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A parabolic-elliptic chemo-repulsion system in 2D domains with nonlinear production

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Abstract

In this paper we analyze a parabolic-elliptic chemo-repulsion system with superlinear production term in two-dimensional domains. Under the injection/extract chemical substance on a subdomain $\omega \subset \Omega \subset \mathbb{R}^2$, we prove the existence and uniqueness of global-in-time strong solutions at finite time.

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1. Introduction

One very interesting feature of living organisms is their interaction with the environment in which they reside. Frequently, the form of interaction involves the movement of living organisms generated by an external stimulus, the response to such stimulus is called *taxis*. The process which leads to taxis it is divided into three steps [18]: first, the cell detects the extracellular signal by specific receptors on its surface; then, the cell processes the signal and, finally, the alters its motile behavior. Depending on the nature of the stimulus or signal we have different kinds of taxis, namely: *aerotaxis*, *chemotaxis*, *haptotaxis*, *phototaxis*, among others (see [18, 21]). In particular, the chemotaxis phenomenon is understood as the movement of living organisms induced by the presence of certain chemical substances. In 1970, Keller and Segel [12] proposed a mathematical model that describes the chemotactic aggregation of cellular slime molds which preferentially towards relatively high concentrations of a chemical secreted by the amoebae themselves, such phenomenon is called *chemo-attraction*. In contrast, the phenomenon is called *chemo-repulsion*, if a region of high chemical concentration generate a repulsive effect on the organisms.

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We are interested in studying a chemo-repulsion model with nonlinear chemical signal production term given by the following system of partial differential equations

$$\begin{cases} \partial_t u - \Delta u = \nabla \cdot (u \nabla v) \text{ in } (0,T)\times\Omega =: Q, \\ \alpha_v \partial_t v - \Delta v + v = u^p \text{ in } (0,T)\times\Omega =: Q, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0,\ \dfrac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$, $(0,T)$ is a time interval, with $0 < T < \infty$, and the parameter α_v is a nonnegative real number. The unknowns are cell density $u := u(t,x) \geq 0$ and chemical concentration $v := v(t,x) \geq 0$. The term u^p, $p > 1$, represents the chemical signal production term.

System (1.1) with $\alpha_v = 1$ and linear production term (i.e. in $(1.1)_2$, $p = 1$) has been studied by Ciéslak et al. [3] and Tao [23]. In [3], the authors proved the existence and uniqueness of smooth classical solutions in 2D domains and the existence of weak solutions in spaces of dimension 3 and 4. In [23], the author delimits his analysis to a convex domain $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, and changes the chemotactic term $\nabla \cdot (u\nabla v)$ by $\nabla \cdot (\chi(u)\nabla v)$, where $\chi(u)$ is an adequate smooth function. With this modification, Tao proves the existence of a unique global-in-time classical solution of system (1.1), and that the corresponding solution (u,v) converges to $(\overline{u}_0, \overline{u}_0)$ as $t \to \infty$, where $\overline{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0\, dx$. As far as we known, in the case of chemo-repulsion model with superlinear signal production term (1.1), with $\alpha_v = 1$, the literature is scarce. Indeed, we known the studies developed by Guillén-González et al. [9–11]. In [9,10] the authors prove the existence of global-in-time weak solutions in 3D domains with quadratic production term ($p = 2$) and global-in-time strong solutions assuming a regularity criteria, which is satisfied in 1D and 2D domains. They also analyze some numerical schemes to approximate weak solutions. In [11], the authors prove the existence of weak solutions considering $p \in (1,2)$ and propose some fully discrete finite element approximations of system (1.1).

In this work we are interested in studying the parabolic-elliptic system related to problem (1.1) considering a proliferation/degradation coefficient of a chemical substance which acts on a subdomain $\omega \subset \Omega$. Specifically, we consider a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary of class $C^{2,1}$ and a time interval $(0,T)$, with $0 < T < \infty$. Then we will analyze the following system of partial differential equations in the time-space region $Q := (0,T)\times\Omega$:

$$\begin{cases} \partial_t u - \Delta u = \nabla \cdot (u\nabla v), \\ -\Delta v + v = u^p + f\, v\, 1_\omega, \end{cases} \tag{1.2}$$

where $\omega \subset \Omega$ is a subdomain, f denotes a proliferation/degradation coefficient which acts on the subdomain ω and 1_ω is the characteristic function of ω. The term u^p, $p > 1$, is the nonlinear chemical signal production term.

