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$STCR$-Lightlike Product Manifolds of an Indefinite Kaehler Statistical Manifold with a Quarter Symmetric Non-Metric Connection

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ABSTRACT

The present work aims to introduce a novel class of submanifolds, namely STCR-lightlike submanifolds, for an indefinite Kaehler statistical manifold with a quarter symmetric non-metric connection. The characterization theorems on totally umbilical and totally geodesic STCR-lightlike submanifolds with respect to the integrability of distributions have been established. Some conditions for a STCR-lightlike submanifold to be a STCR-lightlike product manifold have been derived.

Keywords: *STCR lightlike submanifolds, indefinite Kaehler statistical manifold, totally geodesic foliation, integrability.*
AMS Subject Classification (2020): *Primary: 53C15 ; Secondary: 53C40; 53C55; 53B05;*

1. Introduction

The geometry of lightlike submanifolds of semi-Riemannian manifold introduced by Duggal and Bejancu [8] is a prime field of study. Various classes like CR-lightlike submanifolds, SCR-lightlike submanifolds and GCR lightlike submanifolds of an indefinite Kaehler manifold have been studied extensively by many geometers [21], [10], [11], [9] et al. But these classes do not contain real lightlike curves. So, [22], [23] introduced transversal lightlike submanifolds and screen transversal lightlike submanifolds of an indefinite Kaehler manifold and also the subclasses called radical ST-lightlike submanifolds and ST-anti invariant lightlike submanifolds. Further, as a generalization of CR-lightlike submanifolds and screen transversal lightlike submanifolds, a new notion termed as Screen Transversal Cauchy-Riemann (STCR) lightlike submanifolds was introduced by [7].

Statistical manifolds, which analyze the geometric structures on sets of certain probability distributions were initiated by [20] and thereafter developed by various researchers [1], [2], [12] and [17] et al. In this context, the lightlike theory of statistical manifolds has been investigated by [3], [4], and many others. Further, by consolidating the notion of statistical manifold with an indefinite Kaehler manifold, several findings have been demonstrated for the CR-lightlike submanifolds and hypersurfaces of an indefinite Kaehler statistical manifold by [15], [18], [19].

[13] introduced a quarter symmetric linear connection as: A linear connection $\bar{\nabla}$ on a Riemannian manifold $(\tilde{M}, \tilde{g})$ is said to be a quarter symmetric connection if its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(X,Y) = \pi(Y)\phi(X) - \pi(X)\phi(Y), \tag{1.1}$$

where ϕ is a (1,1)-tensor field and π is a 1-form associated with a smooth unit vector field ζ, called the characterstic vector field, by $\pi(X) = \tilde{g}(X,\zeta)$. If the linear connection $\bar{\nabla}$ is not a metric connection, then

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$\bar{\nabla}$ is called a quarter symmetric non-metric connection. A significant number of properties on lightlike submanifolds of an indefinite Kaehler manifold with quarter symmetric non-metric connection have been developed by [5], [6],[14],[16].

Keeping the aforementioned theory in focus, this paper introduces the concept of STCR-lightlike submanifolds for an indefinite Kaehler statistical manifold with a quarter symmetric non-metric connection . Some charaterizations pertaining to the integrability of distributions for totally umbilical and totally geodesic STCR lightlike submanifolds have been developed. Various results related to the geometry of STCR-lightlike product manifolds have been given.

2. Preliminaries

Definition 2.1. A pair $(\bar{\nabla}, \tilde{g})$ is called a **statistical structure** on a semi-Riemannian manifold $\tilde{M}$ such that for all $X, Y, Z \in \Gamma(T\tilde{M})$

1. $\bar{\nabla}_X Y - \bar{\nabla}_Y X = [X, Y]$;
2. $(\bar{\nabla}_X \tilde{g})(Y, Z) = (\bar{\nabla}_Y \tilde{g})(X, Z)$ hold.

Then $(\tilde{M}, \tilde{g}, \bar{\nabla})$ is said to be an **indefinite statistical manifold**. Moreover, there exists $\bar{\nabla}^*$ which is a dual connection of $\bar{\nabla}$ with respect to $\tilde{g}$, satisfying

$$X\tilde{g}(Y, Z) = \tilde{g}(\bar{\nabla}_X Y, Z) + \tilde{g}(Y, \bar{\nabla}^*_X Z).$$

Also $(\bar{\nabla}^*)^* = \bar{\nabla}$. If $(\tilde{M}, \tilde{g}, \bar{\nabla})$ is an indefinite statistical manifold, then $(\tilde{M}, \tilde{g}, \bar{\nabla}^*)$ is also a statistical manifold. Hence, the indefinite statistical manifold is denoted by $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*)$.

Following [8], some basic facts about the lightlike theory of submanifolds are as as below:

Consider $(\tilde{M}, \tilde{g})$ as an $(m+n)$-dimensional semi-Riemannian manifold with semi-Riemannian metric $\tilde{g}$ and of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$.
Let (M, g) be a m-dimensional lightlike submanifold of $\tilde{M}$. In this case, there exists a smooth distribution $Rad(TM)$ on M of rank $r > 0$, known as Radical distribution on M such that $Rad(TM_p) = TM_p \cap TM_p^{\perp}, \forall\, p \in M$ where TM_p and $TM_p^{\perp}$ are degenerate orthogonal spaces but not complementary. Then M is called an r-lightlike submanifold of $\tilde{M}$. Now, consider $S(TM)$, known as Screen distribution, as a complementary distribution of radical distribution in TM i.e., $TM = Rad(TM) \perp S(TM)$ and $S(TM^{\perp})$, called screen transversal vector bundle, as a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$ i.e., $TM^{\perp} = Rad(TM) \perp S(TM^{\perp})$. As $S(TM)$ is non degenerate vector subbundle of $T\tilde{M}|_M$, we have $T\tilde{M}|_M = S(TM) \perp S(TM)^{\perp}$ where $S(TM)^{\perp}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\tilde{M}|_M$. Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\tilde{M}|_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$. Then we have $tr(TM) = ltr(TM) \perp S(TM^{\perp})$, $T\tilde{M}|_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp})$.

