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Research Article

Singular perturbations arising in complex Newton's method

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Abstract

We examine the resulting dynamics when Newton's method is applied to perturbations on polynomials that have a multiple root. Specifically, we consider the case where Newton's method is applied to the polynomial family $(z^2 + c)(z - 1)$.

Mathematics Subject Classification (2020). 37F05, 37F10

Keywords. Julia set, Newton method, rational iteration, singular perturbations

1. Introduction

Iteration of rational maps in one complex variable has been widely studied in recent decades continuing the remarkable papers of P.Fatou and G.Julia who introduced normal families and Montel's Theorem to the subject at the begin of the twentieth century. Indeed, these maps are the natural family of functions when iteration of holomorphic maps are on the Riemann sphere $\mathbb{C}_\infty$. In recent years, much attention has been paid to families of rational maps that arise as singular perturbations of polynomials. These are families of rational maps that depend on a parameter A and have the property that, when $A = 0$, the map involved is a polynomial of degree n, but for all other parameters, the maps are rational with higher degree. When the parameter A becomes non-zero, the dynamics of these maps usually go through a substantial transformation. Most of the study of these singular perturbed rational maps has centered on families of the form $F_A(z) = z^n + A/z^d$ where $A \in \mathbb{C}$, n and d are positive integers.

Our main aim in this paper is to describe what happens when Newton's method is applied to the complex polynomial $F_c(z) = (z^2 + c)(z - 1)$ when the parameter c is non-zero but quite small. We shall write $F_c(z) = P_c(z)p(z)$. In this case, the map F_c is called a singular perturbation of $z^2p(z)$. The reason for the interest in such a perturbation arises because the rational map given by Newton's method, namely $N_{F_c} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by $N_{F_c}(z) = z - \frac{F_c(z)}{F_c'(z)}$ has a different degree when c becomes non-zero.

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Complex rational maps are naturally more complicated than polynomials. For simplicity, we shall consider the simplest possible case where $z \to z^2$. For this map, it is well known that the Julia set is unit circle, but when we perturb that map to one of the form $F_A(z) = z^2 + A/z^2$ for $A \neq 0$ the degree of $F_A(z)$ increases from 2 to 4, the origin becomes a pole of order 2 and Julia sets change dramatically. See Figures 1 and 2 [4, p.14]

Figure 1. $A = 0$

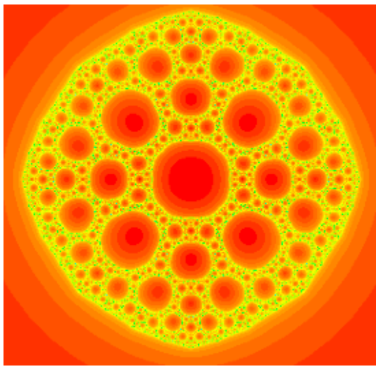

Figure 2. $A = -1/16$

In complex dynamics, the object of central interest in the dynamical plane is the Julia set. For the family P_c there is an open neighborhood of ∞ in the Riemann sphere consisting of points whose orbits tend to ∞. The set of all points whose orbits tend to ∞ is called the basin of ∞. Then the Julia set, denoted by $\mathcal{J}(P_c)$, is the boundary of this basin. There are other equivalent definitions of Julia set. For instance, the Julia set is also the closure of the set of repelling periodic points. Therefore, arbitrarily close to any point in the Julia set, we have both escaping and periodic points, so the Julia set is the place where chaos occurs for these maps. In fact, via Montel's Theorem, given any point in the Julia set, then any open neighborhood of this point, no matter how small, is eventually mapped over the entire complex plane (minus at most one point). Thus the family of iterates of P_c on the Julia set is very sensitive to initial conditions. The filled Julia set is, by definition, the set of all points whose orbits do not tend to ∞. $\mathcal{J}(P_c)$ is also the boundary of the filled Julia set. The Fatou set is then complement of $\mathcal{J}(P_c)$ in Riemann sphere. This is where the dynamical behavior is relatively tame [3, p.269], [4, p.233].

