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Research Article

Ideal and factor conditions for crossed modules and algebra-algebroids

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Abstract

In this paper, using the equivalence between the category of crossed modules of algebras and the category of algebra-algebroids, we will explore the notions of ideality and factors for these algebraic structures. We give the structure of a two sided ideal of an algebra-algebroid and the notion of quotient algebra-algebroid. By considering a two sided ideal of an algebra-algebroid, we show that the crossed module corresponding to this ideal is a crossed ideal of the crossed module corresponding to the algebra-algebroid. Conversely, by taking a crossed ideal of a crossed module, we also show that the corresponding algebra-algebroid to this crossed ideal is an ideal of the algebra-algebroid corresponding to the crossed module.

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1. Introduction

A crossed module, [10], in the category of associative algebras is a triple (C, R, ∂), where R is an **k**-algebra, C is an R-algebra with the action of R on C, and $\partial : C \to R$ is a map of R-algebras satisfying the rules: $\partial(c) \cdot c' = cc' = c \cdot \partial(c')$ for all $c, c' \in C$, where the action of $r \in R$ on $c \in C$ is denoted by $r \cdot c$. Crossed modules are generalisation of both modules and ideals and any algebra is a crossed module, so it is of structures to crossed modules. For example, if I is a two sided ideal of an algebra R, then (I, R, inc) is a crossed module, where inc is the inclusion map from I to R. Furthermore, if $\partial : C \to R$ is a crossed module of algebras, then $\partial(C)$ is a two sided ideal of R.

Recall from [4] that an R-category A is a category equipped with an R-module structure on each hom set such that the composition is R-bilinear, where R is a commutative ring. Using the enriched categories, we can say that an R-category is a category which is enriched over closed category of R-modules. An R-algebroid, [7], A is a small R-category.

In [10], Porter defined the structure of R-algebroids in a bit different setting from [7]. In his definition, an R-algebroid A on a fixed set of objects A_0, and the hom sets are disjoint family of R-modules. Porter proved that in the category of R-algebroids over a fixed set, any internal category is an internal groupoid. It is well known that groups

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are appropriately generalised to groupoids, and similarly Mosa in [7], explained that algebras are appropriately generalised to R-algebroids. He also introduced the notion of crossed modules of R-algebroids and gave some results using the well known properties of crossed modules of groups. In [9] Norrie defined the notion of subcrossed module, normal subcrossed module and quotient crossed module of groups. The categorical equivalence between crossed modules of groups and G-groupoids which are called group-groupoids in [1] or 2-groupoids, is well known by [2]. By this equivalence comparing the associated objects, in the paper [8] the authors constructed the notions of normal subgroup-groupoid and quotient group-groupoid. In this mentioned paper, [8], they have defined the notions of normal subgroup-groupoid and quotient group-groupoid interpreting the group-groupoids corresponding respectively to the normal subcrossed modules and quotient crossed modules. In [11], Porter introduced the notion of crossed module over groups with operations. He also proved that the category of crossed modules of groups with operations is equivalent to that of internal groupoids within groups with operations. Using these results, Şahan and Mucuk [13], in the category of groups with operations, have defined the notions subcrossed module and quotient crossed module of a crossed module over groups with operations. Therefore they have eliminated a gap in the expositions of the equivalence between crossed modules of groups and G-groupoids (or group-groupoids), and groups with operations by discussing the notions of normality and quotient structures for these algebraic settings.

In this paper, we have defined two sided ideal of algebra-algebroids and quotient algebra-algebroids. We prove that if A' is a two sided ideal of an algebra-algebroid A, then the crossed module corresponding to A' is a crossed ideal of the crossed module corresponding to A. Conversely, we show that if C' is a crossed ideal of C, and B' is the corresponding algebra-algebroid to C', then B' is an ideal of the algebra-algebroid B corresponding to the crossed module C.

2. Preliminaries

In this chapter, we begin by recalling the notions of R-algebroids and crossed modules over R-algebras. Mitchell [4–6] has given the category of R-algebroids, and proved some important results on these materials. These algebraic structures have been studied in several papers for examples [4–7, 10]. The following definition has been given by Mitchell. (cf [4]).

Suppose that R is a commutative ring. Recall from [7] that an R-category is a category equipped with an R-module structure over hom sets such that the category composition is R-bilinear. An R-algebroid is a small R-category.