Theorem 2.1. *[8] Let $(M, g, S(TM), S(TM^{\perp}))$ be an r- lightlike submanifold of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$.Then there exists a complementary vector bundle $ltr(TM)$ called a lightlike transversal bundle of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and basis of $\Gamma(ltr(TM)|_U)$ consisting of smooth sections $\{N_1, \cdots, N_r\}$ $S(TM^{\perp})^{\perp}|_U$ such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad i, j = 0, 1, \cdots, r$$

where $\{\xi_1, \cdots, \xi_r\}$ is a lightlike basis of $\Gamma(RadTM)|_U$.

Let (M, g) be a lightlike submanifold of an indefinite statistical manifold $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*)$. From the theory of lightlike submanifolds of an indefinite statistical manifold, the Gauss and Weingarten formulae developed on its structure are as below:

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}^*_X Y = \nabla^*_X Y + h^{*l}(X, Y) + h^{*s}(X, Y), \tag{2.1}$$

$$\bar{\nabla}_X V = -A_V X + D^l_X V + D^s_X V, \quad \bar{\nabla}^*_X V = -A^*_V X + D^{*l}_X V + D^{*s}_X V, \tag{2.2}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \bar{\nabla}^*_X N = -A^*_N X + \nabla^{*l}_X N + D^{*s}(X, N), \tag{2.3}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \quad \bar{\nabla}^*_X W = -A^*_W X + \nabla^{*s}_X W + D^{*l}(X,W). \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(tr(TM))$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Now, the concept of indefinite statistical manifold and (2.1), (2.2), (2.3), (2.4), implies

$$\tilde{g}(h^s(X,Y),W) + \tilde{g}(Y, D^{*l}(X,W)) = \tilde{g}(Y, A^*_W X), \tag{2.5}$$

$$\tilde{g}(h^l(X,Y),\xi) + \tilde{g}(Y, \nabla^*_X \xi) + \tilde{g}(Y, h^{*l}(X,\xi)) = 0,$$

$$\tilde{g}(D^s(X,N),W) = \tilde{g}(N, A^*_W X),$$

$$\tilde{g}(A_N X, PY) = \tilde{g}(N, \bar{\nabla}^*_X PY),$$

and

$$\tilde{g}(A_N X, N') + \tilde{g}(A^*_{N'} X, N) = 0.$$

From the theory of non-degenerate submanifolds of a statistical manifold, it is known that submanifold of a statistical manifold is a statistical manifold but this is not true for lightlike submanifolds since the definition of statistical manifold and (2.1) implies

$$(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = \tilde{g}(Y, h^l(X,Z)) - \tilde{g}(X, h^l(Y,Z)),$$

and

$$Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla^*_X Z) = \tilde{g}(h^l(X,Y),Z) + \tilde{g}(Y, h^{*l}(X,Z)).$$

Considering the projection morphism P of the tangent bundle TM to the screen distribution, we have the following decomposition w.r.t ∇ and ∇^*:

$$\nabla_X PY = \nabla'_X PY + h'(X,PY), \quad \nabla^*_X PY = \nabla^{*\prime}_X PY + h^{*\prime}(X,PY), \tag{2.6}$$

$$\nabla_X \xi = -A'_\xi X + \nabla'^t_X \xi, \quad \nabla^*_X \xi = -A^{*\prime}_\xi X + \nabla^{*\prime t}_X \xi, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$.

Using (2.1),(2.2),(2.5) and (2.7), we obtain

$$\tilde{g}(h^l(X,PY),\xi) = g(A^{*\prime}_\xi X, PY), \quad \tilde{g}(h^{*l}(X,PY),\xi) = g(A'_\xi X, PY), \tag{2.8}$$

$$\tilde{g}(h'(X,PY),N) = g(A^*_N X, PY), \quad \tilde{g}(h^{*\prime}(X,PY),N) = g(A_N X, PY), \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$. As h^l and h^{*l} are symmetric, so from (2.8), we obtain

$$g(A'_\xi PX, PY) = g(PX, A'_\xi PY), \quad g(A^{*\prime}_\xi PX, PY) = g(PX, A^{*\prime}_\xi PY).$$

Let $\bar{\nabla}^\circ$ be the Levi-Civita connection w.r.t $\tilde{g}$. Then, we have $\bar{\nabla}^\circ = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$.

For a statistical manifold $(\tilde{M}, \tilde{g}, \bar{\nabla}, \bar{\nabla}^*)$, the difference $(1,2)$ tensor K of a torsion free affine connection $\bar{\nabla}$ and Levi-Civita connection $\bar{\nabla}^\circ$ is defined as

$$K(X,Y) = K_X Y = \bar{\nabla}_X Y - \bar{\nabla}^\circ_X Y, \tag{2.10}$$

Since $\bar{\nabla}$ and $\bar{\nabla}^\circ$ are torsion free, we have

$$K(X,Y) = K(Y,X), \quad \tilde{g}(K_X Y, Z) = \tilde{g}(Y, K_X Z), \tag{2.11}$$

for any $X, Y, Z \in \Gamma(TM)$.

Also, from (2.10), we have

$$\tilde{g}(\bar{\nabla}_X Y, Z) = \tilde{g}(K(X,Y),Z) + \tilde{g}(\bar{\nabla}^\circ_X Y, Z). \tag{2.12}$$

Definition 2.2. [15] A triplet $(\bar{\nabla} = \bar{\nabla}^\circ + K, \tilde{g}, \bar{J})$ is called an indefinite Kaehler statistical structure on $\tilde{M}$ if

(i) $(\tilde{g}, \bar{J})$ is an indefinite Kaehler structure on $\tilde{M}$

(ii)$(\bar{\nabla}, \tilde{g})$ is a statistical structure on $\tilde{M}$

and the condition

$$K(X, \bar{J}Y) = -\bar{J}K(X,Y),$$

holds for any $X, Y \in \Gamma(T\tilde{M})$.

Then $(\tilde{M}, \bar{\nabla}, \tilde{g}, \bar{J})$ is called an indefinite Kaehler statistical manifold. If $(\tilde{M}, \bar{\nabla}, \tilde{g}, \bar{J})$ is an indefinite Kaehler statistical manifold, then so is $(\tilde{M}, \bar{\nabla}^*, \tilde{g}, \bar{J})$.

3. STCR-lightlike submanifold

Sahin et.al [7] introduced screen transversal Cauchy Riemann lightlike submanifolds of an indefinite Kaehler manifold. So motivated, we introduce a STCR-lightlike submanifold of an indefinite Kaehler statistical manifold and elaborate its structure with an example.

