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Mckean-Type Estimates for the First Eigenvalue of the p-Laplacian and (p, q)-Laplacian Operators on Finsler Manifolds

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

In this paper, we use Hessian comparison and volume comparison theorems to investigate the Mckean-type estimate theorem for the first eigenvalue of p-Laplacian and (p,q)-Laplacian operators on Finsler manifolds.

Keywords: *p-Laplacian operator; (p,q)-Laplacian operator; first eigenvalue.*
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1. Introduction

The study of the first eigenvalues of the Laplacian play an important role in global differential geometry since they reveal important relations between geometry of the manifold and analysis. The first result on this subject, due to Lichnerowicz [4], says that for an n-dimensional smooth compact manifold without boundary, the first eigenvalue λ_1 can be estimated below by $\frac{n}{n-1}K$, provided that its $\mathrm{Ric} \geq K > 0$. After a wile, it had been shown that the first eigenvalue is also related to the diameter of manifolds (see [15, 16]).
We could say that Lichnerowicz-Obata type estimate [15], Li-Yau-Zhong-Yang type estimate [16], and Mckean type estimate [9, 10, 11] for both positive and negative Ricci curvature are the most well known work in this subject.
In Riemannian geometry, Mckean proved that if (M, g) be a complete and simply connected Riemannian n-manifold with sectional curvature $K \leq -a^2$, then $\lambda_1(M) \geq \frac{(n-1)^2a^2}{4}$ (see [5]). Afterward, this result was extended by Ding in [3], stated that for a complete noncompact and simply connected Cartan-Hadamard manifold satisfying $\mathrm{Ric} \leq -a^2$, the first eigenvalue can be estimate below by $\frac{a^2}{4}$. These results were generalized to the Finsler manifolds by Wu-Xin [11]. Recently, the p-Laplacian on a general Finsler manifold $(M, F, d\mu)$, was discussed in [12] and [13]. It is defined as follows:

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u), \qquad 1 < p < \infty,$$

where the gradient ∇ is a nonlinear operator and equality holds in the weak $W^{1,p}(M)$ sence. Lately, Yin and He generalized Cheng type, Cheeger type, Faber-Krahn type and Mckean type inequalities for the Finsler p-Laplacian operator (see [14]). Actually in [14], authors obtained lower bound for the first eigenvalue of p-Laplacian operator considering nonpositive S-curvature and flag curvature $K \leq -a^2$.
In this paper, we want to extend the Mckean type estimate results to the p-Laplacian and the special class of

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the (p,q)-Laplacian operators on Finsler manifolds. Here we study the first eigenvalue under the line integrate curvature bounds.

2. Preliminaries

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x,y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinate on TM with $y = y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \longrightarrow [0,\infty)$ satisfying the following properties:
(i) Regularity: F is C^∞ in $TM \setminus \{0\}$,
(ii) Positive homogeneity: $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda > 0$,
(iii) Strong convexity: The fundamental quadratic form

$$(g_{ij}) := \left([\frac{1}{2}F^2]_{y^i y^j} \right),$$

is positively definite at every point of $TM \setminus \{0\}$.
Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field. Then the covariant derivative of X by $v \in T_xM$ with reference vector $w \in T_xM \setminus \{0\}$ is

$$D_v^w X(x) = \left\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w) v^j X^k(x) \right\} \frac{\partial}{\partial x^i},$$

where Γ^i_{jk} denote the coefficients of the Chern connection.
Given two linearly independent vectors $V, W \in T_xM \setminus \{0\}$, flag curvature $K(V,W)$ is defined as follows:

$$K(V,W) := \frac{g_V(R^V(V,W)W,V)}{g_V(V,V)g_V(W,W) - g_V(V,W)^2},$$

where R^V is the Chern curvature:

$$\mathbf{R}^V(X,Y)Z = \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z.$$