The goal of this paper is to investigate the dynamics and the Julia sets of the Newton iteration function, $N_{F_c}(z)$, applied to the polynomial $F_c(z) = (z^2 + c)(z - 1)$. We shall pay specific attention to one special critical point and see how the orbit of this point affects the dynamics of the rational map.

The dynamics of Newton's method always presents difficult problems, even when applied to polynomials in one variable. Iteration of the Newton's method function often allows one to find the roots of the corresponding polynomial, but this is not always the case. Specifically, let $N_f(z)$ be Newton iteration map corresponding to the function f. By starting with initial seed z_0 iteration gives the sequence $z_0, z_1 = N_f(z_0), z_2 = N_f(z_1) = N_f^2(z_0), \cdots$ which hopefully converges to a root ζ of f. That certainly happens most of the time, but other things can happen. For example, consider $f(x) = x^{1/3}$, this function is not differentiable at the root $x = 0$. Note that $N_f(x) = -2x$ and $|N_f'(0)| > 1$ and all sequences tend to ∞. Hence we may have no convergence if there is no differentiability. Convergence of Newton-iteration towards a fixed point represents the simplest possible behavior for N_f viewed as a dynamical system. In some cases the convergence of Newton's method is guaranteed, as per Kantorovitch's theorem [6, p.197, 206, 207], [4, p.173].

We shall think of the Newton's method function as being defined on the whole Riemann sphere, i.e., the complex numbers together with the point infinity, $\mathbb{C} \cup \{\infty\}$. The orbit of a point ξ could converge to a cycle, or it could wander chaotically about Riemann sphere, or it could behave in other ways. A point $\xi \in \mathbb{C}$ is called a periodic point of period n if $N_f^n(\xi) = \xi$ and $N_f^k(\xi) \neq \xi$ for all $k < n$, where $k, n \in \mathbb{N}$. If $n = 1$, we say that ξ is a fixed point of N_f and, as is well known, such points correspond to the roots of f. The derivative of $N_f(z)$ is $N_f'(z) = \frac{f(z)f''(z)}{[f'(z)]^2}$ and therefore, the simple roots of $f(z)$ are super-attracting fixed points of $N_f(z)$, i.e., the derivative of N_f at this point is 0. Other types of fixed points may arise. For example, the fixed point is attracting if $|N_f'(\xi)| < 1$; it is rationally indifferent(or parabolic, or neutral) if $|N_f'(\xi)| = e^{2\pi it}$ with some $t \in \mathbb{Q}$; and it is irrationally indifferent if $|N_f'(\xi)| = e^{2\pi it}$ with some $t \in \mathbb{R} \setminus \mathbb{Q}$. Using the Taylor's series for $N_f(z)$, it can be shown that $N_f(z)$ will be linearly convergent at an attracting fixed point and at least quadratically convergent at a super-attracting fixed point. The point at ∞ is always a repelling fixed point with derivative $d/(d-1)$, where d is the degree of f, so large values of z will tend to move away from infinity under iteration[4, p.139]. A point is a critical point if the derivative of the map vanishes at this point. Critical points of N_f are solutions of $N_f'(z) = 0$, i.e., zeroes and inflection points of f. The critical point is non-degenerate if $N_f''(z) \neq 0$ and it is degenerate if $N_f''(z) = 0$. For example, $f(x) = x^n$ has a degenerate critical point at 0 when $n > 2$, but has a non-degenerate when $n = 2$. Note that degenerate critical points may be maxima, minima, or saddle points as in the case of $f(x) = x^3$ [4, p.92], [3, p.80, 88, 310].