Definition 3.1. A real lightlike submanifold M of an indefinite Kaehler statistical manifold $\tilde{M}$ is a $STCR$ (Screen transversal Cauchy Riemann) lightlike submanifold if the following conditions are satisfied:

1. There exist two subbundles E_1 and E_2 of Rad(TM) such that

$$Rad(TM) = E_1 \oplus E_2, \quad \bar{J}(E_1) \subset S(TM), \quad \bar{J}(E_2) \subset S(TM^{\perp}), \tag{3.1}$$

2. There exist two subbundles $E_\circ$ and E' of S(TM) such that

$$S(TM) = \{\bar{J}E_1 \oplus E'\} \perp E_\circ, \quad \bar{J}(E_\circ) = E_\circ, \ \bar{J}(E') = L_1 \perp S, \tag{3.2}$$

   where $E_\circ$ is a non-degenerate distribution on M, L_1 and S are vector subbundles of $ltr(TM)$ and $S(TM^{\perp})$ respectively.

Thus we have following decomposition

$$TM = E \oplus \bar{E}, \tag{3.3}$$

where

$$E = E_\circ \oplus E_1 \oplus \bar{J}E_1, \tag{3.4}$$

and

$$\bar{E} = E_2 \oplus \bar{J}L_1 \oplus \bar{J}S, \tag{3.5}$$

It is clear that E is invariant and $\bar{E}$ is anti-invariant. Thus, we have

$$ltr(TM) = L_1 \oplus L_2, \quad \bar{J}L_1 \subset S(TM), \quad \bar{J}L_2 \subset S(TM^{\perp}),$$

and

$$S(TM^{\perp}) = \{\bar{J}E_2 \oplus \bar{J}L_2\} \perp S.$$

We denote the projections from $\Gamma(TM)$ to $\Gamma(E_\circ), \Gamma(\bar{J}E_1), \Gamma(\bar{J}L_1), \Gamma(\bar{J}S), \Gamma(E_1)$ and $\Gamma(E_2)$ by $P_\circ$, P_1, P_2, P_3, S_1 and S_2 respectively. Also, the projections from $\Gamma(tr(TM))$ to $\Gamma(\bar{J}E_2), \Gamma(\bar{J}L_2), \Gamma(S), \Gamma(L_1)$ and $\Gamma(L_2)$ are denoted by R_1, R_2, R_3, Q_1 and Q_2, respectively. Therefore

$$X = PX + QX = P_\circ X + P_1 X + P_2 X + P_3 X + S_1 X + S_2 X, \tag{3.6}$$

and

$$\bar{J}X = TX + wX, \tag{3.7}$$

for $X \in \Gamma(TM)$, where $PX \in \Gamma(E)$, $QX \in \Gamma(\bar{E})$ and TX and wX are respectively the tangential and transversal parts of $\bar{J}X$. Applying $\bar{J}$ to (3.6) and denoting $\bar{J}P_\circ$, $\bar{J}P_1$, $\bar{J}P_2$, $\bar{J}P_3$, $\bar{J}S_1$, $\bar{J}S_2$ by $T_\circ$, T_1, w_L, w_S, $T_{\bar{1}}$, $w_{\bar{2}}$, respectively, we have

$$\bar{J}X = T_\circ X + T_1 X + T_{\bar{1}} X + w_L X + w_S X + w_{\bar{2}} X, \tag{3.8}$$

for $X \in \Gamma(TM)$, where $T_\circ X \in \Gamma(E_\circ)$, $T_1 X \in \Gamma(E_1)$, $T_{\bar{1}} X \in \Gamma(\bar{J}E_1)$, $w_L X \in \Gamma(L_1)$, $w_S X \in \Gamma(S)$, and $w_{\bar{2}} X \in \Gamma(\bar{J}E_2)$. Also, for any $V \in \Gamma(tr(TM))$,

$$V = R_1 V + R_2 V + R_3 V + Q_1 V + Q_2 V, \tag{3.9}$$

Denote $\bar{J}R_1$, $\bar{J}R_2$. $\bar{J}R_3$, $\bar{J}Q_1$, $\bar{J}Q_2$ by B_2, C_1, $B_{\bar{S}}$, $B_{\bar{L}}$, C_2, respectively so that

$$\bar{J}V = B_2 V + B_{\bar{S}} V + B_{\bar{L}} V + C_1 V + C_2 V. \tag{3.10}$$

where BV and CV are sections of TM and $tr(TM)$, respectively.

Inspired by [7], we consider the following example:

Example 3.1. Let $\tilde{M} = (R_4^{12}, \tilde{g})$ be an indefinite Kaehler manifold, where $\tilde{g}$ is of signature $(-,-,-,-,+,+,+,+,+,+,+,+)$ with respect to the basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}, \partial x_{11}, \partial x_{12}\}$. If $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$ is the standard coordinate sysytem of R_4^{12} , then by setting $\bar{J}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, -x_{10}, x_9, -x_{12}, x_{11})$, we have $\bar{J}^2 = -I$.

Following definition (2.2), the triplet $(\bar{\nabla} = \bar{\nabla}^\circ + K, \tilde{g}, \bar{J})$ where K satisfies (2.11), defines an indefinite Kaehler statistical structure on $\tilde{M}$.

Consider a submanifold M of R_4^{12} given by the equations:

$$x_1 = \sin u_2, \quad x_2 = -\cos u_2, \quad x_3 = u_1, \quad x_4 = u_3 - \frac{u_4}{2}, \quad x_5 = u_2,$$

$$x_6 = 0, \quad x_7 = u_1, \quad x_8 = u_3 + \frac{u_4}{2}, \quad x_9 = u_5 + u_7, x_{10} = u_6 - u_7,$$

$$x_{11} = u_5 - u_7, \quad x_{12} = u_6 + u_7.$$

Here TM is spanned by Z_1,Z_2,Z_3,Z_4,Z_5,Z_6,Z_7 where

$$Z_1 = \partial x_3 + \partial x_7, \quad Z_2 = \cos u_2\, \partial x_1 + \sin u_2\, \partial x_2 + \partial x_5, \quad Z_3 = \partial x_4 + \partial x_8,$$

$$Z_4 = \frac{1}{2}\{-\partial x_4 + \partial x_8\}, \quad Z_5 = \partial x_9 + \partial x_{11}, \quad Z_6 = \partial x_{10} + \partial x_{12},$$

$$Z_7 = \partial x_9 - \partial x_{10} - \partial x_{11} + \partial x_{12},$$

We see that M is 2-lightlike with $RadTM = Span\{Z_1, Z_2\}$ and $\bar{J}Z_1 = Z_3$. Thus, $E_1 = Span\{Z_1\}$ and $E_2 = Span\{Z_2\}$. Also, $\bar{J}Z_5 = Z_6 \in \Gamma(S(TM))$ implies that $E_\circ = Span\{Z_5, Z_6\}$.