Now we will give a detailed definition of an R-algebroid A on a set of objects A_0 as follows:

A directed graph A over a set A_0 consists of the functions $s,t : A \to A_0,\ \varepsilon : A_0 \to A$ called the source, target and identity maps, respectively, such that the condition $s\varepsilon = t\varepsilon = \mathrm{id}_{A_0}$ is satisfied. We show a directed graph A over the set A_0 as diagramatically

$$\begin{tikzcd} A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A_0 . \arrow[l, bend left, "\varepsilon"] \end{tikzcd}$$

For $x, y \in A_0$, we can write the set

$$A(x,y) = \{a \in A :\ s(a) = x,\ t(a) = y\}$$

and show 1_x for $\varepsilon(x)$, where $\varepsilon(x)$ is an arrow from x to x. Let $a \in A(x,y)$. Then we can write this arrow by $a : x \to y$.

The following definition can be found in [7].

Definition 2.1. An R-algebroid $(A, A_0, s, t, \varepsilon, +, \circ)$ is a directed graph

$$\begin{tikzcd} A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A_0 \arrow[l, bend left=50, "\varepsilon"] \end{tikzcd}$$

together with the following properties; $(x, y \in A_0)$

(1) On each hom set $A(x, y)$, there is an R-module structure,
(2) There is an R-bilinear function, called composition, denoted by

$$\begin{array}{rcl} \circ : A(x,y) \times A(y,z) & \longrightarrow & A(x,z) \\ (a,b) & \longmapsto & a \circ b. \end{array}$$

Then, for $a : x \to y$ and $b : y \to z$, we have $s(a \circ b) = s(a)$ and $t(a \circ b) = t(b)$. This composition is associative and the elements 1_x, $x \in A_0$, act as identities for the composition. Equivalently, we can explain this statements as follows:

If $a \in A(x, y)$, that is, $a : x \to y$, then we have $1_x \circ a = a \circ 1_y = a$. Consequently, such a composition in the directed graph (A, A_0), makes into a small category. The zero elements of $A(x, y)$ are denoted 0_{xy} or 0. From the bilinearity, we can write $a \circ 0 = 0 \circ a = a$.

Now suppose that $(A, A_0, s, t, \varepsilon, \circ)$ and $(B, B_0, s', t', \varepsilon', \circ')$ are R-algebroids. A morphism f from A to B consists of the pairs of morphisms (f_1, f_0) such that the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "f_1"'] & A_0 \arrow[l, bend right=50, "\varepsilon"'] \arrow[d, "f_0"] \\ B \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] & B_0 \arrow[l, bend left=50, "\varepsilon'"] \end{tikzcd}$$

that is, $f_0 s = s' f_1,\ f_0 t = t' f_1,\ \varepsilon f_0 = f_1 \varepsilon$ and where f_1 is an homomorphism of R-modules.

Definition 2.2. ([7]) For an R-algebroid A over the set A_0, a two sided ideal I in A is a family of submodules

$$\{I(x, y) \subseteq A(x, y)\}_{x,y \in A_0}$$

such that I satisfies the axiom: if $a \in I(x, y),\ b \in A(z, x),\ c \in A(y, w)$, then $b \circ a \in I(z, y)$ and $a \circ c \in I(x, w)$.

Let A be an R-algebroid over A_0. A subalgebroid A' is a disjoint family of R-submodules $\{A'(x, y) \subseteq A(x, y)\}_{x,y \in A_0}$ with units and each R-bilinear function

$$A'(x, y) \times A'(y, z) \to A'(x, z)$$

is the restriction of the R-bilinear function of A.

We can give the definition of factor algebroid from [7].

Definition 2.3. Suppose that I is a two sided ideal in A. It can be defined the factor R-module $A(x, y)/I(x, y)$ for all $x, y \in A_0$. Then there is an R-bilinear morphism

$$A(x, y)/I(x, y) \times A(y, z)/I(y, z) \to A(x, z)/I(x, z)$$

and associativity holds. Then one can get an R-algebroid A/I when is the family of factor R-modules

$$\{A(x, y)/I(x, y)\ :\ x, y \in A_0\}.$$

3. Algebra-Algebroids and crossed modules of algebras

In this section, we explore an equivalence between the category of algebra-algebroids and the category of crossed modules of algebras.