Then the Ricci curvature of V for (M,F) is:

$$Ric(V) = \sum_{i=1}^{n-1} K(V, e_i),$$

here $e_1, \cdots, e_{n-1}, \frac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V, namely, one has $\mathrm{Ric}(\lambda V) = \mathrm{Ric}(V)$ for any $\lambda > 0$.
For a given volume form $d\mu = \sigma(x)dx$ and a vector $y \in T_xM \setminus \{0\}$, the distortion of $(M, F, d\mu)$ is defined by

$$\tau(V) := \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma}.$$

Considering the rate of changes of the distortion along geodesics, leads to the so-called S-curvature as follows

$$S(V) := \frac{d}{dt}[\tau(\gamma(t), \dot\gamma(t)]_{t=0},$$

where $\gamma(t)$ is the geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = V$.
Now we can introduce the weighted Ricci curvature on the Finsler manifolds, which was defined by Ohta in [6].

Definition 2.1. ([6]) Let $(M, F, d\mu)$ be a Finsler n-manifold with volume form $d\mu$. Given a vector $V \in T_xM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = x$, $\dot\gamma(0) = V$. Define

$$\dot S(V) := F^{-2}(V) \frac{d}{dt}[S(\gamma(t), \dot\gamma(t)]_{t=0}.$$

Then the weighted Ricci curvatures of M defined as follows

$$\begin{aligned} Ric_n(V) &:= \begin{cases} Ric(V)+\dot{S}(V), & for\ S(V)=0,\\ -\infty, & otherwise, \end{cases}\\ Ric_N(V) &:= Ric(V)+\dot{S}(V)-\frac{S(V)^2}{(N-n)F(V)^2}, \quad \forall N\in(n,\infty),\\ Ric_\infty(V) &:= Ric(V)+\dot{S}(V). \end{aligned}$$

For a smooth function $u: M \longrightarrow \mathbb{R}$ and any point $x \in M$, the gradient vector of u at x is defined by

$$\nabla u(x)=\begin{cases} g^{ij}(x,\nabla u)\dfrac{\partial u}{\partial x_j}\dfrac{\partial}{\partial x_i}, & du(x)\neq 0,\\ 0, & du(x)=0. \end{cases}$$

So the gradient vector field of a differentiable function f on M by the Legendre transformation $\mathcal{L}: T_xM \to T_x^*M$ is defined as

$$\nabla u := \mathcal{L}^{-1}(du).$$

Let $\mathfrak{M}=\{x\in M : \nabla u|_x \neq 0\}$. We define the Hessian $H(u)$ of u on $\mathfrak{M}$ as follows:

$$H(u)(X,Y) := XY(u)-\nabla_X^{\nabla u}Y(u), \quad \forall X,Y\in\Gamma(TM|_{\mathfrak{M}}).$$

Fix a volume form $d\mu$, the divergence $\operatorname{div}(X)$ of X is defined as:

$$d(X\rfloor d\mu)=\operatorname{div}(X)d\mu.$$

For a given smooth function $u: M\longrightarrow\mathbb{R}$, the Laplacian Δu of u is defined by $\Delta u=\operatorname{div}(\nabla u)=\operatorname{div}(\mathcal{L}^{-1}(du))$. The Finsler p-Laplacian of a smooth function $u: M\to\mathbb{R}$ can be defined by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Since the gradient operator ∇ is not a linear operator in general, the Finsler p-Laplacian is greatly different from the Riemannian p-Laplacian.
Given a vector field V such that $V\neq 0$ on $M_u=\{x\in M; du(x)\neq 0\}$ the weighted gradient vector and the weighted p-Laplacian on the weighted Riemannian manifold (M,g_V) are defined by

$$\nabla^V u := \begin{cases} g^{ij}(V)\dfrac{\partial u}{\partial x^j}\dfrac{\partial}{\partial x^i}, & on\ M_u,\\ 0, & on\ M\setminus M_u, \end{cases} \qquad \Delta_p^V u := div(|\nabla^V u|^{p-2}\nabla^V u).$$

Here we note that $\nabla^V u=\nabla u$, $\Delta_p^V u=\Delta_p u$.

