Identification of a time-dependent diffusivity coefficient in heat-like space-time fractional differential equations

Mine Aylin Bayrak[1,*], Ali Demir[1]

[1] *Department of Mathematics, Kocaeli University, Izmit, Kocaeli, Turkey*

Abstract

The goal of this research is to reveal the unknown time dependent diffusion coefficient in space-time fractional differential equations by means of fractional Taylor series method. Unlike most methods used in inverse problems, using no over-measured data is a substantial advantage of this method. As a result, the unknown diffusion coefficient could be determined with high precision. Illustrative examples shows that the retrieved unknown coefficient and the solution of the problem are in a high agreement with the exact solution of the corresponding the inverse problems.

Keywords: Space-time fractional partial differential equations, Fractional Taylor series method, Inverse problems, Heat equation, Thermal diffusivity.

1. Introduction

Nonlocal properties of fractional derivatives make fractional differential equations a substantial tool in modelling of diverse processes. Therefore this subject gains a growing attention of scientist in various research areas [1–12]. As a result, determination of unknown parameters in fractional differential equations with or without additional measured data becomes one of the main challenges in inverse problems [13–16].
In this research, our focus is on establishing time dependent diffusivity coefficient and the solution of the mathematical problem including heat-like differential equation by employing fractional Taylor series method. Unlike many methods in inverse problems, this method does not require any local or nonlocal over-measured data. Hence, this prohibits the error in determination of unknown coefficient and solution. Moreover, having a Dirichlet boundary condition at the final point is enough for acquiring the unknown

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coefficient. The other boundary and initial conditions ensure the uniqueness of the unknown coefficient of a given heat-like differential equations [17]. All these properties make this method much more valuable.
The main goal in this article is to reveal the unknown coefficient of the following governing space-time fractional heat equation:

$$D_t^\alpha u(x,t) = a(t) D_x^\beta (D_x^\beta u(x,t)), 0 < x < l, 0 < t < T, 0 < \alpha, \beta \leqslant 1, \tag{1}$$

where $u(x,t)$ and $a(t) > 0$ represent the temperature and thermal diffusivity, respectively. Associated to (1) the prescribed initial condition is

$$u(x,0) = \varphi(x), 0 \leqslant x \leqslant l, \tag{2}$$

and the prescribed Dirichlet boundary conditions are

$$u(0,t) = \mu_1(t), 0 < t \leqslant T, \tag{3}$$

$$u(1,t) = \mu_2(t), 0 < t \leqslant T. \tag{4}$$

Having the condition $a(t) > 0$ makes the problem (1)-(4) well-posed.
$u(x,t)$ must be continuously differentiable with respect to time variable t and two times continuously differentiable with respect to space variable x, i.e.

$$W_2^1[0,T] \cap W_2^2[0,l]$$

where the Banach spaces $W_2^1[0,T]$ and $W_2^2[0,l]$ are defined as follows:

$$\begin{aligned} W_2^1[0,T] &= \{u = u(.,t) : u, u' \epsilon AC[0,T]\}, \\ W_2^2[0,l] &= \{u = u(x,.) : u, u' \epsilon AC[0,l]\}, \end{aligned} \tag{5}$$

Notice that, $AC[0,T]$ and $AC[0,l]$ denote the space of absolutely continuous functions. Moreover, the diffusivity $a(t)$ belong to space of continuous function $C[0,T]$.

2. Preliminaries

Essential concepts and features of fractional derivatives are presented in this section [1–4].
Definition 1. The Riemann-Liouville fractional integral of order α ($\alpha \geqslant 0$) is

given as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt,\ \alpha > 0,\ x > 0, \tag{6}$$

$$J^0 f(x) = f(x). \tag{7}$$

Definition 2. The α^{th} order derivative of $u(x,t)$ in Liouville-Caputo sense is given as

$$D_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} \frac{\partial^n u(x,\xi)}{\partial t^n} d\xi, & n-1 < \alpha < n, \\ \frac{\partial^n u(x,t)}{\partial t^n} & , \alpha = n \in N. \end{cases} \tag{8}$$

Definition 3. An (α, β)-fractional Taylor series is defined as follows [18]:

$$\sum_{i+j=0}^{\infty} g_{i,j} t^{i\alpha} x^{j\beta} = \underbrace{g_{0,0}}_{i+j=0} + \underbrace{g_{1,0} t^{\alpha} + g_{0,1} x^{\beta}}_{i+j=1} + ... + \underbrace{\sum_{k=0}^{n} g_{n-k,k} t^{(n-k)\alpha} x^{k\beta}}_{i+j=n} + ... \tag{9}$$

where $g_{i,j}, i, j \epsilon N$ are the coefficients of the series.
We have the following fractional form of Taylor's formula that is related to (9)