Definition 3.1. An internal category in the category of R-algebras is called an algebra-algebroid. So an algebra-algebroid (A, A_0) is an algebroid A in which the set of objects A_0 and the set of morphisms A are R-algebras, the source, target and unit maps are homomorphisms of R-algebras.

For an algebra-algebroid (A, A_0) for $a, b \in A$, the multiplication of A is denoted by $a.b$ while the algebroid composition is denoted by $a \circ b$ with $t(a) = s(b)$, the additive inverse of a is denoted by $-a$. In an algebra-algebroid, the interchange law is given by

$$(a \circ b) \cdot (c \circ d) = (a \cdot c \circ b \cdot d)$$

for $a, b, c, d \in A$.

A morphism of algebra-algebroid is a morphism of algebroids in which each morphism is an homomorphism of R-algebras. We will denote the category of algebra-algebroids by **AlgAlgoid**.

Recall that if M is R-algebra, the maps

$$\begin{array}{rcl} R \times M & \longrightarrow & M \\ (r, m) & \longmapsto & r \cdot m \end{array} \quad , \quad \begin{array}{rcl} M \times R & \longrightarrow & M \\ (m, r) & \longmapsto & m \cdot r \end{array}$$

are left and right actions of R on M if and only if

1.	$k(r \cdot m) = (kr) \cdot m = r \cdot (km)$	,	$k(m \cdot r) = (km) \cdot r = m \cdot (kr)$
2.	$r \cdot (m + m') = r \cdot m + r \cdot m'$	,	$(m + m') \cdot r = m \cdot r + m' \cdot r$
3.	$(r + r') \cdot m = r \cdot m + r' \cdot m$	,	$m \cdot (r + r') = m \cdot r + m \cdot r'$
4.	$r \cdot (mm') = (r \cdot m)m' = m(r \cdot m')$	,	$(mm') \cdot r = m(m' \cdot r) = (m \cdot r)m'$
5.	$(rr') \cdot m = r \cdot (r' \cdot m)$	,	$m \cdot (rr') = r(m \cdot r) \cdot r'$

for all $k \in \mathbf{k},\ m, m' \in M,\ r, r' \in R$.

Definition 3.2. Let R be a $\mathbf{k}$-algebra with identity. A precrossed module of commutative algebras is an R-algebra C, together with the action of R on C and R-algebra homomorphism $\partial : C \to R$, such that $c \in C$, $r \in R$

CM1. $\partial(r \cdot c) = r\partial(c), \partial(c \cdot r) = \partial(c)r$

This is a crossed R-module if an addition, for all $c, c' \in C$,

CM2. $\partial(c) \cdot c' = cc' = c \cdot \partial(c')$

The last condition is called the Peiffer identity. We denote such a crossed module by (C, R, ∂).

A morphism in the category of crossed module of R-algebras is a pair of morphisms of R-algebras $(f, g) : (C, R, \partial) \to (C', R', \partial')$, where $f : C \to C'$ and $g : R \to R'$ such that $f(r \cdot c) = g(r)f(c)$ similarly $f(c.r) = f(c)g(r)$ for all $c \in C,\ r \in R$ and we have $\partial' f(c) = g\partial(c)$.

We can define the category of crossed modules denoting it by **CM**.

The following proposition is well-known, we give a sketch of proof to use it in the next sections.

Proposition 3.3. *The category of crossed modules of R-algebras is equivalent to that of algebra-algebroids.*

Proof. First we give a construction of the functor

$$\delta : \mathbf{AlgAlgoid} \to \mathbf{CM}$$

from the category of algebra-algebroids to the category of crossed modules. Let (A, A_0) be an algebra-algebroid. We define $\delta(A, A_0)$ by a crossed module (C, R, ∂), where $C =$

Ker$s$, $R = A_0$ and $\delta = \partial\,|_{\mathrm{Ker}s}$ is the restriction of the target map. Then clearly, C and R are algebras and the target map is a morphism of algebras. The action of R on C can be given by

$$c.r = c\varepsilon(r), \qquad r.c = \varepsilon(r)c$$

for $r \in R$ and $c \in C$, and then we obtain

$$\begin{aligned} t(r \cdot c) &= t(\varepsilon(r))t(c) \\ &= rt(c) \end{aligned}$$

similarly

$$t(c \cdot r) = t(c)r$$

and

$$c \cdot t(c') = cc' = t(c) \cdot c',$$

for all $r \in R,\ c, c' \in C$. Then, (C, R, ∂) is a crossed module of R-algebras.