2.1. *Eigenvalues of (p,q)-Laplacian*

In this paper, we introduced a class of (p,q)-Laplacian on Finsler manifolds which had been defined in [10] for $\mathbb{R}^N$ and in [4] for the Riemannian case.

$$\Delta_p u+\Delta_q u=\operatorname{div}((|\nabla u|^{p-2}+|\nabla u|^{q-2})\nabla u), \tag{2.1}$$

where $u\in W=W_0^{1,p}(M)\cap W_0^{1,q}(M)$, $1<q<p<\infty$. We say that λ is an eigenvalue of (2.1) if there exists $u\in W$, $u\neq 0$ such that

$$-\Delta_p u-\Delta_q u=\lambda|u|^{p-2}u, \tag{2.2}$$

or

$$\int_M |\nabla u|^{p-2}|u.|vd\mu+\int_M |\nabla u|^{q-2}\nabla u.\nabla v d\mu=\lambda\int_M |u|^{p-2}uvd\mu, \tag{2.3}$$

for any $v\in W^{1,p}(M)\cap W^{1,q}(M)$. The first positive eigenvalue $\lambda_{1,p,q}(M)$ of (2.1) is obtained as follows:

$$\lambda_{1,p,q}(M)=\inf\left\{\int_M |\nabla u|^p d\mu+\int_M |\nabla u|^q d\mu : \int_M |u|^p d\mu=1\right\}. \tag{2.4}$$

A volume form $d\mu$ on Finsler manifold (M,F) in local coordinates $(U,x_1,...,x_n)$ can express as $d\mu=\sigma_B(x)dx^1\wedge ...\wedge dx^n$ and $d\mu=\sigma_H(x)dx^1\wedge ...\wedge dx^n$, where $\sigma_B(x)$ and σ_H are so-called Busemann-Hausdorff volume form and Holmes-Thompson volume form respectively. We recall maximal and minimal volume forms for Finsler manifolds from [7]. Let

$$dV_{max}=\sigma_{max}dx^1\wedge ...\wedge dx^n$$

and

$$dV_{min}=\sigma_{min}dx^1\wedge ...\wedge dx^n.$$

Here $\sigma_{max}(x):=\max_{y\in T_xM\setminus\{0\}}\sqrt{\det(g_{ij}(x,y)}$ and $\sigma_{min}(x):=\min_{y\in T_xM\setminus\{0\}}\sqrt{\det(g_{ij}(x,y)}$. We may use the notation dV_{ext} (means extreme volume form) for both maximal and minimal volume forms dV_{max} and dV_{min}. The uniformity function $\mu:M\to\mathbb{R}$ is

$$\mu(x):=\max_{y,z,u\in T_xM\setminus\{0\}}\frac{g_y(u,u)}{g_z(u,u)}.$$

The uniformity constant is $\mu_F=\max_{x\in M}\mu(x)$, so it is obvious that

$$\mu^{-1}F^2(u)\le g_y(u,u)\le \mu F^2(u).$$

Fix $x\in M$, the indicatrix at x is $I_x=\{v\in T_xM:F(v)=1\}$, then for $v\in I_x$, the cut-value $c(v)$ is defined by

$$c(v):=\sup\{t>0:d(x,exp_x(tv))=t\},$$

and the tangential cut locus $C(x)$ is defined by

$$\mathbf{C}(x):=\{c(v)v:c(v)<\infty,v\in I_x\}.$$

Let the cut locus and injectivity radius of x denote by $C(x)=exp_x\mathbf{C}(x)$ and $i_x=\{c(v),v\in I_x\}$, respectively. we know for sure that $C(x)$ has zero Hausorff measure in M. As well we set $\mathbf{D}_x=\{tv:0\le t<c(v),v\in I_x\}$ and $D_x=exp_x\mathbf{D}_x$. The largest star-shape domain with respect to the origin of T_xM is $\mathbf{D}(x)$ and $D_x=M\setminus C(x)$. Now considering polar coordinate on D_x for any q the polar coordinates are defined by $(r,\theta)=(r(q),\theta^1(q),...,\theta^{n-1}(q))$, where $r(q)=F(v)$ and is just the distance function with respect to x, $\theta^\alpha(q)=\theta^\alpha(u)$, here $v=exp_x^{-1}(q)$ and $u=\frac{v}{F(v)}$. We take $T=d(exp_x)(\frac{\partial}{\partial r})$ as the unit radial coordinate vector which is orthogonal to coordinate vectors ∂_α respect to g_T. These vectors defined as follows:

$$\begin{aligned}\partial_\alpha|_{exp_x(ru)} &= s(exp_x)\left(\frac{\partial}{\partial\theta^\alpha}\right)\Big|_{exp_x(ru)}\\ &= d(exp_x)_{ru}\left(r\frac{\partial}{\partial\theta^\alpha}\right)=rd(exp_x)_{ru}\left(\frac{\partial}{\partial\theta^\alpha}\right),\end{aligned}$$

for $\alpha=1,...,n-1$, so $T=\nabla r$. Taking $\tilde{g}=g_{\nabla r}$ as the singular Riemannian metric on $D(x)$, we get

$$\tilde{g}=dr^2+\tilde{g}_{\alpha\beta}d\theta^\alpha d\theta^\beta,\qquad \tilde{g}_{\alpha\beta}=g_{\nabla r}(\partial_\alpha,\partial_\beta).$$

Let $h(r)=trace_{g_{\nabla r}}H(r)$, from [8, 11], we have

$$\frac{\partial h}{\partial r}+\frac{h^2}{n-1}\le -\mathrm{Ric}(\nabla r),\qquad \frac{\partial}{\partial r}(\log\tilde{\sigma})=h.$$

where $\tilde{\sigma}(r,\theta)=\sqrt{\det(\tilde{g}_{\alpha\beta})}$. Assume that $\sigma_c(r)=\mathfrak{s}_c(r)^{n-1}$, and $h_c(r)=(n-1)\mathfrak{ct}_c(r)$, where

$$\mathfrak{s}_c(r)=\begin{cases}\frac{sin(\sqrt{c}r)}{\sqrt{c}}, & c>0,\\ r, & c=0,\\ \frac{sinh(\sqrt{-c}r)}{\sqrt{-c}}, & c<0,\end{cases}\qquad \mathfrak{ct}_c(r)=\begin{cases}\sqrt{c}cot(\sqrt{c}r), & c>0,\\ \frac{1}{r}, & c=0,\\ \sqrt{-c}coth(\sqrt{-c}r), & c<0.\end{cases}$$

Then

$$(\log\sigma_c)^{'}=h_c,\quad h_c^{'}+\frac{h_c^2}{n-1}=1(n-1)c.$$

We may need the following theorems for distance function $r(x,.)$ which was stated in [9].

Theorem 2.1. *Let (M, F) be a forward complete Finsler n-manifold admits non-positive flag curvature K. Suppose that $r = d(x, .)$ is smooth at $y \in M$ and γ as an unique minimal normal geodesic from x to y. Then for any $c < 0$, $l \geq 1$, we have*

$$\begin{aligned} trac_{g_{\nabla r}} H(r)(y) &\geq \sqrt{-c}coth(\sqrt{-c}r(y)) \\ &-\left[(2l-1)\int_\gamma (\max\{\mathrm{Ric}(\gamma^{'}(t)) - c, 0\})^l dt\right]^{\dfrac{1}{2l-1}}. \end{aligned}$$

Theorem 2.2. *Let (M, F) be a forward complete Finsler n-manifold admits non-positive flag curvature K. Suppose that $r = d(x, .)$ is smooth at $y \in M$, and γ be the unique minimal normal geodesic from x to y. Then for any $c < 0$, $l \geq 1$, and $X \in T_yM$ with $g_{\nabla r}(\nabla r, X) = 0$ and $g_{\nabla r}(X, X) = 1$, we have*

$$\begin{aligned} H(r)(X, X) &\geq \sqrt{-c}coth(\sqrt{-c}r(y)) \\ &-\left[(2l-1)\int_\gamma (\max\{\overline{K}(\nabla r) - c, 0\})^l dt\right]^{\dfrac{1}{2l-1}}. \end{aligned}$$