Theorem 1.1. *[P.Fatou] Every attracting cycle for a polynomial or a rational function attracts at least one critical point.*

Proof. See [2, p.79]. □

By the Riemann Hurtwitz relation:

Theorem 1.2. *A non-constant rational map with degree d has exactly $2d-2$ critical points in $\mathbb{C}_\infty$, counted with multiplicity [1, p.43].*

We are interested in the dynamics of Newton's method on Riemann sphere. We can always conjugate $N_f(z)$ by an invertible linear(Möbius) transformation T, so the orbits of $N_f(z)$ will be essentially the same as the orbits of $T \circ N_f \circ T^{-1}$. On the Riemann sphere, the point at ∞ is like any other point. We can conjugate N_f by the transformation $z \to 1/z$ that interchanges 0 and ∞. Therefore the behavior of $N_f(z)$ at ∞ is the same as the behavior of $1/N_f(\frac{1}{z})$ at 0. The basin of attraction of a fixed point υ of the map N_f is the set $\{z | \lim_{n\to\infty} N_f^n(z) = \upsilon\}$, i.e., the set of all points whose orbits converge to υ under the iteration of N_f. This basin may have infinitely many components, and the immediate basin of attraction is the connected component containing the fixed point υ. The rational map N_f divides the Riemann sphere into two invariant sets, the Julia set, $\mathcal{J}(N_f)$, and Julia set's complement. As mentioned earlier, the Julia set consists of points for which the dynamical behavior under iteration of N_f is complicated. Points in the complement of the Julia set will normally converge to a fixed point or an attracting cycle. This complement could also contain a Siegel disk or Herman ring in which the iterations are locally like an irrational rotation of a disk or an annulus.

2. The dynamics of the perturbed map

In this section we consider the dynamics of a special class of rational functions, namely those rational functions that are obtained from Newton's method as applied to a polynomials of the form $F_c(z) = (z^2+c)(z-1)$. There are two reasons to be interested in the collection of Newton iteration maps given by N_{F_c}:

1. These form a natural family of non-polynomial examples, and
2. Their dynamical properties are related to the non-degenerate free critical point.

Proposition 2.1. *Infinity is a repelling fixed point for the Newton's method applied to* $F_c(z) = (z^2+c)p(z)$, *where* $p(z) = z-1$ *and* c *is any constant.*

Proof. The Newton's method function is the rational map:

$$N_{F_c}(z) = z - \frac{P_c(z)p(z)}{P_c'(z)p(z) + P_c(z)p'(z)} = \frac{(z^2-c)p(z) + z(z^2+c)p'(z)}{2zp(z) + (z^2+c)p'(z)} = \frac{2z^3 - z^2 + c}{3z^2 - 2z + c}$$

∞ is a fixed point, since $\lim_{z\to\infty} N_{F_c}(z) = \infty$.

$$N_{F_c}'(z) = \frac{F_c(z)F_c''(z)}{[F_c'(z)]^2} = \frac{[(z^2+c)p(z)][2p(z) + 4zp'(z) + (z^2+c)p''(z)]}{[2zp(z) + (z^2+c)p'(z)]^2}$$

To determine its nature, we map ∞ to 0 via $g(z) = \frac{1}{z}(= v)$ and get the conjugate function, $G(v) = g(N_{F_c}(\frac{1}{v})) = \frac{1}{N_{F_c}(\frac{1}{v})} = \frac{3v-2v^2+cv^3}{2-v+cv^2}$. ∞ is a repelling fixed point, since $G(0) = 0$ and $G'(0) = \frac{3}{2}$. □

We will first consider the dynamics of $F_0(z) = z^2(z-1)$, before the examining the dynamics of F_c when c is small.