Further, the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{2}\{-\partial x_3 + \partial x_7\}, \quad N_2 = \frac{1}{2}\{-\cos u_2 \partial x_1 - \sin u_2 \partial x_2 + \partial x_5\}.$$

Hence, $L_1 = Span\{N_1\}$, $L_2 = Span\{N_2\}$, $S(TM^\perp) = Span\{\bar{J}Z_2, \bar{J}N_2, \bar{J}Z_7\}$, $S = Span\{\bar{J}Z_7 = W\}$ and $E' = Span\{\bar{J}N_1 = Z_4, \bar{J}Z_7 = W\}$.

Therefore M is a proper $STCR$-lightlike submanifold of the indefinite Kaehler statistical manifold R_4^{12}.

4. Quarter symmetric non-metric connection

For a Levi-Civita connection $\bar{\nabla}^\circ$ on an indefinite Kaehler statistical manifold $(\tilde{M}, \bar{J}, \tilde{g})$ where $\bar{\nabla}^\circ = \frac{1}{2}\{\bar{\nabla} + \bar{\nabla}^*\}$, we set

$$\tilde{D}_X Y = \bar{\nabla}_X Y - K(X,Y) + \pi(Y)\bar{J}X, \tag{4.1}$$

and

$$\tilde{D}_X Y = \bar{\nabla}^*_X Y + K(X,Y) + \pi(Y)\bar{J}X, \tag{4.2}$$

for any $X, Y \in \Gamma(T\tilde{M})$. Since $\bar{\nabla}$ and $\bar{\nabla}^*$ are torsion free, therefore from the relationship between dual connections, we obtain

$$(\tilde{D}_X \tilde{g})(Y, Z) = -\pi(Y)\tilde{g}(\bar{J}X, Z) - \pi(Z)\tilde{g}(Y, \bar{J}X), \tag{4.3}$$

and

$$\tilde{T}^{\tilde{D}}(X, Y) = \pi(Y)\bar{J}X - \pi(X)\bar{J}Y, \tag{4.4}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$ where $\tilde{T}^{\tilde{D}}$ is a torsion tensor of the connection $\tilde{D}$ and π is a 1-form associated with the vector field U on $\tilde{M}$ by $\pi(X) = \tilde{g}(X, U)$. So, $\tilde{D}$ becomes a quarter symmetric non-metric connection. Since $\tilde{M}$ admits a tensor field $\bar{J}$ of type (1,1), therefore for any $X, Y \in \Gamma(T\tilde{M})$, we have

$$\tilde{D}_X \bar{J}Y = \bar{J}\tilde{D}_X Y + \pi(Y)X + \pi(\bar{J}Y)\bar{J}X, \tag{4.5}$$

Let M be a STCR-lightlike submanifold of an indefinite Kaehler statistical manifold $(\tilde{M}, \tilde{g})$ with quarter symmetric non-metric connection $\tilde{D}$. Let D be the induced linear connection on M from $\tilde{D}$. Therefore the Gauss formula is as follows:

$$\tilde{D}_X Y = D_X Y + \tilde{h}^l(X,Y) + \tilde{h}^s(X,Y), \tag{4.6}$$

for any $X, Y \in \Gamma(TM)$, where $D_X Y \in \Gamma(TM)$ and $\tilde{h}^l$, $\tilde{h}^s$ are lightlike second fundamental form and the screen second fundamental form of M, respectively. Now from (2.1), (4.6) in (4.1), we get

$$D_X Y = \nabla_X Y + \pi(Y)TX - K(X,Y), \tag{4.7}$$

$$\tilde{h}^l(X,Y) = h^l(X,Y) + w_L X\pi(Y), \tag{4.8}$$

$$\tilde{h}^s(X,Y) = h^s(X,Y) + w_s X\pi(Y) + w_{\bar{2}} X\pi(Y). \tag{4.9}$$

Further, using (4.3), (3.8), (4.6) we have

$$(D_X g)(Y,Z) = g(\tilde{h}^l(X,Y),Z) + g(Y, \tilde{h}^l(X,Z)) - \pi(Y)g(TX,Z) - \pi(Z)g(TX,Y), \tag{4.10}$$

and

$$T^D(X,Y) = \pi(Y)TX - \pi(X)TY.$$

for any $X, Y, Z \in \Gamma(TM)$, where T^D is torsion tensor of the induced connection D on M. Hence, the following result holds:

Theorem 4.1. *Let M be a $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$. Then the induced connection D on the lightlike submanifold M is also a quarter symmetric non-metric connection.*

Suppose that $\tilde{h}^l$ vanishes identically on M. Therefore

$$(D_X g)(Y,Z) = -\pi(Y)g(TX,Z) - \pi(Z)g(TX,Y).$$

follows from (4.10).

Consequently, we arrive to the following outcome:

Theorem 4.2. *Let M be a $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$. Then the induced connection D on the lightlike submanifold M is also a quarter symmetric metric connection if and only if $\tilde{h}^l$ vanishes identically on M and the characterstic vector field $\zeta \in \Gamma(S(TM^\perp))$ such that $\pi(X) = g(X,\zeta)$.*

Corresponding to quarter symmetric non-metric connection $\tilde{D}$, the Weingarten formulae are as below:

$$\tilde{D}_X N = -\tilde{A}_N X + \tilde{\nabla}^l_X N + \tilde{D}^s(X,N), \tag{4.11}$$

$$\tilde{D}_X W = -\tilde{A}_W X + \tilde{\nabla}^s_X W + \tilde{D}^l(X,W), \tag{4.12}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$. Using (2.3),(2.4) (4.11),(4.12) and (4.1) and then equating the tangential and transversal parts, we derive

$$\tilde{A}_N X = A_N X - \pi(N)TX + K(X,N), \quad \tilde{A}_W X = A_W X - \pi(W)TX + K(X,W), \tag{4.13}$$

$$\tilde{\nabla}^l_X N = \nabla^l_X N + \pi(N)w_L X, \quad \tilde{\nabla}^s_X W = \nabla^s_X W + \pi(W)w_s X + \pi(W)w_{\bar{2}} X, \tag{4.14}$$

$$\tilde{D}^s(X,N) = D^s(X,N) + \pi(N)w_s X + \pi(N)w_{\bar{2}} X, \quad \tilde{D}^l(X,W) = D^l(X,W) + \pi(W)w_L X. \tag{4.15}$$

