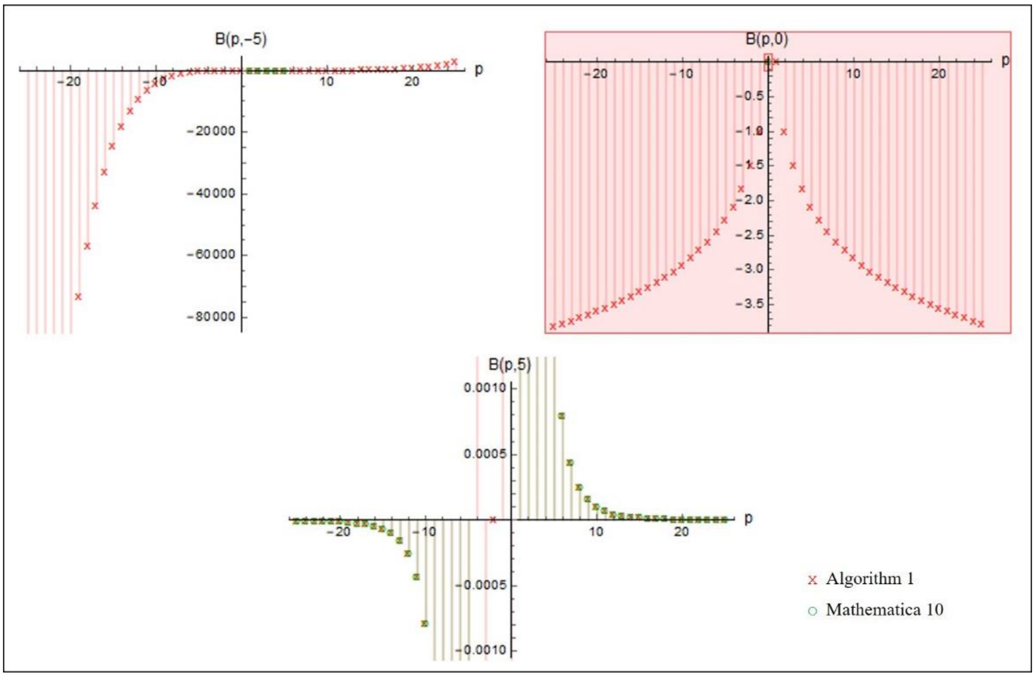

Figure 2. The graphs of beta function for $-25 \le p \le 25$ and some q values

In Figure 3, the graphs are presented for beta functions $B(p,-10)$, $B(p,0)$ and $B(p,3)$ by using both Algorithm 1 and Algorithm 2. These graphs confirm that Algorithm 1 and Algorithm 2 compute exactly the same results for a wide range of p and q integers.

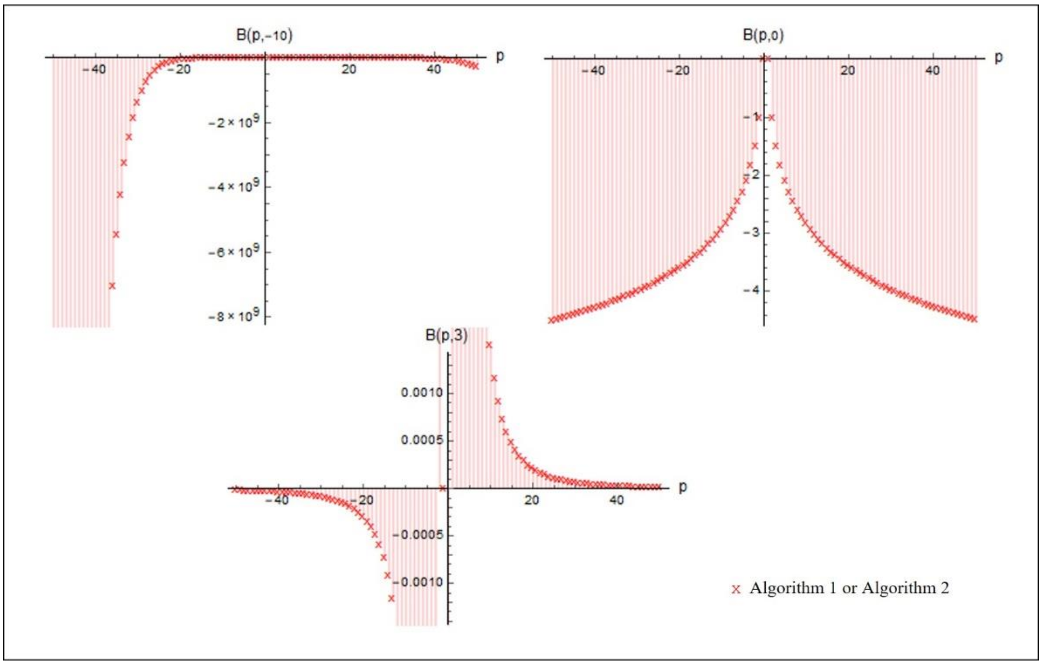

Figure 3. The graphs of the beta function for $-50 \le p \le 50$ and some q values

5. Conclusion and Comment

In this paper, we have created two algorithms for computational processes of beta function. While one of the them is following up equations from previous works, the another is using our own. At the end of this work, we reach two recurrence relations (28) and (29) that yield results consistent with the literature for the beta function with all integers.These new equations follow a different path than the equations in the previous works and they compute beta function without using the factorial and gamma function. Our new algorithm which based on these equations is quite successful to compute beta function especially in the case of negative integers. Eventually, we would like to remark that our relations and algorithm which are described in this paper are very effective and useful in areas where needed the beta function such as computational physics, applied and engineering mathematics, statistics and probability.