2.1. The dynamics of $F_0(z) = z^2(z-1)$ for the case $c = 0$

Newton's method applied to the polynomial function $F_0(z) = z^2(z-1)$ yields the rational map

$$N_{F_0}(z) = \frac{2z^3 - z^2}{3z^2 - 2z}\ .$$

The finite fixed points for $N_{F_0}(z)$ are 0 and 1 which are an attracting fixed point and a super-attracting fixed point, respectively. In addition, ∞ is a repelling fixed point. In Figures 3 and 4, the computer graphics pictures illustrate of $N_{F_0}(z)$ on the dynamical plane. Each color in the picture belongs to a finite root of $N_{F_0}(z)$. In Figure 3, the red area is the basin of attraction for the attracting fixed point 0 and the blue area is the attracting basin for the super-attracting fixed point 1 of $N_{F_0}(z)$. In Figure 4, the same basins are shown when viewed from infinity. It is the simple case $c = 0$ for Newton iteration that has no decorations on the Julia set on the boundary of basin; rather this boundary is a simple closed curve passing through ∞.

The points $0, 1$ and $1/3$ are the critical points for N_{F_0}. The orbits of these points are called the critical orbits and they play a dominant role in determining the structure of the Julia set of N_{F_0}. The goal in this paper is to consider the case where the value of the parameter c becomes non-zero. When this occurs, the dynamical behavior changes dramatically. We will next describe those changes.

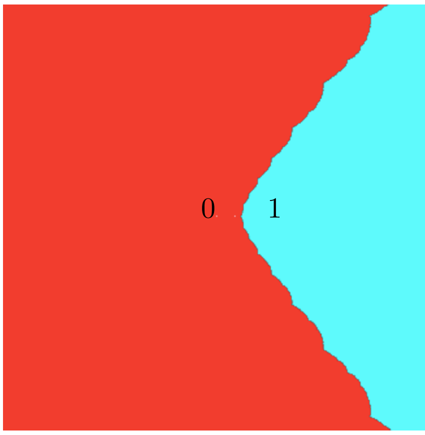

Figure 3

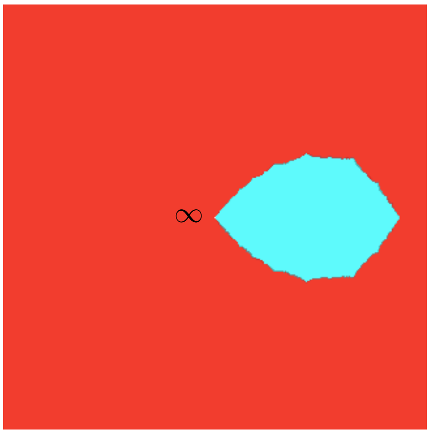

Figure 4

Dynamical plane pictures when $c = 0$.

2.2. The dynamics of $F_c(z) = (z^2 + c)(z - 1)$ for the case $c \neq 0$

We will now consider the case where c is different from 0 but quite small. Newton's method applied to this polynomial $F_c(z) = (z^2 + c)(z - 1)$ where the parameter $c = 0.01$ yields the rational map,

$$N_F(z) = N_{F_{0.01}}(z) = \frac{2z^3 - z^2 + 0.01}{3z^2 - 2z + 0.01}\ .$$

The finite roots of $F(z)$ are $1, \pm 0.1i$ and the real root 1 is a super-attracting fixed point of N_F. ∞ is a repelling fixed point of N_F. The points $1, 1/3$, and $\pm 0.1i$ are critical points for N_F. The critical points $1, 1/3$ are the common critical points for the functions N_{F_0} and N_F with different critical values and also they are non-degenerate critical points. In addition, the common critical point 1 is a super-attracting fixed point for the Newton's maps N_{F_0} and N_F.

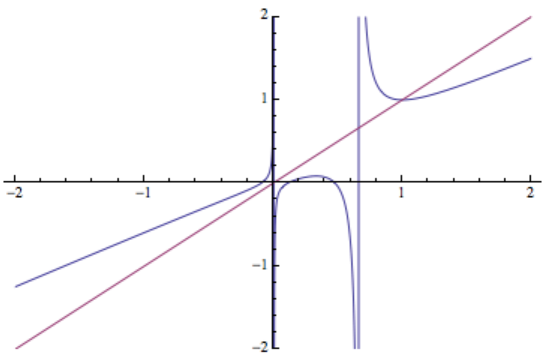

Figure 5

The Newton map for the polynomial $F : z \to z^3 - z^2 + (0.01)z - 0.01$ has only one real root. Left: the graph of N_F on the interval $[-2, 2]$ with the super-attracting fixed point of the Newton map indicated. In Figure 6 the behavior of this Newton map in the complex plane is displayed.