Consider P as the projection of TM on $S(TM)$ so that any $X \in \Gamma(TM)$ can be written as $X = PX + \sum_{i=1}^r \eta_i(X)\xi_i$, where $\{\xi_i\}_{i=1}^r$ is a basis for $Rad(TM)$. Therefore, for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(RadTM)$, we have

$$D_X PY = D'_X PY + \tilde{h}'(X,PY), \quad D_X \xi = -\tilde{A}'_\xi X + \tilde{\nabla}'^t_X \xi, \tag{4.16}$$

where $(D'_X PY, \tilde{A}'_\xi X)$ and $(\tilde{h}'(X,PY), \tilde{\nabla}'^t_X \xi)$ belong to $S(TM)$ and $Rad(TM)$ respectively. Thus we have

$$D'_X PY = \nabla'_X PY + \pi(PY)PTX - K(X,PY), \tag{4.17}$$

$$\tilde{h}'(X,PY) = h'(X,PY) + \pi(PY)\sum_{i=1}^{r}\eta_i(TX)\xi, \tag{4.18}$$

and

$$\tilde{A}'_\xi X = A'_\xi X - \pi(\xi)PTX + K(X,\xi), \tag{4.19}$$

$$\tilde{\nabla}'^t_X \xi = \nabla'^t_X \xi + \pi(\xi)\sum_{i=1}^{r}\eta_i(TX)\xi_i, \tag{4.20}$$

where $\eta_i(X) = \tilde{g}(X, N_i)$. Further, using (2.9),(4.18) and (4.13), we derive

$$\tilde{g}(\tilde{h}'(X,PY), N_j) = g(\tilde{A}'_{N_j}X, PY) + \pi(N_j)g(PTX,PY) + \tilde{g}(K(X,N),PY) + \pi(PY)\eta_j(TX),$$

$$\tilde{g}(\tilde{h}^l(X,PY),\xi) = g(\tilde{A}'_\xi X, PY) - \pi(\xi)g(PTX,PY) - \tilde{g}(K(X,\xi),PY) + \pi(Y)g(w_L X,\xi),$$

Also, for induced connection D' of D, we get

$$(D'_X g)(PY,PZ) = -\pi(PY)g(PTX,PZ) - \pi(PZ)g(PY,PTX).$$

Since $\bar{M}$ is an indefinite Kaehler statistical manifold, the ensuing lemmas are obtained using (4.5), (3.8), (3.10) and (4.6).

Lemma 4.1. *For a $STCR$-lightlike submanifold M of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$, we have*

$$\begin{aligned} D_X TY - TD_X Y = \tilde{A}_{w_L Y}X + \tilde{A}_{w_s Y}X + \tilde{A}_{w_{\bar{2}} Y}X + \pi(\bar{J}Y)TX + B\tilde{h}^l(X,Y) \\ + B\tilde{h}^s(X,Y) + \pi(Y)X, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \tilde{D}^l(X, w_s Y) + \tilde{D}^l(X, w_{\bar{2}}Y) = w_L(D_X Y) - \tilde{\nabla}^l_X(w_L Y) - \tilde{h}^l(X,TY) + C_1\tilde{h}^s(X,Y) \\ + C_1\tilde{h}^l(X,Y) + \pi(\bar{J}Y)w_L X, \end{aligned} \tag{4.22}$$

$$\begin{aligned} \tilde{D}^s(X, w_L Y) = w_s(D_X Y) + w_{\bar{2}}(D_X Y) - \tilde{\nabla}^s_X(w_s Y) - \tilde{\nabla}^s_X(w_{\bar{2}}Y) - \tilde{h}^s(X,TY) \\ + C_2\tilde{h}^s(X,Y) + C_2\tilde{h}^l(X,Y) + \pi(\bar{J}Y)w_s X + \pi(\bar{J}Y)w_{\bar{2}}X, \end{aligned} \tag{4.23}$$

for any $X, Y \in \Gamma(TM)$.

Lemma 4.2. *Let M be a $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$. Then*

$$D_X BV - B\tilde{\nabla}^t_X V = -T\tilde{A}_V X + \tilde{A}_{C_1 V}X + \tilde{A}_{C_2 V}X + \pi(\bar{J}V)TX + \pi(V)X, \tag{4.24}$$

$$\tilde{h}^l(X,BV) = -\tilde{\nabla}^l_X C_1 V - \tilde{D}^l(X, C_2 V) + C_1\tilde{\nabla}^t_X V - w_L\tilde{A}_V X + \pi(\bar{J}V)w_L X, \tag{4.25}$$

$$\begin{aligned} \tilde{h}^s(X,BV) = -w_s\tilde{A}_V X - w_{\bar{2}}\tilde{A}_V X + C_2\tilde{\nabla}^t_X V + \pi(\bar{J}V)w_s X + \pi(\bar{J}V)w_{\bar{2}}X \\ -\tilde{\nabla}^s_X C_2 V - \tilde{D}^s(X, C_1 V), \end{aligned} \tag{4.26}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(tr(TM))$.

Definition 4.1. Let M be a lightlike submanifold of a indefinite Kaehler statistical manifold $\tilde{M}$. Then M is said to be a totally umbilical with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$) if $h(X,Y) = H\bar{g}(X,Y)$ (resp. $h^*(X,Y) = H^*\bar{g}(X,Y)$) for all $X, Y \in \Gamma(TM)$, where $H \in \Gamma(tr(TM))$ (resp. $H^* \in \Gamma(tr(TM))$) stands for transversal curvature vector fields of M in $\bar{M}$ with respect to $\bar{\nabla}$ (resp. $\bar{\nabla}^*$).