We complete system (1.2) with initial condition for the cell density

$$u(0,x) = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{1.3}$$

and non-flux boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \quad \frac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega. \tag{1.4}$$

For equation $(1.2)_2$, notice that, in the subdomain ω, where $f \geq 0$ then we inject chemical substance and, where $f \leq 0$, we extract chemical substance. Also, system (1.2)-(1.4) with $f \equiv 0$ and $p = 1$ has been studied by Mock in [16,17]. In these works, the author proved the existence and uniqueness of global-in-time classical solutions by using continuity arguments, under the assumptions that the initial data u_0 is strictly positive and twice continuously differentiable at $x \in \Omega$. Furthermore, the author proved that the solutions are uniformly bounded and converge with an exponential rate to the steady-state. However, the theory employed by Mock cannot be applied here, because the coefficient f is only a function belongs to the space $L^q(\omega)$, with $2 < q < \infty$ (see Theorem 2.7, below). In order to overcome this difficulty, we will apply the Leray-Schauder fixed-point arguments and parabolic and elliptic regularity theorems to obtain our results. The parabolic-parabolic system related to (1.2)-(1.4) has been studied by Guillén-González et al. [6–8]. In [6,7] the authors consider the linear case ($p = 1$) and, under minimal assumptions, they achieved their results by using energy estimates, fixed-point theorems and bootstrapping arguments via L^p-regularity of the parabolic heat-Neumann problem (see Theorem 2.4, below). In particular, in [6] the authors proved the existence and uniqueness of global-in-time strong solutions in two-dimensional domains. In [7] is proved the existence of weak solutions in 3D domains and establish a regularity criterion to get global-in-time strong solutions. In [8], the authors consider the superlinear case with $p \in (1, 2]$ and prove the existence and uniqueness of global-in-time strong solutions. The stationary case have been consider in [15]. In our case, due to the elliptic nature of equation $(1.2)_2$, the analysis is more complicated. Technically speaking, to obtain appropriate estimates for v, it requires us to improve a smallness condition for the function f (see estimate (2.10) below). Moreover, to improve the regularity of the solution (u, v) of system (1.2)-(1.4), we carefully combine elliptic and parabolic results for the heat-Neumann problem. The key point in our analysis is to control the integral $(\int_\Omega u^p\,dx)^2 = \|u\|^{2p}_{L^p}$. We have managed to control it in terms of $\|\nabla(u^{p/2})\|^2$ using the Gagliardo-Nirenberg (Lemma 2.3) and Young inequalities, which has only been possible in 2D domains and for $p \in (1, 2)$ (see estimate (3.16), below); so the case $p \geq 2$ remains open.

The paper is organized as follows: In Section 2 we fix the notation, introduce the functional spaces to be used, give the concept of strong solutions of problem (1.2)-(1.4) and we establish two regularity (parabolic and elliptic) results for Neumann problems that will be used throughout this work. In Section 3, we prove the existence and uniqueness of strong solutions of system (1.2)-(1.4), applying the Leray-Schauder fixed-point theorem, and establish that allows us deduce that the pair (u, v) does not blow-up at finite time.