$$\begin{aligned} u(x,t) &= \sum_{i+j=0}^{\infty} \frac{D_t^{i\alpha} D_x^{j\beta} (u(x,t))|_{(x,t)=(0,0)}}{\Gamma(i\alpha+1)\Gamma(j\beta+1)} t^{i\alpha} x^{j\beta} \\ &= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{D_t^{i\alpha} D_x^{j\beta} (u(x,t))|_{(x,t)=(0,0)}}{\Gamma(j\beta+1)} x^{j\beta} \right) \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \\ &= \sum_{i=0}^{\infty} u_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha+1)}, \qquad x \in I \subset R, \qquad 0 \leqslant t < R, \end{aligned} \tag{10}$$

where $u_i(x)$ are the coefficients of the series (10) and R is the radius of convergence. Hence approximation of $u(x,t)$ can be rewritten as

$$u_n(x,t) = \sum_{i=0}^{n} u_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \tag{11}$$

Lemma 4. Let $u(x,t)$ has a fractional Taylor series representation as (9) for $(x,t)\epsilon[0,R_x)\times[0,R_t)$. If $D_t^{r\alpha}D_x^{s\beta}u(x,t)\epsilon\mathcal{C}((0,R_x)\times(0,R_t))$ for $r,s\epsilon N$, then

$$D_t^{r\alpha}u(x,t)=\sum_{i+j=0}^{\infty}g_{i+r,j}\frac{\Gamma((i+r)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta} \tag{12}$$

$$D_x^{s\beta}u(x,t)=\sum_{i+j=0}^{\infty}g_{i,j+s}\frac{\Gamma((j+s)\beta+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}t^{i\alpha}x^{j\beta} \tag{13}$$

Theorem 5. If there exists a constant $0<\gamma<1$ such that

$$\|u_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\|\leqslant\frac{\gamma^n}{1-\gamma}\|u_0(x)\|, n\in N, x\in I\subset R, 0\leqslant t<R, \tag{14}$$

then, the sequence of approximate solution (11) converges to the exact solution.

Proof. For all $x\in I, 0\leqslant t<R$, we have

$$\begin{aligned}\|u_{n+1}(x,t)-u_n(x,t)\|&=\|u_{n+1}(x)\frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\|\\&\leqslant\left\|\sum_{i=n+1}^{\infty}u_i(x,t)\right\|\leqslant\sum_{i=n+1}^{\infty}\|u_i(x,t)\|\end{aligned}$$

based on Theorem 2 in [19], we have

$$\begin{aligned}&\leqslant\sum_{i=n+1}^{\infty}\gamma\|u_{i-1}(x,t)\|\\&\leqslant\sum_{i=n+1}^{\infty}\gamma^2\|u_{i-2}(x,t)\|\\&\leqslant\\&\vdots\\&\leqslant\\&\leqslant\frac{\gamma^{n+1}}{1-\gamma}\|u_0(x,t)\|\\&=\frac{\gamma^{n+1}}{1-\gamma}\|u_0(x)\|.\end{aligned} \tag{15}$$

Since $0 < \gamma < 1$ and $u_0(x)$ is bounded, then

$$\lim_{n\to\infty} \|u(x,t) - u_n(x,t)\| = 0. \tag{16}$$

This completes the proof.

3. Fractional Taylor series method

In order to determine the thermal diffusion coefficient $a(t)$ of time in the space-time fractional diffusion problem (1)-(4), in the series form we plug the fractional Taylor series of $u = u(x,t)$ and $a = a(t)$ into (1)-(4) which leads to:

$$\sum_{i+j=0}^{\infty} g_{i+1,j} \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)} t^{i\alpha} x^{j\beta}$$
$$= \sum_{k=0}^{\infty} a_k \Big\{ \sum_{i+j=0}^{\infty} g_{i,j+2} \frac{\Gamma((j+2)\beta+1)}{\Gamma(i\alpha+1)\Gamma(j\beta+1)\Gamma(k\alpha+1)} t^{(i+k)\alpha} x^{j\beta} \Big\} \tag{17}$$

Making two series on both sides of above equation equal to each other, the unknown coefficients in the fractional Taylor series of $a(t)$ are acquired.

4. Illustrative Examples

In this section, we illustrate two examples of inverse problems about determination of unknown time dependent thermal diffusivity.