In Figure 6, the computer graphics picture illustrates how points behave under iteration of $N_F(z)$ in dynamical plane. First, we will make clear the fact that we are dealing with the complex plane, the x-axis is the real direction and y-axis is the imaginary direction. The Newton map, N_F, for the polynomial $F : z \to z^3 - z^2 + (0.01)z - 0.01$ has degree 3. Since the function has three roots,

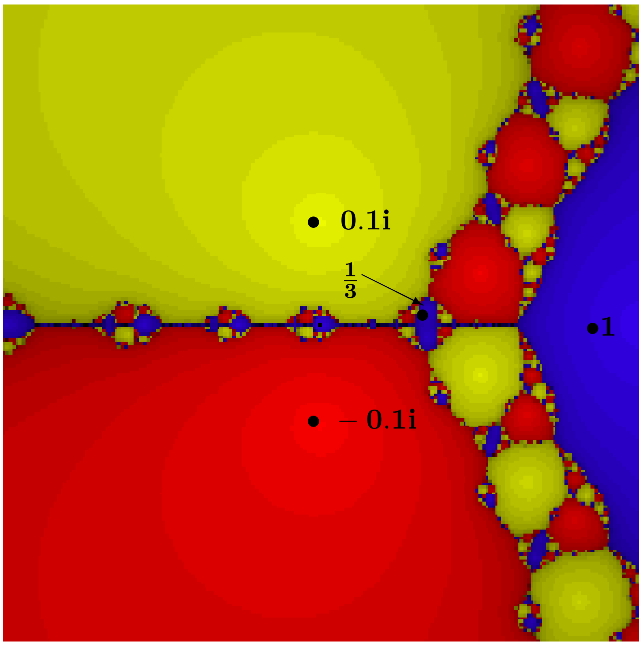

Figure 6

the graph of the complex plane is divided into three parts, each of which is a basin of attraction for a root. Colors indicate to which of the three roots a given starting point converges. These are finite roots of Newton's iteration which are contained in the Fatou set. The blue area is the basin of super-attracting fixed point for the map N_F, $\mathcal{A}_{N_F}(1) = \{z \in \mathbf{C} : N_F^n(z) \to 1, n \to \infty\}$. The shading of the colors indicates the speed of convergence to the roots. The boundary of Newton basin is the Julia set on which N_F is chaotic. $\mathcal{A}_{N_F}(1) = \mathbf{C} \setminus \mathcal{K}_{N_F}$, $\mathcal{K} = \{z \in \mathbf{C} : N_F^n(z) \nrightarrow 1, n \to \infty\}$, $\partial\mathcal{K} = \partial\mathcal{A}_{N_F}(1) = \mathcal{J}(N_F)$. In addition, the free critical point 1/3 is lies on the real axis and in a pre-image of the immediate basin of 1. Every root can be connected to ∞ within its basin of attraction. Note that there are no black regions in the basins, so Newton's map does not fail anywhere on that basin. Boundaries of basins will usually be complicated fractals - the decorations on the boundary of the three immediate basins correspond to their pre-images. Notice that the immediate basin of attraction is a connected component containing the fixed points of N_F. It is no longer just a simple closed curve as in the case $c = 0$.

One of the most important goals of Newton's method is to approximate the roots of a function for which initial values will this method converge? Will it converge to a root, and if so, to which root? In Figure 6, the speed of convergence for Newton's map of the function $(z^2 + 0.01)(z - 1)$ is clearly observed.