Now, we define the functor

$$\eta : \mathbf{CM} \to \mathbf{AlgAlgoid}.$$

Suppose that (C, R, ∂) is a crossed module. We give $\eta(C, R, \partial)$ as an algebra-algebroid (A, A_0), where $A_0 = R,\ A_1 = C \ltimes R$ is the semidirect product of C and R with the operations

$$\begin{aligned} (c, r) + (c', r') &= (c + c', r + r') \\ (c, r) \cdot k &= (ck, rk) \\ k \cdot (c, r) &= (kc, kr) \\ (c, r) \cdot (c', r') &= (cc' + r \cdot c' + c \cdot r', rr') \end{aligned}$$

for all $r, r' \in R,\ c, c' \in C,\ k \in \mathbf{k}$. The structual maps of algebra-algebroid are given by $s(c, r) = r,\ t(c, r) = \partial(c) + r$ and the composition of algebroid can be defined by

$$(c, r) \circ (c', r') = (c + c', r')$$

if $r' = \partial(c) + r$. This composition can be denoted by the following way

$$\begin{tikzcd} r \arrow[r] \arrow[rrr, bend right] & \partial(c) + r & r' \arrow[r] & \partial(c') + r'. \end{tikzcd}$$

Therefore we have

$$\begin{aligned} s((c, r) \circ (c', r')) &= r = s(c, r) \\ t((c, r) \circ (c', r')) &= \partial(c') + \partial(c) + r \\ &= \partial(c') + r' \\ &= t(c', r'). \end{aligned}$$

One can easily check the interchange law, using the conditions CM1. and CM2. □

3.1. Factor crossed modules and factor algebra-algebroids

In the section, we define the notions of subalgebra-algebroid and two sided ideal of an algebra-algebroid by using the corresponding crossed modules cases discussed by Shammu in his Ph.D. thesis [12].

Definition 3.4. Suppose that (C, R, ∂) is a crossed module. A subcrossed module (C', R', ∂') of (C, R, ∂) is a crossed module satisfying the following conditions

(1) C' is a subalgebra of C and R' is a subring of R.
(2) The action of R' on C' is the restriction of the action of R on C.
(3) ∂' is the restriction of ∂ to C'.

Definition 3.5. A subcrossed module (C', R', ∂') of the crossed module (C, R, ∂) is called a crossed ideal such that the following conditions hold.

CI1. For all $c \in C$ and $c' \in C'$, $cc', c'c \in C'$ and for all $r \in R$, $r' \in R'$, $rr', r'r \in R'$.

CI2. For all $c \in C$ and $r' \in R'$, $r' \cdot c, c \cdot r' \in C'$.

CI3. C' is closed under action of R, i.e. $r \cdot c', c' \cdot r \in C'$ for all $r \in R$ and $c' \in C'$.

The following diagram is commutative

$$\begin{tikzcd} C' \arrow[r, "\partial'"] \arrow[d, hook, "u"'] & R' \arrow[d, hook, "v"] \\ C \arrow[r, "\partial"'] & R \end{tikzcd}$$

where u and v are inclusion.

Theorem 3.6. *Let (C', R', ∂') be a crossed ideal in (C, R, ∂). Then the morphism of factor algebras $\eta : C/C' \to R/R'$ is a crossed module structure.*