Here $H(r)(X, Y) = g_{\nabla r}(\nabla_X^{\nabla r}\nabla r, Y)$ and $\overline{K}(\nabla r) = \max_{g_{\nabla r}(\nabla r, E)=0} K(\nabla r; E)$, due to the local frame $E_1, ..., E_{n-1}, E_n = \nabla r$ such that E_i, $1 \leq i \leq n-1$ are eigenvectors of $H(r)$ with eigenvalues λ_i.

Theorem 2.3 (Volume comparison)**.** *Let (M, F) be a forward complete Finsler n-dimensional manifold with $K > 0$, and $c < 0$, $l \geq 1$. Then*
(i) Suppose that there is $C > 0$ so that the radial flag curvature at $x \in M$ satisfies

$$\int_\gamma (\max\{\overline{K}(\nabla r) - c, 0\})^l dt \leq C,$$

for any minimal normal geodesic γ issuing from x, then

$$\frac{vol_{ext}(B_x(r))}{V_{c,\Lambda,n}(r)} \leq \max_{x \in B_x(R)} \mu(x)^{\dfrac{n}{2}} . \frac{vol_{ext}(B_x(R))}{V_{c,\Lambda,n}(R)},$$

holds for any $r < R \leq i_x$, where $\Lambda = -(n-1)[(2l-1)C]^{\dfrac{1}{2l-1}}$ and

$$V_{c,\Lambda,n}(R) = vol(\mathbb{S}^{n-1}(1)) \int_0^R e^{\Lambda t} \mathfrak{s}_c(t)^{n-1} dt.$$

(ii) Suppose that there is $C > 0$ such that the radial Ricci curvature at $x \in M$ satisfies

$$\int_\gamma (\max\{\mathrm{Ric}(\lambda r) - c, 0\})^l dt \leq C,$$

for any minimal normal geodesic γ issuing from x, then

$$\frac{vol_{ext}(B_x(r))}{V_{c,\Lambda,2}(r)} \leq \max_{x \in B_x(R)} \mu(x)^{\dfrac{n}{2}} . \frac{vol_{ext}(B_x(R))}{V_{c,\Lambda,2}(R)},$$

holds for any $r < R \leq i_x$, where $\Lambda = -[(2l-1)C]^{\dfrac{1}{2l-1}}$.

3. Main results

In this section we shall prove some Mckean type theorems for the first eigenvalue of p-Laplacian operator under the line integrate curvature bounds and as a result with the same course we get Mckean type estimate for the first eigenvalue of the (p, q)-Laplacian operator.

Let $(M, F, d\mu)$ be a Finsler n-manifold with volume form $d\mu$, $\Omega \subset M$ a domain with compact closure and nonempty boundary $\partial\Omega$. The first Dirichlet eigenvalue $\lambda_{1,p}(\Omega)$ of Ω with respect to $d\mu$ is defined by:

$$\lambda_{1,p}(\Omega) = \inf\left\{\frac{\int_\Omega (F^*(df))^2 d\mu}{\int_\Omega f^p d\mu} : f \in W_0^{1,p}(\Omega) \setminus \{0\}\right\}, \tag{3.1}$$

here F^* is the dual Finsler metric on T^*M, and $W_0^{1,p}(\Omega) := \{f \in W^{1,p} | f|_{\partial\Omega} = 0\}$, where $W^{1,p}(\Omega)$ be the completion of $C^\infty(\Omega)$.
First we prepare the requirements for proving the main results. Set $B_p(R)$ as the forward geodesic ball with radius R centered at p and $R < i_p$, where i_p denotes the injectivity radius of p. With this notations we state the following lemma:

Lemma 3.1. *Let* $\Omega_\varepsilon(R) = \dfrac{B_p(R)}{B_p(\varepsilon)}$ *and* $r = d_F(p, .)$ *be smooth radial function on* $\Omega_\varepsilon(R)$*. In addition consider* $V = \nabla r$ *as a unit geodesic vector field on* $\Omega_\varepsilon(R)$*, and we can consider the Riemannian metric* $\tilde{g} = g_V$ *on* $\Omega_\varepsilon(R)$*. Then for* $d\mu = dV_{min}$*, we have*

$$\lambda_{1,p}(\Omega_\varepsilon(R)) \geq \frac{1}{\Theta^{(p+n)/2}} \tilde{\lambda}_{1,p}(\Omega_\varepsilon(R)). \tag{3.2}$$

Here $\tilde{\lambda}_{1,p}(\Omega_\varepsilon(R))$ *is the first eigenvalue of* $\Omega_\varepsilon(R)$ *with respect to the* $\tilde{g}$ *and* $\Theta = \max_{x \in B_p(R)} \mu(x)$*.*

Proof. The Legendre transformation $l : TM \longrightarrow T^*M$ is norm preserving and also it preserves the uniformity constant $\mu(x)$, so for any $f \in C_0^\infty(\Omega_\varepsilon(R))$, we obtain

$$\begin{aligned}
(F^*(df))^p(x) &= \big((F^*(df))^2(x)\big)^{\frac{p}{2}} \\
&= \big(g^{*ij}(x, df)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\big)^{\frac{p}{2}} \\
&\geq \big(\frac{1}{\mu^*(x)} g^{*ij}(x, l(V(x)))\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\big)^{\frac{p}{2}} \\
&= \big(\frac{1}{\mu(x)} g^{ij}(x, V(x))\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\big)^{\frac{p}{2}} \\
&= \frac{1}{(\mu(x))^{p/2}}(\|df\|_{\tilde{g}}^2(x))^{\frac{p}{2}} \\
&= \frac{1}{\mu^{p/2}}\|df\|_{\tilde{g}}^p(x).
\end{aligned} \tag{3.3}$$

Using volume comparison 2.3, for $d\mu = dV_{min}$, we get

$$\frac{\int_{\Omega_\varepsilon(R)} (F^*(df))^p dV_{min}}{\int_{\Omega_\varepsilon(R)} f^p dV_{min}} \geq \frac{\int_{\Omega_\varepsilon(R)} (F^*(df))^p dV_{\tilde{g}}}{\Theta^{n/2}\int_{\Omega_\varepsilon(R)} f^p dV_{\tilde{g}}} \geq \frac{1}{\Theta^{(p+n)/2}} \frac{\int_{\Omega_\varepsilon(R)} \|df\|_{\tilde{g}}^p dV_{\tilde{g}}}{\int_{\Omega_\varepsilon(R)} f^p dV_{\tilde{g}}}.$$

Due to the definition of the first eigenvalue of p-Laplacian operator, we get the result. □

Remark 3.1. Results also holds for $d\mu = dV_{max}$, so it holds for $d\mu = dV_{ext}$.

We also need following lemma proving our main results, which was stated in [14] as follows:

Lemma 3.2. *Let* $(M, F, d\mu)$ *be a Finsler manifold,* $\Omega \subset M$ *be a domain with compact closure and nonempty boundary* $\partial\Omega$*. Suppose that* f *is the first Dirichlet eigenfunction of* p*-Laplacian operator in* Ω*, and* X *is a vector field on* Ω *satisfying* $\inf_\Omega \operatorname{div}(X) > 0$*. Then, we have:*
(1) If there is a point $x_0 \in \Omega$ *where* $f(x_0) < 0$*, then*

$$\lambda_{1,p}(\Omega) \geq \left[\frac{\inf_\Omega \operatorname{div}(X)}{p \sup_\Omega F(X)}\right]^p; \tag{3.4}$$