Example 1. Consider the inverse coefficient problem involving space-time fractional differential equations:

$$D_t^{\alpha} u(x,t) = a(t) D_x^{\beta}(D_x^{\beta} u(x,t)), 0 < x < l, 0 < t < 1, \tag{18}$$

$$u(x,0) = E_{\beta}(x^{\beta}), 0 \leqslant x \leqslant l, \tag{19}$$

$$u(0,t) = E_{\alpha}(t^{\alpha} + t^{2\alpha}), 0 < t \leqslant 1, \tag{20}$$

$$u(1,t) = E_{\alpha}(t^{\alpha} + t^{2\alpha}) E_{\beta}(1), 0 < t \leqslant 1, \tag{21}$$

where one parameter Mittag-Leffler function $E_\beta(x^\beta) = \sum\limits_{j=1}^{\infty} \frac{x^{j\beta}}{\Gamma(j\beta+1)}$ is the fractional generalization of the function $exp(x)$. It has the exact solution

$$u(x,t) = E_\alpha(t^\alpha + t^{2\alpha}) E_\beta(x^\beta) \tag{22}$$

and

$$a(t) = 1 + 2\frac{t^\alpha}{\Gamma(\alpha+1)}. \tag{23}$$

We establish the thermal diffusivity $a(t)$ in fractional Taylor series form as follows:

$$a(t) = \sum_{k=0}^{\infty} a_k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, 0 < \alpha \leq 1. \tag{24}$$

which leads to Eq. (16), with the initial coefficients

$$g_{0,j} = \frac{1}{\Gamma(j\beta+1)}, \tag{25}$$

$$g_{0,0} = 1. \tag{26}$$

The coefficients $g_{i,j}$ are acquired by equating two series in Eq. (16), which allow us to form the solution of Eq.(18) as follows:

$$\begin{aligned} u(x,t) = &\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{x^\beta}{\Gamma(\beta+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\beta}}{\Gamma(2\beta+1)} \\ &+ \frac{x^{3\beta}}{\Gamma(3\beta+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \Gamma(2\alpha+1) \Bigg(\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^3} \\ &- \frac{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1) - \Gamma(\alpha+1)\Gamma(3\alpha+1) + (\Gamma(2\alpha+1))^2}{(\Gamma(\alpha+1))^3\Gamma(2\alpha+1)} \\ &+ \frac{(1 + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)})}{\Gamma(2\alpha+1)} \Bigg) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} (1 + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}) \\ &+ \frac{t^\alpha x^{2\beta}}{\Gamma(\alpha+1)\Gamma(2\beta+1)} + \frac{t^\alpha x^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &+ \frac{t^{2\alpha} x^\beta}{\Gamma(2\alpha+1)\Gamma(\beta+1)} (1 + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}) + 1 \end{aligned} \tag{27}$$

In order to establish the thermal diffusivity $a(t)$, the Dirichlet boundary condition at $x = 1$ taken into account in (20) which produces the coefficients a_k as follows:
$a_0 = 1$,
$a_1 = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}$,
$a_2 = -\frac{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)-\Gamma(\alpha+1)\Gamma(3\alpha+1)+(\Gamma(2\alpha+1))^2}{(\Gamma(\alpha+1))^3}$,
$a_3 = -\left(\frac{\Gamma(\alpha+1)(\Gamma(3\alpha+1))^2-(\Gamma(\alpha+1))^3\Gamma(4\alpha+1)-(\Gamma(\alpha+1))^4\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^4\Gamma(2\alpha+1)}\right)$
$- \left(\frac{-(\Gamma(2\alpha+1))^2\Gamma(3\alpha+1)+\Gamma(\alpha+1)(\Gamma(2\alpha+1))^2\Gamma(3\alpha+1)+(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^4\Gamma(2\alpha+1)}\right)$,
$\vdots$

As a result, the unknown thermal diffusivity $a(t)$ is determined in the series of as follows:

$$\begin{aligned} a(t) = 1 &+ \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \left(\frac{\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^2}\right. \\ &\left. - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} - \frac{(\Gamma(2\alpha+1))^2}{(\Gamma(\alpha+1))^3}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &- \left(\frac{\Gamma(\alpha+1)(\Gamma(3\alpha+1))^2 - (\Gamma(\alpha+1))^3\Gamma(4\alpha+1) - (\Gamma(\alpha+1))^4\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^4\Gamma(2\alpha+1)}\right. \\ &+ \frac{(\Gamma(2\alpha+1))^2\Gamma(3\alpha+1) - \Gamma(\alpha+1)(\Gamma(2\alpha+1))^2\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^4\Gamma(2\alpha+1)} \\ &\left. - \frac{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{(\Gamma(\alpha+1))^4\Gamma(2\alpha+1)}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots \end{aligned} \tag{28}$$

It can be concluded from Table 1 that absolute error increases as the orders of time and space fractional derivatives decrease. Similarly, the absolute error increases as time variable increases. In Fig. 1, the graphs of diffusivity $a(t)$ for various orders α of time fractional derivative are presented. Moreover, in Figs. 2-4, 3D graphics of the solution for different values of α and β are illustrated.