Also, M is totally umbilical with respect to $\bar{\nabla}$ (respectively $\bar{\nabla}^*$) if and only if on each co-ordinate neighbourhood, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ ($H^{*l} \in \Gamma(ltr(TM))$ and $H^{*s} \in \Gamma(S(TM^\perp))$ respectively) such that $h^l(X,Y) = H^l\bar{g}(X,Y)$, $h^s(X,Y) = H^s\bar{g}(X,Y)$ and $h^{*l}(X,Y) = H^{*l}\bar{g}(X,Y)$, $h^{*s}(X,Y) = H^{*s}\bar{g}(X,Y)$ respectively with respect to $\bar{\nabla}(respectively\ \bar{\nabla}^*)$.

Also, a $STCR$ lightlike submanifold of a indefinite Kaehler statistical manifold $\tilde{M}$ with quarter symmetric non-metric connection is said to be a totally umbilical if there exist smooth vector fields $\tilde{H}^l \in \Gamma(ltr(TM))$ and $\tilde{H}^s \in \Gamma(S(TM^\perp))$ such that $\tilde{h}^l(X,Y) = \tilde{H}^l g(X,Y)$ and $\tilde{h}^s(X,Y) = \tilde{H}^s g(X,Y)$.

Definition 4.2. A $STCR$ lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection is said to be a totally geodesic if $\tilde{h}(X,Y)=0$. It is simple to verify that M is totally geodesic if $\tilde{h}^l(X,Y)=0$, $\tilde{h}^s(X,Y)=0$ for any $X,Y\in\Gamma(TM)$.

Theorem 4.3. *Let M be a totally umbilical $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$ such that $\tilde{H}^s$ has no component in $\bar{J}E_2$. Then $E_\circ$ is integrable.*

Proof. Let $X,Y\in\Gamma(E_\circ)$ and $N\in\Gamma(L_2)$, then

$$\tilde{g}([X,Y],N)=\tilde{g}(\bar{\nabla}_XY-\bar{\nabla}_YX,N),$$

The symmetric property of difference (1,2) tensor K and (4.1) give

$$\tilde{g}([X,Y],N)=\tilde{g}(\bar{J}\tilde{D}_XY-\pi(Y)\bar{J}^2X-\bar{J}\tilde{D}_YX+\pi(X)\bar{J}^2Y,\bar{J}N),$$

Further from the definition of $STCR$ lightlike submanifold and using (4.5), we obtain

$$\tilde{g}([X,Y],N)=\tilde{g}(\tilde{h}^s(X,\bar{J}Y)-\tilde{h}^s(Y,\bar{J}X),\bar{J}N),$$

M being totally umbilical lightlike submanifold implies that

$$\tilde{g}([X,Y],N)=(g(X,\bar{J}Y)-g(Y,\bar{J}X))\tilde{g}(\tilde{H}^s,\bar{J}N).$$

Hence, the concept of $STCR$ lightlike submanifolds and the hypothesis leads to the required result.
. ∎

Theorem 4.4. *Let $\tilde{M}$ be an indefinite Kaehler statistical manifold with a quarter symmetric non-metric connection $\tilde{D}$ and M be a totally umbilical $STCR$-lightlike submanifold of $\tilde{M}$. If the distribution $E_\circ$ is integrable , then M is totally geodesic $STCR$ lightlike submanifold of $\tilde{M}$ with respect to $\tilde{D}$.*

Proof. For any $X,Y\in\Gamma(E_\circ)$ and from (4.23), we obtain

$$w_s(D_XY)+w_{\bar{2}}(D_XY)-w_s(D_YX)-w_{\bar{2}}(D_YX)=\tilde{h}^s(X,TY)-\tilde{h}^s(Y,TX),$$

Using the fact that M is a totally umbilical lightlike submanifold, we get

$$w_s[X,Y]+w_{\bar{2}}[X,Y]=(\tilde{g}(X,\bar{J}Y)-\tilde{g}(Y,\bar{J}X))\tilde{H}^s,$$

Since $E_\circ$ is integrable and if we take $X=\bar{J}$Y, then $2\tilde{g}(Y,Y)\tilde{H}^s=0$. Using the non-degeneracy of $E_\circ$, we get $\tilde{H}^s=0$. Now, for any $X,Y\in\Gamma(E_\circ)$, we have

$$w_L(D_XY)-w_L(D_YX)=\tilde{h}^l(X,TY)-\tilde{h}^l(Y,TX),$$

from (4.22).
As M is a totally umbilical lightlike submanifold, it follows that

$$w_L[X,Y]=(\tilde{g}(X,\bar{J}Y)-\tilde{g}(Y,\bar{J}X))\tilde{H}^l.$$

The non-degeneracy of $E_\circ$ implies $\tilde{H}^l=0$. Hence the result. ∎

Theorem 4.5. *Let M be a totally umbilical $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$. If M is totally geodesic, then $\tilde{h}'=0$ for any $X,Y\in\Gamma(E_\circ)$ and $N\in\Gamma(L_2)$.*

Proof. From (4.6), we have

$$\tilde{g}(\tilde{h}^s(X,\bar{J}Y),\bar{J}N)=\tilde{g}(\tilde{D}_X\bar{J}Y,\bar{J}N),$$

for any $X,Y\in\Gamma(E_\circ)$. Then (4.5),(4.6) and (4.16) imply

$$\tilde{g}(\tilde{h}^s(X,\bar{J}Y),\bar{J}N)=\tilde{g}(\tilde{h}'(X,Y),N).$$

Thus, the result follows using the given hypothesis. ∎

Theorem 4.6. *For a totally umbilical $STCR$-lightlike submanifold M of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$, the subbundle E_2 of $Rad(TM)$ is always integrable for any $X \in \Gamma(E_\circ)$.*

Proof. For any $\xi_1, \xi_2 \in \Gamma(E_2)$ and $X \in \Gamma(E_\circ)$, we have

$$\tilde{g}([\xi_1, \xi_2], X) = \tilde{g}(\bar{\nabla}_{\xi_1}\xi_2 - \bar{\nabla}_{\xi_2}\xi_1, X),$$
$$= \tilde{g}(\bar{J}\bar{\nabla}_{\xi_1}\xi_2, \bar{J}X) - (\bar{J}\bar{\nabla}_{\xi_2}\xi_1, \bar{J}X),$$

From definition (2.2), we get

$$\tilde{g}([\xi_1, \xi_2], X) = \tilde{g}(\bar{\nabla}^*_{\xi_1}\bar{J}\xi_2, \bar{J}X) - (\bar{\nabla}^*_{\xi_2}\bar{J}\xi_1, \bar{J}X),$$