(2) If there is a point $x_0 \in \Omega$ *such that* $f(x_0) > 0$*, then*

$$\lambda_{1,p}(\Omega) \geq \left[\frac{\inf_\Omega \operatorname{div}(X)}{p \sup_\Omega \overleftarrow{F}(X)}\right]^p. \tag{3.5}$$

Now we are able to prove:

Theorem 3.1. *Let (M,F) be a forward complete noncompact and simply connected Finsler n-manifold with nonpositive flag curvature and finite uniformity constant μ_F, $x \in M$, and $c < 0$, $l \geq 1$. Suppose that there exists $C > 0$ with*

$$[(2l-1)C]^{\frac{1}{2l-1}} < \sqrt{-c},$$

such that the radial flag curvature and radial Ricci curvature satisfy in the following respectively:

$$\int_\gamma (\max\{\overline{\mathbf{K}}(\nabla r) - c, 0\})^l dt \leq C, \tag{3.6}$$

and

$$\int_\gamma (\max\{\mathbf{Ric}(\nabla r) - c, 0\})^l dt \leq C, \tag{3.7}$$

for any minimal normal geodesic γ with beginning point x. Then we have

$$\lambda_{1,p}(M) \geq \frac{(p+n-3)^p[\sqrt{-c} - [(2l-1)C]^{1/(2l-1)}]^p}{p^p \mu_F^{(n+p)/2}}.$$

Proof. Since (M,F) is a forward complete noncompact and simply connected Finsler manifold with nonpositive flag curvature, by Cartan-Hadamard theorem $r = d_F(x,.)$ is smooth on $M \setminus \{x\}$. For a unit geodesic vector field $V = \nabla r$, we have

$$dr(X) = g_V(V,X) = \tilde{g}(V,X) = \tilde{g}(\tilde{\nabla} r, X),$$

so $\nabla r = \tilde{\nabla} r$, furthermore for the Chern connection ∇^V, we have

$$g_V(\nabla^V_X V, Z) = g_V(\tilde{\nabla}^V_X, Z),$$

that means $\nabla^V_X V = \tilde{\nabla}_X V$ and thus for any $Y \in TM$, we get

$$\tilde{H}(r)(X,Y) = g_V(\tilde{\nabla}_X V, Y) = g_V(\nabla^V_X V, Y) = H(r)(X,Y).$$

Namely

$$\tilde{\Delta}_p r = \tilde{\operatorname{div}}(|\tilde{\nabla} r|^{p-2}\tilde{\nabla} r) = \operatorname{div}(|\nabla r|^{p-2}\nabla r) = \Delta_p r,$$

here $\tilde{H}, \tilde{\Delta}, \tilde{\operatorname{div}}$ are Hessian, Laplacian and divergence with respect to $\tilde{g}$. Using Theorem 2.1 and Theorem 2.2, we obtain

$$\begin{aligned}
\Delta_p r &= \operatorname{div}(|\nabla r|^{p-2}\nabla r)\\
&= |\nabla r|^{p-2}\Delta r + (p-2)Hess(\nabla r, \nabla r)|\nabla r|^{p-4}\\
&\geq (n-1)[\sqrt{-c} - [(2l-1)C]^{1/(2l-1)}] + (p-2)[\sqrt{-c} - [(2l-1)C]^{1/(2l-1)}]\\
&= (p+n-3)[\sqrt{-c} - [(2l-1)C]^{1/(2l-1)}].
\end{aligned}$$

By applying Lemma 3.2 and equations (3.4) and (3.5) for $V = \nabla r$, we conclude

$$\lambda_{1,p}(\Omega_\varepsilon(R)) \geq \frac{1}{\mu_F^{(p+n)/2}}\tilde{\lambda}_{1,p}(\Omega_\varepsilon(R)) \geq \frac{(p+n-3)^p[\sqrt{-c} - [(2l-1)C]^{1/(2l-1)}]^p}{p^p \mu_F^{(n+p)/2}}.$$