Table 1: The table of absolute errors $E(\alpha, \beta)$ of Example 1 at $x = 0.5$ for $n = 6$.

t	$Exact$	$E(1,1)$	$E(1,0.9)$	$E(1,0.7)$	$E(0.9,1)$	$E(0.9,0.7)$	$E(0.7,1)$	$E(0.7,0.7)$
0.1	1.84043	5.49e-05	8.24e-05	4.18e-04	1.62e-04	7.74e-04	1.53e-03	3.76e-03
0.2	2.09591	8.29e-04	1.02e-03	2.41e-03	2.00e-03	4.77e-03	1.19e-02	2.23e-02
0.3	2.43485	4.41e-03	5.19e-03	9.39e-03	9.24e-03	1.80e-02	4.14e-02	7.24e-02
0.4	2.88462	1.49e-02	1.73e-02	2.80e-02	2.82e-02	5.07e-02	1.04e-01	1.77e-01
0.5	3.48279	3.92e-02	4.55e-02	6.97e-02	6.88e-02	1.19e-01	2.20e-01	3.67e-01
0.6	4.28032	8.82e-02	1.02e-01	1.52e-01	1.46e-01	2.48e-01	4.16e-01	6.86e-01
0.7	5.34550	1.78e-01	2.06e-01	3.03e-01	2.80e-01	4.71e-01	7.28e-01	1.19e+00
0.8	6.76897	3.33e-01	3.86e-01	5.62e-01	5.02e-01	8.39e-01	1.21e+00	1.95e+00
0.9	8.66984	5.88e-01	6.83e-01	9.86e-01	8.56e-01	1.42e+00	1.91e+00	3.06e+00
1	11.20328	9.92e-01	1.15e-01	1.66e+00	1.40e+00	2.31e+00	2.93e+00	4.64e+00

Example 2. Consider the inverse coefficient problem involving space-time fractional differential equations:

$$D_t^{\alpha} u(x,t) = a(t) D_x^{\beta}(D_x^{\beta} u(x,t)), 0 < x < 1, 0 < t < 1, \tag{29}$$

with initial condition

$$u(x,0) = E_{\beta}(x^{\beta}), 0 \leqslant x \leqslant 1, \tag{30}$$

and boundary conditions

$$u(0,t) = \begin{cases} E_{\alpha}(t^{3\alpha} - \frac{t^{\alpha}}{4}), 0 < t \leqslant \frac{1}{2}, \\ E_{\alpha}(-t^{3\alpha} + \frac{t^{\alpha}}{4}), \frac{1}{2} < t < 1. \end{cases} \tag{31}$$

$$u(1,t) = \begin{cases} E_{\alpha}(t^{3\alpha} - \frac{t^{\alpha}}{4}) E_{\beta}(1), 0 < t \leqslant \frac{1}{2}, \\ E_{\alpha}(-t^{3\alpha} + \frac{t^{\alpha}}{4}) E_{\beta}(1), \frac{1}{2} < t < 1. \end{cases} \tag{32}$$

which has the exact solution

$$u(x,t) = \begin{cases} E_{\alpha}(t^{3\alpha} - \frac{t^{\alpha}}{4}) E_{\beta}(x^{\beta}), 0 < t \leqslant \frac{1}{2}, \\ E_{\alpha}(-t^{3\alpha} + \frac{t^{\alpha}}{4}) E_{\beta}(x^{\beta}), \frac{1}{2} < t < 1. \end{cases} \tag{33}$$

and

$$a(t) = \begin{cases} \frac{1}{4} - 6\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, 0 < t \leqslant \frac{1}{2}, \\ -\frac{1}{4} + 6\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \frac{1}{2} < t < 1. \end{cases} \tag{34}$$

Figure 1: The graphics of thermal diffusivity $a(t)$ for Example 1 .

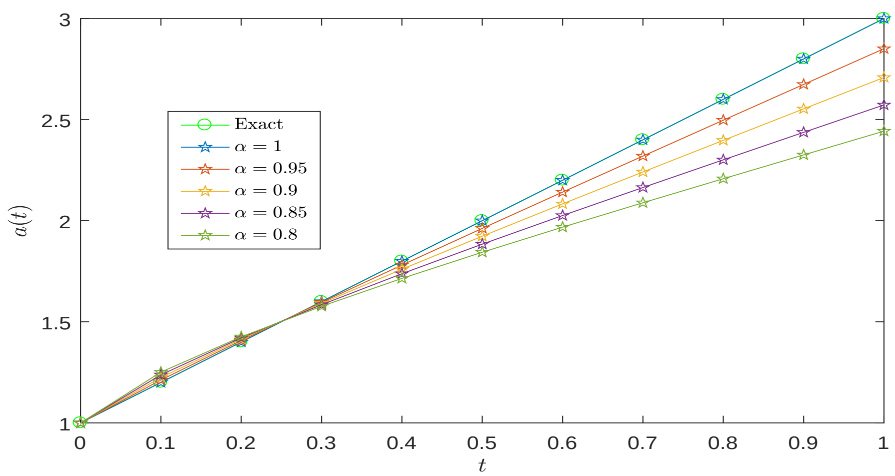


Figure 2: The graphics of exact solution $u(x,t)$ for Example 1 .