Now (4.1) and (2.11) imply

$$\tilde{g}([\xi_1, \xi_2], X) = -\tilde{g}(\bar{J}\xi_2, \tilde{D}_{\xi_1}\bar{J}X) + \tilde{g}(\bar{J}\xi_1, \tilde{D}_{\xi_2}\bar{J}X),$$

Further, using (4.6), we derive

$$\tilde{g}([\xi_1, \xi_2], X) = -\tilde{g}(\bar{J}\xi_2, \tilde{h}^s(\xi_1, \bar{J}X)) + \tilde{g}(\bar{J}\xi_1, \tilde{h}^s(\xi_2, \bar{J}X)),$$

Since M is totally umbilical, therefore

$$\tilde{g}([\xi_1, \xi_2], X) = -\tilde{g}(\xi_1, \bar{J}X)\tilde{g}(\bar{J}\xi_2, \tilde{H}^s) + \tilde{g}(\xi_2, \bar{J}X)\tilde{g}(\bar{J}\xi_1, \tilde{H}^s),$$

As $\xi_1, \xi_2 \in \Gamma(E_2)$ and $X \in \Gamma(E_\circ)$, we obtain

$$\tilde{g}([\xi_1, \xi_2], X) = 0.$$

Thus our assertion follows. □

Theorem 4.7. *Let M be a proper totally umbilical $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$. Then $\bar{J}\tilde{H}^s = U$ for any $X, Y \in \bar{J}S$.*

Proof. Let $X, Y \in \bar{J}S$. Then using (4.21)

$$-TD_XY = \tilde{A}_{wY}X + B\tilde{h}^l(X,Y) + B\tilde{h}^s(X,Y) + \pi(Y)X,$$

Taking the inner product on both sides with respect to X, we have

$$\tilde{g}(\tilde{A}_{\bar{J}Y}X, X) = \tilde{g}(\tilde{h}^s(X,Y), \bar{J}X) - \tilde{g}(X,X)\pi(Y),$$

Now, (4.9) and (4.13) imply

$$\tilde{g}(A_{\bar{J}Y}X, X) + \tilde{g}(K(X, \bar{J}Y), X) = \tilde{g}(h^s(X,Y), \bar{J}X), \tag{4.27}$$

Further, using (2.5), (4.9) for dual connections of the indefinite Kaehler statistical manifold $\tilde{M}$, we have $\tilde{g}(h^{*s}(X,X), \bar{J}Y) = \tilde{g}(X, A_{\bar{J}Y}X)$,

$$\tilde{g}(\tilde{h}^s(X,X), \bar{J}Y)) - \pi(X)\tilde{g}(X,Y) + \tilde{g}(K(X, \bar{J}Y), X) = \tilde{g}(\tilde{h}^s(X,Y), \bar{J}X)$$
$$-\tilde{g}(X,X)\pi(Y),$$

From the concept of a totally umbilical lightlike submanifold, we get

$$\tilde{g}(X,X)\tilde{g}(\tilde{H}^s, \bar{J}Y) - \pi(X)\tilde{g}(X,Y) + \tilde{g}(K(X, \bar{J}Y), X) = \tilde{g}(X,Y)\tilde{g}(\tilde{H}^s, \bar{J}X)$$
$$-\tilde{g}(X,X)\pi(Y),$$

Interchanging Y by X and subtracting these equations, we obtain

$$\tilde{g}(\bar{J}\tilde{H}^s - U, X)(\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}(X,Y)^2) = \tilde{g}(X,X)\tilde{g}(K(Y, \bar{J}X), Y)$$
$$-\tilde{g}(X,Y)\tilde{g}(K(X, \bar{J}Y), X),$$

Since $X, Y \in \bar{J}S$ and M is a Kaehler statistical manifold, it follows that

$$\tilde{g}(\bar{J}\tilde{H}^s - U, X)(\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}(X,Y)^2) = 0.$$

So, using the non-degeneracy of S , we get the desired result. □

5. STCR-lightlike product manifolds

Definition 5.1. A $STCR$ lightlike submanifold M of an indefinite Kaehler statistical manifold $\tilde{M}$ is called a $STCR$-lightlike product manifold if E and $\bar{E}$ define totally geodesic foliations in M.

Theorem 5.1. *Let M be a $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$. Then the distribution E defines a totally geodesic foliation in M if and only if $\tilde{h}(X, \bar{J}Y) = 0$ for any $X, Y \in \Gamma(E)$.*

Proof. From the concept of $STCR$-lightlike submanifold, the distribution E defines a totally geodesic foliation in M, if and only if, $D_XY \in \Gamma(E)$ for $X, Y \in \Gamma(E)$ or $\tilde{g}(D_XY, \bar{J}\xi) = \tilde{g}(D_XY, \bar{J}W) = \tilde{g}(D_XY, N_2) = 0$ for $\xi_1 \in \Gamma(E_1), N_2 \in \Gamma(L_2), W \in \Gamma(S)$. Thus from definition (3.1) and equations (4.5), (4.6), we have

$$\tilde{g}(D_XY, \bar{J}\xi_1) = \tilde{g}(\tilde{D}_XY, \bar{J}\xi_1) = -\tilde{g}(\tilde{D}_X\bar{J}Y), \xi_1),$$

$$\tilde{g}(D_XY, \bar{J}\xi) = -\tilde{g}(\tilde{h}^l(X, \bar{J}Y), \xi_1),$$

Also,

$$\tilde{g}(D_XY, N_2) = \tilde{g}(\tilde{D}_X\bar{J}Y), \bar{J}N_2) = \tilde{g}(\tilde{h}^s(X, \bar{J}Y), \bar{J}N_2),$$

Similarly,

$$\tilde{g}(D_XY, \bar{J}W) = \tilde{g}(\tilde{D}_XY, \bar{J}W) = -\tilde{g}(\tilde{D}_X\bar{J}Y), W) = -\tilde{g}(\tilde{h}^s(X, \bar{J}Y), W).$$

Therefore, the distribution E defines a totally geodesic foliation in M, if and only if, $\tilde{h}(X, \bar{J}Y) = 0$ for $X, Y \in \Gamma(E)$. □

Theorem 5.2. *Let M be a $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$. Then the distribution $\bar{E}$ defines a totally geodesic foliation in M if and only if $\tilde{A}_{wY}X + \pi(Y)X \in \Gamma(\bar{E})$ for any $X, Y \in \Gamma(\bar{E})$.*