Theorem 2.2. *The immediate basin of an attracting fixed point or cycle of N_F contains at least one critical point of N_F [7, p.66], [5, p.296].*

Remember that the point 1/3 is the free critical point for the Newton's map. That is the point whose fate essentially determines everything in complex dynamical behavior of N_F. How is this? The key to the answer is the parameter c after changing the parameter from 0 to any constant on a circle in complex plane we see the periodic channels leading to ∞. In order to explain this we change the parameter c from real to complex. For example, in Figures $7-8$, the value of parameter

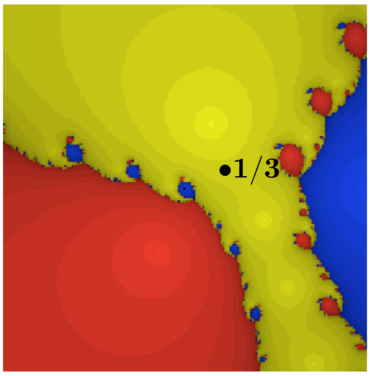

Figure 7

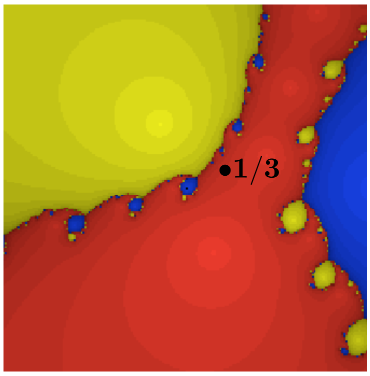

Figure 8

is $c_1 = 0.1 - 0.1i$, $c_2 = 0.1 + 0.1i$, respectively. In Figure 7, the three roots of the function $F_{c_1}(z) = z^3 - z^2 + (0.1 - 0.1i)z - (0.1 - 0.1i)$ are $\pm 0.143912 \pm 0.347434i, 1$, and in Figure 8, the three roots of the function $F_{c_2}(z) = z^3 - z^2 + (0.1+0.1i)z - (0.1+0.1i)$ are $\pm 0.143912 \pm 0.347434i, 1$. These are finite fixed points of Newton's iteration which are contained in the Fatou set. Since the function has three roots, the graph of the complex plane is divided three parts, each of which is a basin for a root. The boundary of the basin is the fractal which is the Julia set. By the definition of Julia set, Newton's method does not converge on the boundary points, but it is chaotic. The Newton iteration functions for both values c_1 andc_2 have critical points 1 and 1/3. In Figures 7, the yellow area and in Figures 8, the red area goes to infinity and contains the free critical point.

Corollary 2.3. *The non-degenerate free critical point plays vital role in determining the dynamics of the rational map which arising in complex Newton's method is applied to polynomial family* $F_c(z) = (z^2 + c)(z - 1)$, *where* c *is a complex (or non-complex) parameter.*

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References

[1] A. Beardon, *Iteration of Rational Functions*, (Springer-Verlag), 1991.

[2] B. Branner, *The Mandelbrot Set by Bodil Branner, Chaos and Fractals The Mathematics Behind the Computer Graphics*, Proceeding of symposia in applied mathematics **39**, 1989.

[3] R.L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, Redwood City, California, 1989.

[4] R.L. Devaney, *A First Course In Chaotic Dynamical Systems*, Second Edition, Theory and Experiment, CRC Press, Taylor and Francis Group, 2020.

[5] A. Douady and J.H. Hubbard, *On the Dynamics of Polynomial-like mappings*, Ann. Sci. École Norm. Sup., **18**, 287-343, 1985.

[6] J.H. Hubbard and B.B. Hubbard *Vector Calculus Linear Algebra, and Differential Forms*, Prentice Hall.Upper Saddle River, New Jersey 07458, 1990.

[7] L. Keen, *Julia sets Chaos and Fractals, the Mathematics behind the Computer Graphics*, ed. Devaney and Keen, Proc. Symp. Appl. Math., Amer. Math. Soc. **39**, 57-75, 1989.