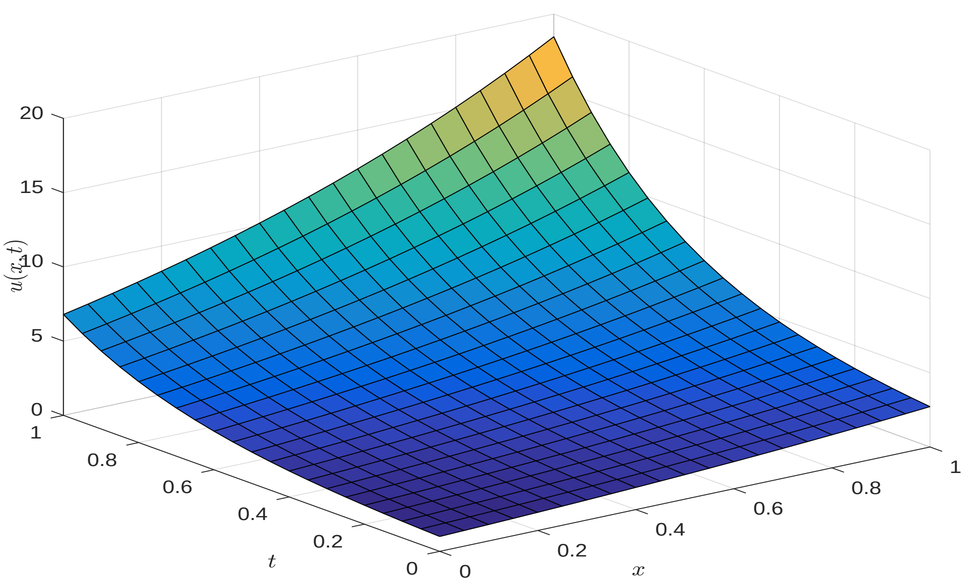


Figure 3: The graphics of approximate solution $u(x,t)$ of Example 1 for $\alpha = \beta = 1$.

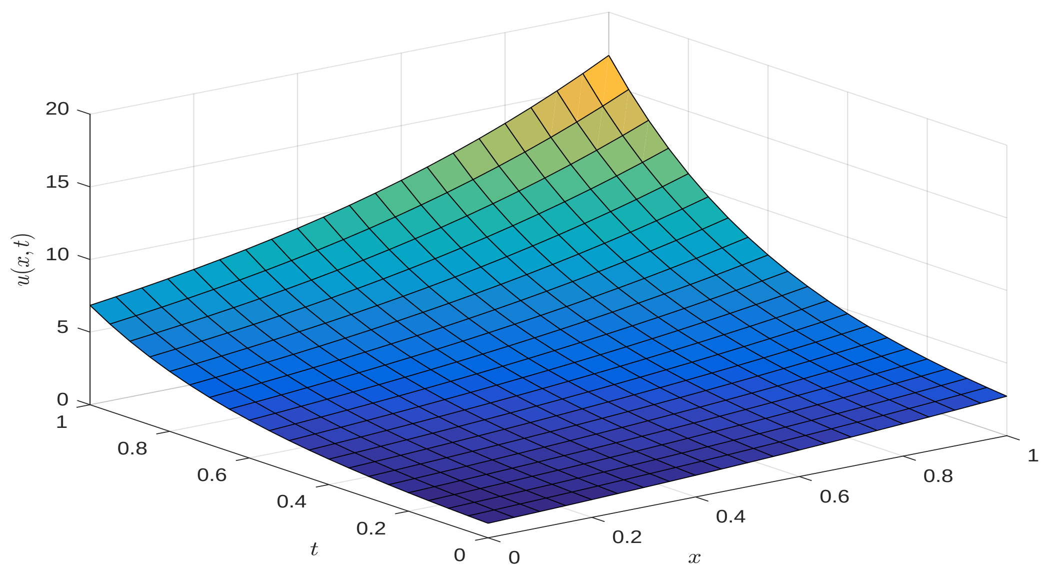
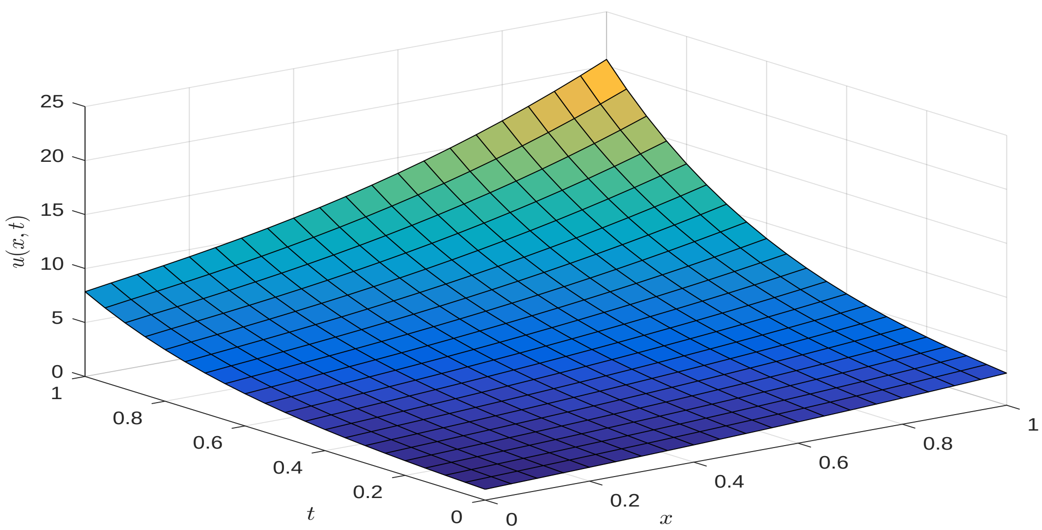