Proof. Since M is a $STCR$-lightlike submanifold of $\tilde{M}$, the distribution $\bar{E}$ defines a totally geodesic foliation in M, if and only if, $D_XY \in \Gamma(\bar{E})$ for $X, Y \in \Gamma(\bar{E})$. From (4.21), we get

$$-B\tilde{h}^l(X, Y) - B\tilde{h}^s(X, Y) = A_{w_LY}X + A_{w_sY}X + A_{w_{\bar{2}}Y}X + \pi(Y)X,$$

which implies

$$-B\tilde{h}(X, Y) = \tilde{A}_{wY}X + \pi(Y)X.$$

Thus, the proof is completed. □

Theorem 5.3. *Let M be a $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$. Then M is $STCR$ lightlike product manifold if the tensor field T is parallel with respect to the induced connection i.e. $(D_XT)Y = 0$ for any $X, Y \in \Gamma(TM)$.*

Proof. For $X, Y \in \Gamma(E)$ and from (4.21)

$$\pi(\bar{J}Y)TX + B\tilde{h}^l(X, Y) + B\tilde{h}^s(X, Y) + \pi(Y)X = 0,$$

using the hypothesis. Therefore, we get

$$\tilde{g}(B\tilde{h}^s(X, Y), N_2) = 0,$$

Also,

$$\tilde{g}(B\tilde{h}^l(X, Y), \xi_1) = 0, \quad \tilde{g}(B\tilde{h}^s(X, Y), W) = 0.$$

for $N_2 \in \Gamma(L_2), \xi \in \Gamma(E_1)$ and $W \in \Gamma(S)$. This implies that E defines a totally geodesic foliation in M. As per the supposition and (4.21), we derive

$$-B\tilde{h}(X, Y) = \tilde{A}_{wY}X + \pi(Y)X.$$

Thus $\bar{E}$ defines a totally geodesic foliation in M. Accordingly, M is a $STCR$ lightlike product manifold. □

However the converse does not hold.
If $\bar{E}$ defines a totally geodesic foliation in M, then $TD_XY = 0$ for $X, Y \in \Gamma(\bar{E})$. Now for $Y \in \Gamma(\bar{E})$, we have $TY = 0$ which implies that $D_XTY = 0$. Hence $(D_XT)Y = 0$, for any $X, Y \in \Gamma(\bar{E})$. Also, since E defines totally geodesic foliation in M, therefore from equation (4.21), we get $(D_XT)Y = \pi(\bar{J}Y)TX + \pi(Y)X \neq 0$. This is the claimed result.

Theorem 5.4. *Let M be a $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$ such that $w(D_XY) = 0$ for any $X, Y \in \Gamma(TM)$. Then M is $STCR$-lightlike product manifold if M is a totally geodesic $STCR$-lightlike submanifold of $\tilde{M}$.*

Proof. For any $X, Y \in \Gamma(E)$, we have

$$\tilde{h}^s(X, TY) - C_2\tilde{h}^s(X, Y) - C_2\tilde{h}^l(X, Y) = 0,$$

using (4.23). As M is totally geodesic $STCR$-lightlike submanifold, then

$$\tilde{g}(\tilde{h}^s(X, TY), W) = \tilde{g}(\tilde{h}^s(X, TY), \bar{J}N_2) = 0,$$

for any $W \in \Gamma(S)$ and $N_2 \in \Gamma(L_2)$. Also, from (4.22), we derive

$$\tilde{g}(\tilde{h}^l(X, TY), \xi) = 0,$$

for any $\xi \in \Gamma(E_1)$. This implies that E defines a totally geodesic foliation in M.
Further, from (2.11),(4.1), (4.2), we obtain

$$\tilde{g}(TD_XY, Z) = -\tilde{g}(Y, \bar{J}\tilde{h}(X, Z)).$$

for any $X, Y \in \Gamma(\bar{E})$ and $Z \in \Gamma(E_\circ)$. Since M is totally geodesic $STCR$-lightlike submanifold and the distribution $E_\circ$ is non-degenerate, therefore $TD_XY = 0$ for $X, Y \in \Gamma(\bar{E})$. Thus, $\bar{E}$ defines a totally geodesic foliation in M. This completes the proof. □

Theorem 5.5. *Let M be a totally umbilical $STCR$-lightlike submanifold of an indefinite Kaehler statistical manifold $\tilde{M}$ with a quarter symmetric non-metric connection $\tilde{D}$. Then M is a $STCR$-lightlike product manifold if and only if $\tilde{h}(X, \bar{J}Y) = 0$ for any $X \in \Gamma(TM)$, $Y \in \Gamma(E)$.*

Proof. Let M be $STCR$-lightlike product manifold it follows that $\tilde{h}(X, \bar{J}Y) = 0$ for any $X, Y \in \Gamma(E)$. Since M is a totally umbilical $STCR$-lightlike submanifold, therefore

$$\tilde{h}(X, \bar{J}Y) = \bar{g}(X, \bar{J}Y)\tilde{H} = 0,$$

for any $X \in \Gamma(\bar{E})$ and $Y \in \Gamma(E)$. So, we obtain $\tilde{h}(X, \bar{J}Y) = 0$ for any $X \in \Gamma(TM)$, $Y \in \Gamma(E)$.
Conversely, if $\tilde{h}(X, \bar{J}Y) = 0$ for any $X, Y \in \Gamma(E)$, then E defines a totally geodesic foliation in M. Now, for $X, Y \in \Gamma(\bar{E})$ and $Z \in \Gamma(E_\circ)$,

$$\tilde{g}(TD_XY, Z) = -\tilde{g}(\tilde{A}_{\bar{J}Y}X, Z) = \tilde{g}(\tilde{D}_X\bar{J}Y, Z),$$

Since $\bar{M}$ is an indefinite Kaehler statistical manifold,

$$\tilde{g}(\bar{J}Y, \bar{\nabla}^*_X Z) - \tilde{g}(K(X, JY), Z),$$

follows from (4.1).
Further from (4.2) and (2.11), we derive

$$\tilde{g}(TD_XY, Z) = -\tilde{g}(Y, \bar{J}\tilde{h}(X, Z)) = 0.$$

Since E is non-degenerate, therefore $TD_XY = 0$, which shows that $\bar{E}$ defines a totally geodesic foliation in M. □

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