2. Preliminaries

In this section we establish some notations. We will use the Lebesgue space $L^s(\Omega)$, $1 \leq s \leq \infty$, with norm denoted by $\|\cdot\|_{L^s}$. In particular, the L^2-norm and its inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Also, we consider the usual Sobolev spaces $W^{m,s}(\Omega) = \{u \in L^s(\Omega) : \|\partial^\alpha u\|_{L^s} < \infty\}$, with norm denoted by $\|\cdot\|_{W^{m,s}}$. When $s = 2$, we denote $H^m(\Omega) := W^{m,2}(\Omega)$ and the respective norm by $\|\cdot\|_{H^m}$. We will use the space $W^{m,s}_{\mathbf{n}}(\Omega) := \{u \in W^{m,s}(\Omega) : \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\}$, $m > 1 + \frac{1}{s}$, with norm $\|\cdot\|_{W^{m,s}_{\mathbf{n}}}$. Also, if X is a Banach space, we will denote by $L^p(X)$ the space of valued functions in X defined on the interval $[0, T]$ that are integrable in the Bochner sense, and its norm will be denoted by $\|\cdot\|_{L^s(X)}$. For simplicity, we will denote $L^s(Q) := L^p(L^s(\Omega))$ for $s \neq \infty$ and its norm by $\|\cdot\|_{L^s(Q)}$. In the case $s = \infty$, $L^\infty(Q) := L^\infty((0, T) \times \Omega)$ and its respective norm will be denoted by $\|\cdot\|_{L^\infty(Q)}$. Also, $C(X) := C([0, T]; X)$ denotes the space of continuous functions from $[0, T]$ into a Banach space X, and its respective norm by $\|\cdot\|_{C(X)}$. Moreover, as usual, the letters C, K, C_1, K_1,..., denote positive constants independent of (u, v), but its value may change from line to line.

We will study the existence of strong solutions of system (1.2)-(1.4). The following definition gives the concept of strong solutions of problem (1.2)-(1.4) in the case of nonlinear production u^p, for $p \in (1, 2)$.

Definition 2.1 (Strong Solutions)**.** Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ of class $C^{2,1}$. Let $f \in L^q(\omega)$, for $2 < q < \infty$, $u_0 \in H^1(\Omega)$ with

$u_0 \geq 0$ a.e. in Ω. We say that a pair (u, v) is a strong solution of problem (1.2)-(1.4) in the time interval $(0, T)$, if $u \geq 0$, $v \geq 0$ in Q

$$u \in S_u := \{u \in L^\infty(H^1(\Omega)) \cap L^2(H^2(\Omega)) \,:\, \partial_t u \in L^2(Q)\}, \tag{2.1}$$
$$v \in S_v := L^q(W^{2,q}(\Omega)), \tag{2.2}$$

(u, v) satisfies pointwisely a.e. $(t, x) \in Q$ the system

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta u &=& \nabla \cdot (u \nabla v), \\ -\Delta v + v &=& u^p + f v \, 1_\omega, \end{array} \right. \tag{2.3}$$

and the initial and boundary conditions (1.3) and (1.4) are satisfied, respectively.

Remark 2.2. System (1.2)-(1.4) is conservative in u. Indeed, integrating $(1.2)_1$ in Ω we have

$$\frac{d}{dt}\left(\int_\Omega u \, dx\right) = 0 \;\Rightarrow\; \int_\Omega u(t) \, dx = \int_\Omega u_0 \, dx := m_0, \;\; \forall t > 0. \tag{2.4}$$

Furthermore, integrating $(1.2)_2$ in Ω we obtain

$$\int_\Omega v \, dx = \int_\Omega u^p \, dx + \int_\omega f v \, dx. \tag{2.5}$$

A key point in our analysis is to estimate the integral $(\int_\Omega u^p \, dx)^2$, which is possible under the constraint $p \in (1, 2)$. The following Gagliardo-Nirenberg interpolation inequality allows us to achieve this purpose.

Lemma 2.3. [20, p.125] *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain. Let $u \in L^r(\Omega)$ and $1 \leq s_1, s_2 \leq \infty$ satisfying*

$$\frac{1}{r} = \theta\left(\frac{1}{s_1} - \frac{1}{2}\right) + \frac{1-\theta}{s_2}, \quad \theta \in [0, 1].$$

Then, the following estimate holds

$$\|u\|_{L^r} \leq C_1 \|\nabla u\|_{L^{s_1}}^\theta \|u\|_{L^{s_2}}^{1-\theta} + C_2 \|u\|_{L^{s^*}}, \tag{2.6}$$

where C_1 and C_2 are positive constants which depend on Ω, s_1 and s_2, and $s^* > 0$ is arbitrary.