Appendix. The algorithms

This section describes the details of two algorithms that contain the Mathematica codes of calculations. These are presented here:

Algorithm 1:

```
fi[a_]:= Module[{aa},
   If[a==0,bn=0,tot=0;
     For[ix=1,ix<=a,ix++,
          tot=tot +(1/ix)];
    bn=tot];
   aa=bn];
betaf[k_,l_]:=Module[{jj},
   s=Max[k,l];
   c=Min[k,l];
   If[And[s==0,c==0],bet=0,If[And[c==0,s>=1],bet=-fi[s-1]]];
   If[And[c<0,s==0],bet=-fi[Abs[c]]];
   If[And[s>0,c>0],bet=Gamma[s]*Gamma[c]/Gamma[s+c]];
   If[And[s>0,c<0],
     If[s<=Abs[c],bet=Power[-1,s]*((s-1)!*(Abs[c]-s)!)/Abs[c]!,
           bet=Power[-1,Abs[c]]*(s-1)!/(Abs[c]!*(s-Abs[c])!)*
               (fi[Abs[c]]-fi[s-Abs[c]-1])]];
   If[And[s<0,c<0],
               toti=0;
               For[i=0,i<=Abs[s]-1,i++,
                    coef=-Binomial[Abs[c]+i,i]/(Abs[s]-i);
               toti=toti+coef];
               totj=0;
               For[j=0,j<=Abs[c]-1,j++,
                    coef=-Binomial[Abs[s]+j,j]/(Abs[c]-j);
                    totj=totj+coef];
      bet=toti+totj];
   jj=bet];
p=Input["Please enter integer p"];q=Input["Please enter integer
q"];
Print["p=",p,"  q=",q];
Print["Beta Function (Algorithm 1)=", betaf[p, q]];
Print["Beta Function (Mathematica)=", Beta[p, q]];
p = -3   q = -2
```

Beta Function (Algorithm 1)= $-\frac{37}{3}$

Beta Function (Mathematica)= ComplexInfinity

Algorithm 2:

```
betaf[px_,qx_]:= Module[{dd},
   m=Max[px,qx];
   n=Min[px,qx];
   If[px<qx,m=qx;n=px];
   If[m>=0,kup=m;
           If[n>=0,lup=-n,lup=n],
                      kup=Abs[m];
                      lup=n+1];
   f=(Abs[m]+1)*(Abs[n]+1);
   Array[cx,f];
   If[And[n==0,Or[m==0,m==1]],bet=0,
    cx[0,0]=0;
    cx[0,1]=0;
    For[k=0,k<=kup,k++,
     For[l=0,l>=lup,l--,
      If[m>=0,
           If[n>=0,
                a=Min[k,Abs[l]];
                b=Max[k,Abs[l]];
                If[a==0,
                      If[b>1,cx[a,b]=cx[a,b-1]+1/(1-b)],
                           If[a==1,cx[a,b]=1/b,
                                 cx[a,b]=(a-1)/(a+b-1)*cx[a-
                           1,b]]],
                a=k;
                b=l;
                If[a==0,If[b=!=0,cx[a,b]=cx[a,b+1]+1/b;
                                      cx[a,b]=cx[a,b]],
                If[b==0,cx[a,b+1]=1/a];
                cx[a,b]=cx[a-1,b]-cx[a-1,b+1]]],
           a=-k;
           b=l-1;
           If[a==0,If[b=!=0,cx[a,b]=cx[a,b+1]+1/b;
                           cx[b,a]=cx[a,b]],
                      cx[a,b]=cx[a+1,b]+cx[a,b+1]]];
           beta=cx[a,b];
           If[And[k==kup,l==lup],bet=beta]]]];
   If[And[m>0,n<0],
    If[m+n>0,
     bet=cx[m,n]/(m+n)]];
   dd=bet];
p=Input["Please enter integer p"];q=Input["Please enter integer
q"];
Print["p =",p,"  q =",q];
Print["Beta Function (Algorithm 2)=", betaf[p, q]];
Print["Beta Function (Mathematica)=", Beta[p, q]];
p = -3   q = -2
```

```
Beta Function (Algorithm 2)= -37/3
Beta Function (Mathematica)= ComplexInfinity
```

Author Statement

Sılay Aytaç Yükçü : Conceptualization, Methodology, Software, Validation, Investigation, Visualization, Review and Editing.

Acknowledgment

As the authors of this study, we declare that we do not have any support and thank you statement.

Conflict of Interest

As the authors of this study, we declare that we do not have any conflict of interest statement.

Ethics Committee Approval and Informed Consent

As the authors of this study, we declare that we do not have any ethics committee approval and/or informed consent statement.

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