Proof. Consider the following diagram

$$\begin{tikzcd} C \arrow[r, "q_1"] \arrow[d, "\partial"'] & C/C' \arrow[d, "\eta"] \\ R \arrow[r, "q_2"'] & R/R' \end{tikzcd}$$

in which q_1 and q_2 are canonical homomorphism. Then R acts on C/C' by $r \cdot (c + C') = r \cdot c + C'$ since $r \cdot c \in C$ and R' acts on C/C' by trivially. Because we obtain $r' \cdot (c + C') = r' \cdot c + C' = 0 + C'$ since $r' \cdot c \in C'$ for all $r' \in R'$ and $c \in C$. Therefore R/R' acts on C/C' by

$$(r + R') \cdot (c + C') = r \cdot c + C'$$

and then $\eta : C/C' \to R/R'$ given by $\eta(c + C') = \partial'(c) + R'$ is a well defined homomorphism. Indeed, if

$$\begin{aligned} c_1 + C' = c_2 + C' &\Rightarrow c_1 - c_2 \in C' \\ \partial'(c_1 - c_2) \in R' &\Rightarrow \partial'(c_1) + R' = \partial'(c_2) + R' \end{aligned}$$

and we have $\eta(c_1 + C') = \eta(c_2 + C')$.

We can show the crossed modules conditions as follows:

CM1. For all $r \in R$ and $c \in C$,

$$\begin{aligned} \eta((r + R') \cdot (c + C')) &= \eta(r \cdot c + C') \\ &= \partial'(r \cdot c) + R' \\ &= r\partial'(c) + R' \\ &= r\eta(c + C') \end{aligned}$$

CM2. For all $c_1, c_2 \in C$,

$$\begin{aligned} \eta(c_1 + C') \cdot (c_2 + C') &= (\partial'(c_1 + C')) \cdot (c_2 + C') \\ &= \partial'(c_1) \cdot c_2 + C' \\ &= c_1c_2 + C' \\ &= (c_1 + C')(c_2 + C'). \end{aligned}$$

It is called that $(C/C', R/R', \eta)$ is a factor crossed module of (C, R, ∂) by (C', R', ∂'). $\square$

Remark 3.7. For a crossed ideal (C', R', ∂') of the crossed module (C, R, ∂), the following diagram

$$\begin{tikzcd} C' \arrow[r, "\partial'"] \arrow[d, hook, "u"'] & R' \arrow[d, hook, "v"] \\ C \arrow[r, "\partial"'] & R \end{tikzcd}$$

together with the h-map given by

$$\begin{array}{rcl} h : R' \times C & \to & C' \\ (r', c) & \mapsto & r' \cdot c \in C' \end{array}$$

from CI3, is a crossed square in the sense of [3].

Proposition 3.8. *Let*

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r] \arrow[d, "\partial'"] & C \arrow[r] \arrow[d, "\partial"] & P' \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & R' \arrow[r] & R \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence of crossed modules of algebras. Then (C', R', ∂') is a crossed ideal of (C, R, ∂) and we have the short exact sequence of algebras

$$0 \longrightarrow C' \rtimes R' \longrightarrow C \rtimes R \longrightarrow P' \rtimes P \longrightarrow 0$$

and $C' \rtimes R'$ is an ideal of $C \rtimes R$.

The isomorphism theorem for crossed modules was given by [12] as follows:

Theorem 3.9. *Let $(f, g) : (M, N, \delta) \to (C, R, \partial)$ be a morphism of crossed modules and* $\mathrm{Ker} f = M'$, $\mathrm{Ker} g = N'$. *Then*

$$(M/M', N/N', \eta) \cong (\mathrm{Im} M, \mathrm{Im} N, \partial)$$

By considering subcrossed module and crossed ideals, to obtain a neat definition of subalgebra-algebroids and two sided ideal algebroid, we will give the following theorem.

Theorem 3.10. *We suppose that (C', R', ∂') is a subcrossed module of the crossed module (C, R, ∂) and (A', A'_0), (A, A_0) are the algebra-algebroids corresponding to these crossed modules respectively. Then A' is a subalgebroid of A, A'_0 is a subalgebra of A_0 and the algebra of morphisms A' is a subalgebra of A.*

Theorem 3.11. *Let (C', R', ∂') be a subcrossed ideal of (C, R, ∂) and let (B', B'_0), (B, B_0) be the algebra-algebroids corresponding to these crossed modules respectively. Then (B', B'_0) is a subalgebroid of (B, B_0), B'_0 is an ideal of B_0 and B' is an ideal of B.*