In particular, for $r = 4$ and $s_1 = s_2 = 2$; from Lemma 2.3, we have that $\theta = \frac{1}{2}$ and

$$\|u\|_{L^4} \leq C(\|\nabla u\|^{1/2} \|u\|^{1/2} + \|u\|), \text{ with } C := \max\{C_1, C_2\};$$

which is a generalized version of the classical interpolation inequality in 2D domains (see, for instance, [2, p. 314])

$$\|u\|_{L^4} \leq C \|u\|^{1/2} \|u\|_{H^1}^{1/2} \qquad \forall u \in H^1(\Omega). \tag{2.7}$$

Also, frequently we will use the following equivalent norms in the Sobolev spaces $H^1(\Omega)$ and $H^2(\Omega)$ (see [19], for more details):

$$\|u\|_{H^1} \equiv \left(\|\nabla u\|^2 + \left(\int_\Omega u \, dx\right)^2\right)^{1/2} \qquad \forall u \in H^1(\Omega), \tag{2.8}$$

$$\|u\|_{H^2} \equiv \left(\|\Delta u\|^2 + \left(\int_\Omega u \, dx\right)^2\right)^{1/2} \qquad \forall u \in H^2_{\mathbf{n}}(\Omega). \tag{2.9}$$

We will apply the following result concerning to parabolic regularity.

Theorem 2.4. (Parabolic-Regularity) [4, Theorem 10.22, p. 344] *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$, $u_0 \in \widetilde{W}^{2-2/s,s}(\Omega)$ and $g \in L^s(Q)$, with $1 < s < \infty$ $(s \neq 3)$. Then, there exists a unique solution u of problem*

$$\begin{cases} \partial_t u - \Delta u &= g \quad in \quad Q, \\ u(0,x) &= u_0(x) \quad in \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &= 0 \quad on \quad (0,T) \times \partial\Omega, \end{cases}$$

such that

$$u \in C(\widetilde{W}^{2-2/s}(\Omega)) \cap L^s(W^{2,s}(\Omega)), \ \partial_t u \in L^s(Q).$$

Moreover, there exists a positive constant $C := C(\Omega, T)$ which satisfies the following estimate

$$\|u\|_{C(\widetilde{W}^{2-2/s})} + \|u\|_{L^s(W^{2,s})} + \|\partial_t u\|_{L^s(Q)} \leq C(\|u_0\|_{\widetilde{W}^{2-2/s}} + \|g\|_{L^s(Q)}).$$

Here, the space $\widetilde{W}^{2-2/s,s}(\Omega) := W^{2-2/s,s}(\Omega)$ if $s < 3$ and $\widetilde{W}^{2-2/s,s}(\Omega) := W_{\mathbf{n}}^{2-2/s,s}(\Omega)$ if $s > 3$.

Remark 2.5. In Theorem 2.4, the case $s = 3$ implies that $u \in C(X_{3,3}) \cap L^3(W^{2,3}(\Omega))$ with $\partial_t u \in L^3(Q)$, for a certain interpolation space $X_{3,3}$ (see [4, Theorem 10.22], for details). The description of the space $X_{3,3}$ is not clear in terms of $\widetilde{W}^{2-2/s,s}(\Omega)$ or another Sobolev spaces.

Furthermore, by adapting [5, Theorem 2.4.2.7] we deduce the following result on elliptic regularity for a Neumann problem.

Theorem 2.6. (Elliptic-Regularity) *Let $\Omega \in C^{1,1}$ be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, and $h \in L^s(\Omega)$ with $1 < s < \infty$. Then the problem*

$$\begin{cases} -\Delta u + u &= h \quad in \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &= 0 \quad on \quad \partial\Omega, \end{cases}$$

admits a unique solution $u \in W^{2,s}(\Omega)$. Moreover, there exists a positive constant $C := C(\Omega)$ such that

$$\|u\|_{W^{2,s}} \leq C\|h\|_{L^s}.$$

Now we will enunciate the main result of this paper.