Figure 4: The graphics of approximate solution of $u(x,t)$ of Ex. 1 for $\alpha = \beta = 0.9$.



We retrieve the unknown thermal diffusivity coefficient $a(t)$ in fractional Taylor series form as follows:

$$a(t)=\sum_{k=0}^{\infty}a_k\frac{t^{k\alpha}}{\Gamma(k\alpha+1)},0<\alpha\leq 1. \tag{35}$$

which leads to Eq. (17), with the initial coefficients

$$g_{0,j}=\frac{1}{\Gamma(j\beta+1)}, \tag{36}$$

$$g_{0,0}=1. \tag{37}$$

The coefficients $g_{i,j}$ are obtained by equating two series in Eq. (17), which enable us to form the solution $u(x,t)$ of Eq.(26) as follows:

$$u(x,t)=\begin{cases} -\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\beta}}{\Gamma(\beta+1)}+\frac{t^{2\alpha}}{16\Gamma(2\alpha+1)}+\frac{x^{2\beta}}{\Gamma(2\beta+1)} \\ +\frac{x^{3\beta}}{\Gamma(3\beta+1)}-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\Gamma(2\alpha+1)\Big(\frac{1}{64\Gamma(2\alpha+1)} \\ -\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\Big) \\ -\frac{t^{\alpha}x^{2\beta}}{4\Gamma(\alpha+1)\Gamma(2\beta+1)}+\frac{t^{2\alpha}x^{\beta}}{16\Gamma(2\alpha+1)\Gamma(\beta+1)} \\ -\frac{t^{\alpha}x^{\beta}}{4\Gamma(\alpha+1)\Gamma(\beta+1)}+1 & , \quad 0<t\leqslant\frac{1}{2}, \\ \frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\beta}}{\Gamma(\beta+1)}+\frac{t^{2\alpha}}{16\Gamma(2\alpha+1)}+\frac{x^{2\beta}}{\Gamma(2\beta+1)} \\ +\frac{x^{3\beta}}{\Gamma(3\beta+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\Gamma(2\alpha+1)\Big(\frac{1}{64\Gamma(2\alpha+1)} \\ +\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\Big) \\ +\frac{t^{\alpha}x^{2\beta}}{4\Gamma(\alpha+1)\Gamma(2\beta+1)}+\frac{t^{2\alpha}x^{\beta}}{16\Gamma(2\alpha+1)\Gamma(\beta+1)} \\ +\frac{t^{\alpha}x^{\beta}}{4\Gamma(\alpha+1)\Gamma(\beta+1)}+1 & , \quad \frac{1}{2}<t<1. \end{cases} \tag{38}$$

In order to determine the unknown coefficient $a(t)$ the boundary condition at $x=1$ into account in (32) produce the coefficients a_k as follows:

$$a(t)=\begin{cases} a_0=-\frac{1}{4}, \\ a_1=0, \\ a_2=\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}, \\ a_3=\frac{(\Gamma(3\alpha+1))^2-\Gamma(2\alpha+1)\Gamma(4\alpha+1)+\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{4(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)}, \\ \vdots & , \quad 0<t\leqslant\frac{1}{2}, \\ a_0=\frac{1}{4}, \\ a_1=0, \\ a_2=-\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}, \\ a_3=\frac{(\Gamma(3\alpha+1))^2-\Gamma(2\alpha+1)\Gamma(4\alpha+1)+\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{4(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)}, \\ \vdots & , \quad \frac{1}{2}<t<1. \end{cases} \tag{39}$$

As a result, the unknown coefficient $a(t)$ is determined in the series of as follows:

$$a(t)=\begin{cases} -\frac{1}{4}+\left(\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ +\left(\frac{(\Gamma(3\alpha+1))^2-\Gamma(2\alpha+1)\Gamma(4\alpha+1)+\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{4(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...,0<t\leqslant\frac{1}{2}, \\ \frac{1}{4}-\left(\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ +\left(\frac{(\Gamma(3\alpha+1))^2-\Gamma(2\alpha+1)\Gamma(4\alpha+1)+\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{4(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)}\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...,\frac{1}{2}<t<1. \end{cases} \tag{40}$$

It can be concluded from Table 2 that absolute error increases as the orders of time and space fractional derivatives decrease. Similarly, the absolute error increases as time variable increases. In Fig. 5, the graphs of diffusivity $a(t)$ for various orders α of time fractional derivative are presented. Moreover, in Figs. 6-8, 3D graphics of the solution for different values of α and β are illustrated.