Proof. Suppose that (C', R', ∂') is a crossed ideal of (C, R, ∂), $\mathbf{B}' = (B', B'_0)$ is a subalgebroid of $\mathbf{B} = (B, B_0)$. We have that $Ob(\mathbf{B}') = B'_0$ is an ideal of $Ob(\mathbf{B}) = B_0$, from CI1. of Definition 3.5. We only need to checks that $B' = C' \rtimes R$ is an ideal of $B = C \rtimes R$. This is obtained by Proposition 3.8. Using definition of crossed ideal, we can this calculation as follows: For $(c, r) \in C \rtimes R$ and $(c', r') \in C' \rtimes R'$,

$$(c, r).(c', r') = (cc' + c.r' + r.c', rr').$$

Since (C', R', ∂') is a crossed ideal of (C, R, ∂), by the condition CI1, we have $cc' \in C$ and by the condition CI2 and CI3, $c \cdot r', r \cdot c' \in C'$, because R' is an ideal of R it follows that $rr' \in R'$. Thus, we have $(c, r).(c', r') \in C' \rtimes R'$, and similarly $(c', r').(c, r) \in C' \rtimes R'$. So $B' = C' \rtimes R$ is an ideal of $B = C \rtimes R$ as required. $\square$

Now we can give the definitions of subalgebra-algebroid and two sided ideal of an algebroid as follows.

Definition 3.12. Let (A, A_0) be an algebra-algebroid and (A', A'_0) is an subalgebroid of (A, A_0). If A'_0 is a subalgebra of A_0 and the algebra of morphisms of (A', A'_0) is a subalgebra of (A, A_0), then we say that (A', A'_0) is a subalgebra-algebroid of (A, A_0).

Definition 3.13. Let (A, A_0) be an algebra-algebroid and (A', A'_0) is an subalgebra-algebroid of (A, A_0). Then (A', A'_0) is called two sided ideal of (A, A_0), if A'_0 is a two sided ideal of A_0 and the algebra of morphisms of (A', A'_0) is a two sided ideal of (A, A_0).

Proposition 3.14. *Suppose that (A, A_0) and (B, B_0) are algebra-algebroids and $f = (f_1, f_0)$ is a algebra-algebroid morphism as shown in the following diagram*

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & B \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] \\
A_0 \arrow[r, "f_0"'] \arrow[u, bend left=60, "\varepsilon"] & B_0 \arrow[u, bend right=60, "\varepsilon'"']
\end{tikzcd}$$

Then $\mathrm{Ker} f = (\mathrm{Ker} f_1, \mathrm{Ker} f_0)$ *is a two sided ideal of A, A_0.*

Theorem 3.15. *Assume that (A, A_0) is a algebra-algebroid.*

(1) *If (A', A'_0) is a subalgebra-algebroid of (A, A_0), then the crossed module (C', R', ∂') corresponding to (A', A'_0) is a subrossed module of the crossed module (C, R, ∂) corresponding to (A, A_0).*

(2) *If (A', A'_0) is a two sided ideal of the algebroid (A, A_0), then the crossed module (C', R', ∂') corresponding to (A', A'_0) is a crossed ideal of (C, R, ∂) corresponding to (A, A_0).*

Proof. 1. It is clear.

2. Suppose that (A', A'_0) is a two sided ideal of the algebroid (A, A_0) and (C', R', ∂'), (C, R, ∂) are the crossed modules corresponding to these structures. We know that (C', R', ∂') is a subcrossed module of (C, R, ∂). Now, we prove for (C', R', ∂') that the conditions CI1, CI2, CI3 are satisfied.

CI1. From Definition 3.13, we have that R' is a two sided ideal of R.

CI2. For $y \in A'_0 = R'$ and $a \in \mathrm{Ker} s = C$. We have

$$\begin{aligned} s(y.a) &= s(\varepsilon(y)).s(a) \\ &= s(\varepsilon(y))0 \\ &= 0 \end{aligned}$$

so $y.a \in \mathrm{Ker} s$. Since $a \in A$, $\varepsilon(y) \in A'$ and (A', A'_0) is a two sided ideal of (A, A_0), it follows that $\varepsilon(y).a \in A'$. Therefore, $y.a = \varepsilon(y)a \in \mathrm{Ker} s \cap A' = \mathrm{Ker} s' = C'$.