Theorem 2.7. (Strong Solutions) *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ of class $C^{2,1}$. Let $u_0 \in H^1(\Omega)$, with $u_0 \geq 0$ a.e. in Ω and $f \in L^q(\omega)$ for $2 < q < \infty$. If $\|f\|_{L^q(\omega)}$ is small enough such that*

$$\|f\|_{L^q(\omega)} < \beta := \min\left\{\frac{1}{2[p(K_1^2 + K_2^2)]^{1/2}}, \frac{1}{K_3}\right\}, \tag{2.10}$$

where $K_1 := K_1(\Omega)$, $K_2 := |\Omega|^{\frac{1}{2}-\frac{1}{q}}$ and $K_3 := K_3(\Omega)$ are positive constants which depend only on Ω and are given by the injections $H^1(\Omega) \hookrightarrow L^s(\Omega)$, with $1 \leq s < \infty$, $L^q(\Omega) \hookrightarrow L^2(\Omega)$ and $W^{2,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, respectively. Then there exists a unique strong solution (u, v) of system (1.2)-(1.4) in sense of Definition 2.1. Moreover, there exists a positive constant $K := K(m_0, T, \|f\|_{L^q}, \|u_0\|_{H^1}, K_1, K_2, K_3)$ such that

$$\|u\|_{S_u} + \|v\|_{S_v} \leq K. \tag{2.11}$$

3. Proof of Theorem 2.7

In this section we will prove Theorem 2.7. For such effects we will apply the Leray-Schauder fixed-point principle.

3.1. Existence

Throughout this subsection we assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ of class $C^{2,1}$. We consider the following *auxiliary* function spaces:

$$W_u := C(L^2(\Omega)) \cap L^{\frac{2q}{q-2}}(H^1(\Omega)), \qquad W_v := L^q(L^\infty(\Omega)), \tag{3.1}$$

with $2 < q < \infty$.

Lemma 3.1. *The product space $S_u \times S_v$ (defined in (2.1)-(2.2)) is compactly embedded in $W_u \times W_v$.*

Proof. The compact embedding of S_v in W_v is clear. Let $u \in S_u$, then $u \in L^\infty(H^1(\Omega))$ and $\partial_t u \in L^2(Q)$. Since the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, then from [22, Corollary 4] we deduce that S_u is compactly embedded in $C(L^2(\Omega))$. Now, from [14, Théorème 9.6] we can deduce the following injection

$$L^\infty(H^1(\Omega)) \cap L^2(H^2(\Omega)) \hookrightarrow L^{\frac{2q}{q-2}}(H^{\frac{2q-2}{q}}(\Omega)).$$

Also, from [1, p.144] we have the compact embedding $H^{\frac{2q-2}{q}}(\Omega) \hookrightarrow H^1(\Omega)$; thus, using this injection and that $\partial_t u \in L^2(Q)$, from [13, Théorème 5.1] we have that S_u is compactly embedded in $L^{\frac{2q}{q-2}}(H^1(\Omega))$. Consequently, the injection of S_u in W_u is compact. □

In order to prove Theorem 2.7 we will use the Leray-Schauder fixed-point principle, for which we consider the operator $\mathcal{L} : W_u \times W_v \to S_u \times S_v \hookrightarrow W_u \times W_v$ defined by $\mathcal{L}(\overline{u}, \overline{v}) = (u, v)$ with (u, v) satisfying the following linear problem

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta u &=& \nabla \cdot (\overline{u}_+ \nabla v) \quad \text{in} \quad Q, \\ -\Delta v + v &=& \overline{u}^p + f \overline{v}_+ 1_\omega \quad \text{in} \quad Q, \\ u(0, x) &=& u_0(x) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &=& 0, \ \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \end{array} \right. \tag{3.2}$$

where $\overline{u}_+ := \max\{\overline{u}, 0\}$ and $\overline{v}_+ := \max\{\overline{v}, 0\}$, denote the respective positive parts of $\overline{u}$ and $\overline{v}$.

In the following lemmas we will prove that operator $\mathcal{L}$ satisfy the conditions of the Leray-Schauder fixed-point theorem.