Table 2: The table of absolute errors $E(\alpha, \beta)$ of Example 2 at $x = 0.5$ for $n = 6$.

t	$Exact$	$E(1,1)$	$E(1,0.9)$	$E(1,0.7)$	$E(0.9,1)$	$E(0.9,0.7)$	$E(0.7,1)$	$E(0.7,0.7)$
0.1	1.60962	7.07e-06	5.55e-06	1.25e-05	1.96e-05	1.65e-06	1.29e-04	6.85e-05
0.2	1.58091	1.14e-04	9.53e-05	6.11e-05	2.20e-04	1.11e-04	6.21e-04	5.79e-04
0.3	1.57145	4.93e-04	4.21e-04	2.53e-04	7.32e-04	5.08e-04	7.51e-04	3.17e-03
0.4	1.59042	1.13e-03	8.78e-04	1.05e-03	1.20e-03	2.32e-03	1.70e-03	1.20e-02
0.5	1.64870	1.35e-03	5.22e-04	4.27e-03	1.67e-04	8.67e-03	1.13e-02	3.50e-02
0.6	1.54325	3.26e-03	3.87e-03	3.72e-03	7.91e-03	9.19e-03	2.15e-02	3.67e-02
0.7	1.39269	1.56e-02	1.88e-02	2.14e-02	2.38e-02	3.67e-02	3.30e-02	6.81e-02
0.8	1.20161	3.97e-02	4.87e-02	5.77e-02	4.69e-02	8.35e-02	2.67e-02	8.90e-02
0.9	0.97486	7.15e-02	9.08e-02	1.12e-01	6.33e-02	1.38e-01	3.22e-02	5.89e-02
1	0.70567	8.49e-02	1.20e-01	1.60e-01	3.14e-02	1.55e-01	2.06e-01	9.64e-02

5. Conclusion

In this article, inverse problem of determining unknown thermal diffusivity coefficient in mathematical problem including differential equation is taken in hand. Fractional Taylor series method is implemented successfully for establishing time-dependent diffusion coefficient. The considerable advantage of this method is that it doesn't require any over-measured data which allows us to establish the solution of inverse problem more precisely. Taking the Dirichlet boundary condition at final point into account enable us to determine the coefficients in fractional Taylor series of the solution. Future work will be concerned with the construction of the diffusivity coefficient in linear heat-like equations with Neumann boundary conditions.

Figure 5: The graphics of thermal diffusivity $a(t)$ for Example 2 .

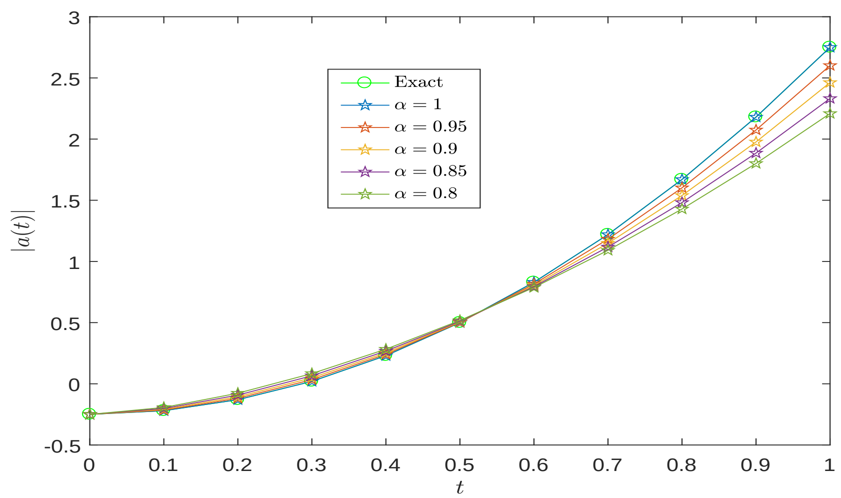


Figure 6: The graphics of exact solution $u(x,t)$ for Example 2 .