CI3. For $x \in A_0 = R$ and $a \in C' = \mathrm{Ker} s \cap A'$. $x.a = \varepsilon(x).a$ where $\varepsilon(x) \in (A, A_0)$. Since (A', A'_0) is a two sided ideal of (A, A_0), we have $\varepsilon(x).a \in A'$. Further

$$\begin{aligned} s(x.a) &= s(\varepsilon(x).a) \\ &= s(\varepsilon(x)).s(a) \\ &= 0_x 0 \\ &= 0_x \end{aligned}$$

so $x.a \in C'$.

Thus we obtain that (C', R', ∂') is a crossed ideal of (C, R, ∂). $\square$

Corollary 3.16. *Let (A, A_0) be an algebra-algebroid and (C, R, ∂) is corresponding crossed module. Then the category of two sided ideal of the algebroid (A, A_0) is equivalent to the category of crossed ideal of the crossed module (C, R, ∂).*

Proposition 3.17. *Let (C, R, ∂) be a crossed module of algebras and (C', R', ∂') a crossed ideal of (C, R, ∂). Then we have*

$$C/C' \rtimes R/R' \cong (C \rtimes R)/(C' \rtimes R').$$

Proof. It can be easily checked that

$$\begin{array}{rcl} \phi : C/C' \rtimes R/R' & \to & (C \rtimes R)/(C' \rtimes R') \\ (a + C', b + R') & \mapsto & (a, b) + (C' \rtimes R') \end{array}$$

is an isomorphism of algebras. □

Definition 3.18. Let (A, A_0) be an algebra-algebroid and (A', A'_0) is a two sided ideal of (A, A_0). Let (C, R, ∂) and (C', R', ∂') be the crossed modules corresponding respectively (A, A_0) and (A', A'_0). Then the algebra-algebroid corresponding to the factor crossed module $(C/C', R/R', \delta)$ is called factor algebra-algebroid and denoted by A/A'.

So the set of objects A/A' is the factor algebra R/R'; and the set of morphisms is the factor algebra A_0/A'_0. For $a \in A/A'$, the source of a is $s(a+R') = s(a)+R'$ and the target of a is $t(a + R') = t(a) + R'$, where

$$(b + R') \circ (a + R') = (b \circ a) + R'$$

in the case of $t(b) = s(a)$. The identity morphism for $x \in R'$ is $\varepsilon(x + R') = \varepsilon(x) + R'$.

Theorem 3.19. *Let (A, A_0) be an algebra-algebroid and (A', A'_0) is a two sided ideal of (A, A_0). Then, there is an algebra-algebroid morphism $f = (f_1, f_0) : (A, A_0) \to (B, B_0)$ such that* $\mathrm{Ker} f = (A', A'_0)$.

Proof. Assume that (A, A_0) be an algebra-algebroid and (A', A'_0) is a two sided ideal of (A, A_0). Let (C, R, ∂) and (C', R', ∂') be the crossed modules corresponding respectively (A, A_0) and (A', A'_0). We can construct the following short exact sequence of crossed modules:

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r] \arrow[d, "\partial'"] & C \arrow[r] \arrow[d, "\partial"] & C/C' \arrow[r] \arrow[d, "\delta"] & 0 \\ 0 \arrow[r] & R' \arrow[r] & R \arrow[r] & R/R' \arrow[r] & 0 \end{tikzcd}$$

By proposition 3.8, we have the short exact sequence of algebras

$$0 \longrightarrow C' \rtimes R' \longrightarrow C \rtimes R \longrightarrow C/C' \rtimes R/R' \longrightarrow 0$$

We have that $\mathrm{Ker} q = C/C' \rtimes R/R'$. Evaluating this is terms of corresponding algebra-algebroids, we obtain that (A', A'_0) is the kernel of $f : (A, A_0) \to (A/A', A_0, A'_0)$. □

Theorem 3.20. *Let (A, A_0) and (B, B_0) be algebra-algebroid and $f = (f_1, f_0) : (A, A_0) \to (B, B_0)$ a morphism of algebra-algebroids. Then the image $f(A, A_0)$ is a subalgebra-algebroid and isomorph to the factor algebra-algebroid A/A', where* $\mathrm{Ker} f = A'$.

Corollary 3.21. *If $f = (f_1, f_0) : (A, A_0) \to (B, B_0)$ a morphism of algebra-algebroids which is surjective on morphisms and* $\mathrm{Ker} f = A'$. *Then the factor algebra-algebroid A/A' is isomorph to the algebra-algebroid (B, B_0).*

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