Lemma 3.2. *The operator $\mathcal{L} : W_u \times W_v \to W_u \times W_v$ is well-defined and completely continuous (compact and continuous).*

Proof. Let $(\overline{u}, \overline{v}) \in W_u \times W_v$. Then, in particular, from Sobolev embeddings we deduce that $\overline{u}(t, \cdot) \in L^{2q}(\Omega)$ for any $t \in (0, T)$; thus, taking into account that $p \in (1, 2)$ we have that $\overline{u}^p \in L^q(\Omega)$. Furthermore, using that $f \in L^q(\omega)$, $q > 2$, and $v \in L^q(L^\infty(\omega))$; then, in particular, we have that $\overline{u}^p(t, \cdot) + f\overline{v}(t, \cdot)1_\omega$ belongs to $L^q(\Omega)$ for any time $t \in (0, T)$. Thus applying Theorem 2.6 (for $s = q > 2$) we deduce that there exists a unique solution $v := v(t, \cdot) \in W^{2,q}(\Omega)$ of elliptic problem

$$\left\{ \begin{array}{rcl} -\Delta v + v &=& \overline{u}^p + f \overline{v}_+ 1_\omega \ \text{ in } \ \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} &=& 0 \ \text{ on } \ \partial\Omega, \end{array} \right.$$

such that $\|v\|_{W^{2,q}} \leq C \left(\|\overline{u}^p\|_{L^q} + \|f\|_{L^q(\omega)} \|\overline{v}\|_{L^\infty} \right)$; which implies

$$\|v\|_{W^{2,q}}^q \leq C \left(\|\overline{u}^p\|_{L^q}^q + \|f\|_{L^q(\omega)}^q \|\overline{v}\|_{L^\infty}^q \right).$$

Then, integrating this last inequality in $(0, T)$ we deduce

$$\|v\|_{L^q(W^{2,q})}^q \leq C \left(\|\overline{u}^p\|_{L^q(Q)}^q + \|f\|_{L^q(\omega)}^q \|\overline{v}\|_{L^q(L^\infty)}^q \right); \tag{3.3}$$

hence, $v \in S_v$.

Now, since $v \in S_v = L^q(W^{2,q}(\Omega))$ ($q > 2$), we have that $\Delta v \in L^q(Q)$ and $\nabla v \in L^q(W^{1,q}(\Omega)) \hookrightarrow L^q(L^\infty(\Omega))$. Also, using that $\overline{u} \in W_u$; then, in particular, $\overline{u}_+ \in L^{\frac{2q}{q-2}}(L^{\frac{2q}{q-2}}(\Omega))$ and $\nabla \overline{u}_+ \in L^{\frac{2q}{q-2}}(L^2(\Omega))$. Thus, we deduce that

$$\nabla \cdot (\overline{u}_+ \nabla v) = \overline{u}_+ \Delta v + \nabla \overline{u}_+ \cdot \nabla v \in L^2(Q).$$

Therefore, applying Theorem 2.4 (for $s = 2$) we conclude that there exists a unique solution to $(3.2)_1$ with $u(x, 0) = u_0(x)$ in Ω and $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $(0, T) \times \partial\Omega$ such that the following estimate holds

$$\begin{aligned} \|u\|_{S_u} &\leq C\left(\|\overline{u}\|_{L^{\frac{2q}{q-2}}(L^{\frac{2q}{q-2}})}\|\Delta v\|_{L^q(Q)} + \|\nabla \overline{u}\|_{L^{\frac{2q}{q-2}}(L^2)}\|\nabla v\|_{L^q(L^\infty)}\right) \\ &\leq C\left(\|u_0\|_{H^1}, \|f\|_{L^q(\omega)}\right). \end{aligned} \tag{3.4}$$

Therefore, operator $\mathcal{L}$ is well-defined from $W_u \times W_v$ to $S_u \times S_v$. The compactness of $\mathcal{L}$ follows from inequalities (3.3) and (3.4).