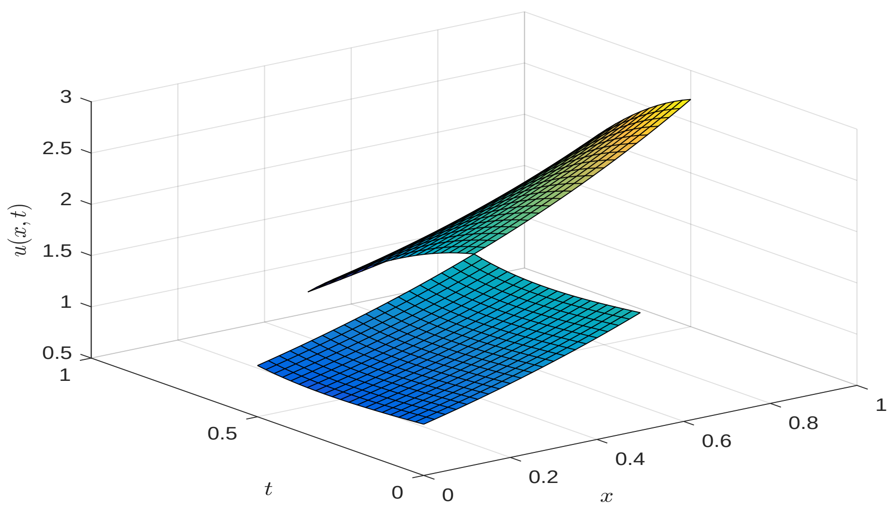


Figure 7: The graphics of approximate solution $u(x,t)$ of Example 2 for $\alpha = \beta = 1$.

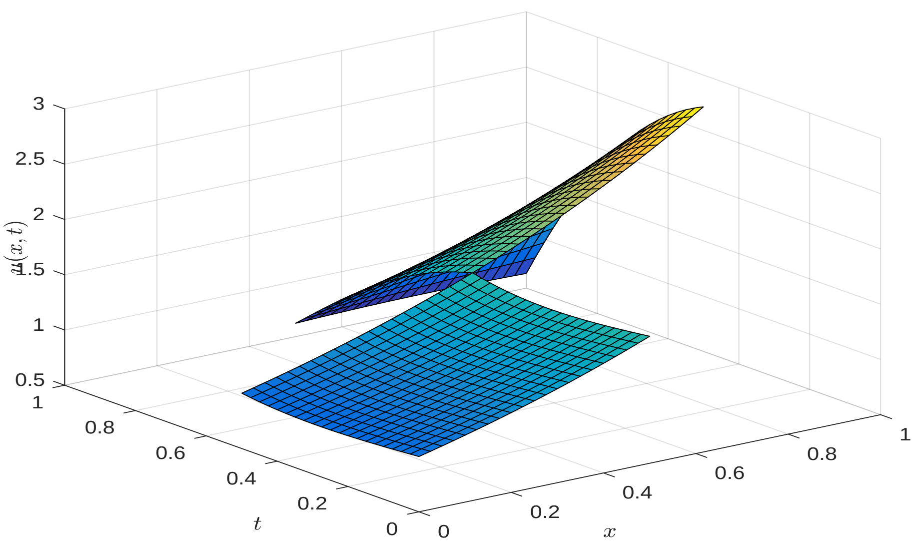
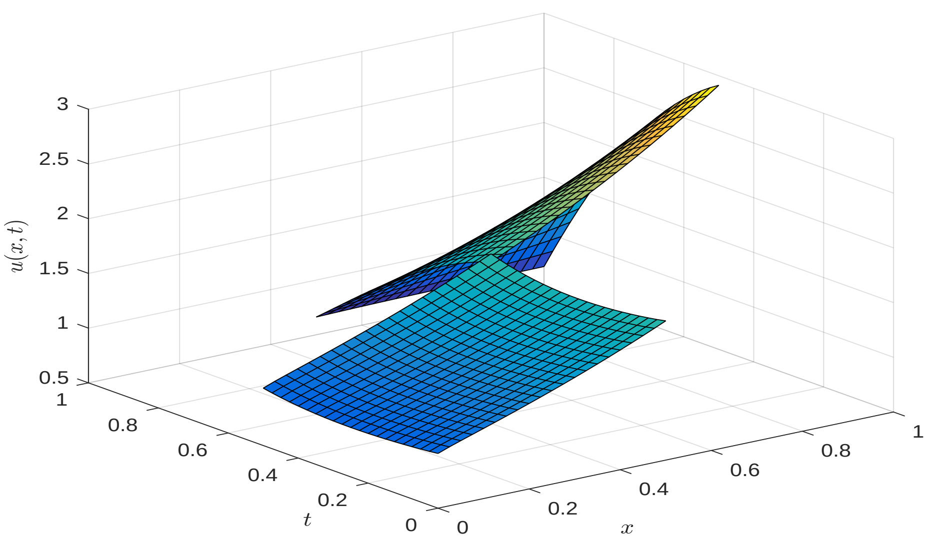


Figure 8: The graphics of approximate solution $u(x,t)$ of Example 2 for $\alpha = \beta = 0.9$.



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