This completes the proof. □

As an important result for the class of (p,q)-Laplacian operator (2.1), we prove:

Theorem 3.2. *Let (M,F) be a forward complete noncompact and simply-connected Finsler n-manifold with nonpositive flag curvature and finite uniformity constant μ_F. Suppose that there is $C > 0$ with $[(2l-1)C]^{1/(2l-1)} < \sqrt{-c}$, such that radial flag curvature and radial Ricci curvature at $x \in M$ satisfy in (3.6) and (3.7) respectively. Then for (2.4) with $p > q$, we have:*

$$\begin{aligned}
\lambda_{1,p,q}(M) \geq\ & \frac{(p+n-3)^p[\sqrt{-c} - [(2p-1)C]^{1/(2p-1)}]^p}{p^p \mu_F^{(n+p)/2}}\\
&+ \frac{(q+n-3)^q[\sqrt{-c} - [(2p-1)C]^{1/(2p-1)}]^q}{q^q \mu_F^{(n+p)/2}},
\end{aligned} \tag{3.8}$$

for $x \in M$, and $c < 0$, $l \geq 1$.

Proof. For any $f \in C_0^\infty(\Omega_\varepsilon(R))$, we conclude from (3.3) that:

$$\frac{\int_{\Omega_\varepsilon(R)}(F^*(df))^p dV_{min}}{\int_{\Omega_\varepsilon(R)} f^p dV_{min}} \geq \frac{1}{\Theta^{(p+n)/2}} \frac{\int_{\Omega_\varepsilon(R)} \|df\|_{\tilde{g}}^p dV_{\tilde{g}}}{\int_{\Omega_\varepsilon(R)} f^p dV_{\tilde{g}}},$$

and we have the same result for q as follows

$$\frac{\int_{\Omega_\varepsilon(R)}(F^*(df))^q dV_{min}}{\int_{\Omega_\varepsilon(R)} f^q dV_{min}} \geq \frac{1}{\Theta^{(q+n)/2}} \frac{\int_{\Omega_\varepsilon(R)} \|df\|_{\tilde{g}}^q dV_{\tilde{g}}}{\int_{\Omega_\varepsilon(R)} f^q dV_{\tilde{g}}},$$

Considering $p > q$ as stated in theorem, we obtain

$$\begin{aligned} & \frac{\int_{\Omega_\varepsilon(R)}(F^*(df))^p dV_{min}}{\int_{\Omega_\varepsilon(R)} f^p dV_{min}} + \frac{\int_{\Omega_\varepsilon(R)}(F^*(df))^q dV_{min}}{\int_{\Omega_\varepsilon(R)} f^p dV_{min}} \\ \geq \quad & \frac{1}{\Theta^{(n+p)/2}} \left(\frac{\int_{\Omega_\varepsilon(R)} \|df\|_{\tilde{g}}^p dV_{\tilde{g}}}{\int_{\Omega_\varepsilon(R)} f^p dV_{\tilde{g}}} + \frac{\int_{\Omega_\varepsilon(R)} \|df\|_{\tilde{g}}^q dV_{\tilde{g}}}{\int_{\Omega_\varepsilon(R)} f^p dV_{\tilde{g}}} \right). \end{aligned} \tag{3.9}$$

So, due to the definition of first eigenvalue of (p, q)-Laplacian (2.4), we gain

$$\lambda_{1,p,q}(M) \geq \frac{1}{\Theta^{(n+p)/2}} \tilde{\lambda}_{1,p,q}. \tag{3.10}$$

Using the same method as in Theorem 3.1, we have

$$\begin{aligned} \Delta_p r + \Delta_q r &= \tilde{\Delta}_p r + \tilde{\Delta}_q r \\ &= (p+n-3)[\sqrt{-c} - [(2l-1)C]^{1/(2l-1)}] \\ &\quad +(q+n-3)[\sqrt{-c} - [(2l-1)C]^{1/(2l-1)}], \end{aligned}$$

substituting this in (3.10) completes the proof. □

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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