Finally, in order to prove the continuity of operator $\mathcal{L}$ we consider $\{(\overline{u}_m, \overline{v}_m)\}_{m\geq 1} \subset W_u \times W_v$ a sequence such that

$$(\overline{u}_m, \overline{v}_m) \to (\overline{u}, \overline{v}) \text{ in } W_u \times W_v, \text{ as } m \to \infty. \tag{3.5}$$

In particular, the sequence $\{(\overline{u}_m, \overline{v}_m)\}_{m\geq 1}$ is bounded in $W_u \times W_v$; thus, from (3.3) and (3.4) we deduce that the sequence $\{(u_m, v_m) =: \mathcal{L}(\overline{u}_m, \overline{v}_m)\}_{m\geq 1}$ is bounded in the product space $S_u \times S_v$. Then, from the compactness of $S_u \times S_v$ in $W_u \times W_v$, we deduce that there exists a subsequence of $\{\mathcal{L}(\overline{u}_m, \overline{v}_m)\}_{m\geq 1}$, still denoted by $\{\mathcal{L}(\overline{u}_m, \overline{v}_m)\}_{m\geq 1}$, and a limit element $(\widehat{u}, \widehat{v}) \in S_u \times S_v$ such that, when m goes to ∞, the following convergence holds

$$\mathcal{L}(\overline{u}_m, \overline{v}_m) \to (\widehat{u}, \widehat{v}) \text{ weak in } S_u \times S_v \text{ and strong in } W_u \times W_v. \tag{3.6}$$

Then, from (3.5) and (3.6) we can pass to the limit in (3.2) when m goes to ∞, with $(u, v) = \mathcal{L}(\overline{u}_m, \overline{v}_m)$ and $(\overline{u}, \overline{v}) = (\overline{u}_m, \overline{v}_m)$, which implies that $\mathcal{L}(\overline{u}, \overline{v}) = (\widehat{u}, \widehat{v})$. Therefore, by the uniqueness of the limit, the whole sequence $\{\mathcal{L}(\overline{u}_m, \overline{v}_m)\}_{m\geq 1}$ converges to $\mathcal{L}(\overline{u}, \overline{v})$ strongly in $W_u \times W_v$. Thus operator $\mathcal{L} : W_u \times W_v \to W_u \times W_v$ is continuous. □

Lemma 3.3. *Let $u_0 \in H^1(\Omega)$, with $u_0 \geq 0$ a.e. in Ω and $f \in L^q(\omega)$, with $2 < q < \infty$. If $\|f\|_{L^q(\omega)}$ is small enough such that*

$$\|f\|_{L^q(\omega)} < \beta, \tag{3.7}$$

where $\beta > 0$ is the constant given in Theorem 2.7. Then the set

$$\mathcal{L}_\alpha := \{(u, v) \in W_u \times W_v : (u, v) = \alpha\mathcal{L}(u, v) \text{ for some } \alpha \in [0, 1]\} \tag{3.8}$$

is bounded in $W_u \times W_v$, independently of the parameter $\alpha \in [0, 1]$. Indeed, the set $\mathcal{L}_\alpha$ is also bounded in $S_u \times S_v$; that is, there exists a positive constant $K := K(m_0, T, \|u_0\|_{H^1}, \|f\|_{L^q(\omega)}, \beta)$ such that all pairs of functions $(u, v) \in \mathcal{L}_\alpha$, for $\alpha \in [0, 1]$, satisfy the estimate

$$\|(u, v)\|_{S_u \times S_v} \leq K. \tag{3.9}$$

Proof. Let $(u, v) \in \mathcal{L}_\alpha$ for $\alpha \in (0, 1]$ (the case $\alpha = 0$ is trivial). Then, from Lemma 3.2 the pair (u, v) belongs to $S_u \times S_v$ and satisfy pointwisely a.e. in Q the following system

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& \nabla \cdot (u_+ \nabla v) & \text{in} \quad Q, \\ -\Delta v + v &=& \alpha u^p + \alpha f v_+ 1_\omega & \text{in} \quad Q, \end{array} \right. \tag{3.10}$$

endowed whit the corresponding initial and boundary conditions. Thus, it suffices to bound $(u, v) \in S_u \times S_v$, independently of $\alpha \in (0, 1]$. The proof is carry out in five steps.

Step 1: $u, v \geq 0$ and u satisfy the mass conservation property (2.4).

Testing $(3.10)_1$ by $u_- := \min\{u, 0\} \le 0$, and taking into account that $u_- = 0$ if $u \ge 0$; $\nabla u_- = \nabla u$ if $u \le 0$ and $\nabla u_- = 0$ if $u > 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|u_-\|^2 + \|\nabla u_-\|^2 = -\int_\Omega u_+ \nabla v \cdot \nabla u_- \, dx = 0.$$

Then, using that $u_0 \ge 0$ a.e. in Ω, we deduce that $u_- \equiv 0$. Hence, $u \ge 0$ a.e. in Q. Similarly, testing $(3.10)_2$ by $v_- \le 0$ we obtain

$$\|\nabla v_-\|^2 + \|v_-\|^2 = \alpha \int_\Omega u^p v_- \, dx + \alpha \int_\omega f v_+ v_- \, dx \le 0;$$

which implies that $v_- \equiv 0$; thus $v \ge 0$ a.e. in Q. Finally, the mass conservation property (2.4) follows integrating $(3.10)_1$ in Ω.

<u>Step 2:</u> v is bounded in $L^2(H^2(\Omega))$.

Testing $(3.10)_1$ by $\frac{\alpha}{p-1}u^{p-1}$ and $(3.10)_2$ by $-\frac{1}{p}\Delta v$, for $p \in (1,2)$; then, integrating by parts in Ω, adding the respective equations and considering that chemotaxis and production terms cancel, we can obtain

$$\frac{\alpha}{p(p-1)}\frac{d}{dt}\|u^{p/2}\|^2 + \frac{4\alpha}{p^2}\|\nabla(u^{p/2})\|^2 + \frac{1}{p}\|\Delta v\|^2 + \frac{1}{p}\|\nabla v\|^2 = -\frac{\alpha}{p}\int_\omega f v \Delta v \, dx. \quad (3.11)$$

Now, taking into account that $H^1(\Omega) \hookrightarrow L^s(\Omega)$, we can fix a positive constant $K_1 := K_1(\Omega)$ such that $\|v\|_{L^s} \le K_1\|v\|_{H^1}$, for $s > 2$. Moreover, from the Hölder and Young inequalities and using that $p \in (1,2)$ and $\alpha \in (0,1]$ we have

$$\begin{aligned} -\frac{\alpha}{p}\int_\omega f v \Delta v \, dx \le \int_\omega |f v \Delta v| \, dx &\le \|f\|_{L^q(\omega)}\|v\|_{L^s}\|\Delta v\| \le K_1\|f\|_{L^q(\omega)}\|v\|_{H^1}\|\Delta v\| \\ &\le \frac{3}{4p}\|\Delta v\|^2 + \frac{p}{3}K_1^2\|f\|^2_{L^q(\omega)}\|v\|^2_{H^1} \\ &\le \frac{3}{4p}\|\Delta v\|^2 + K_1^2\|f\|^2_{L^q(\omega)}\|v\|^2_{H^1}, \end{aligned} \quad (3.12)$$

where $\frac{1}{p} + \frac{1}{s} = \frac{1}{2}$. Thus, from (3.11) and (3.12) we deduce that

$$\frac{\alpha}{p(p-1)}\frac{d}{dt}\|u^{p/2}\|^2 + \frac{4\alpha}{p^2}\|\nabla(u^{p/2})\|^2 + \frac{1}{4p}\|\Delta v\|^2 + \frac{1}{p}\|\nabla v\|^2 \le K_1^2\|f\|^2_{L^q(\omega)}\|v\|^2_{H^1}. \quad (3.13)$$

On the other hand, integrating in Ω the second equation in (3.10); then, multiplying the respective equality by $\int_\Omega v \, dx$ and applying the Young inequality, we have

$$\begin{aligned} \left(\int_\Omega v \, dx\right)^2 &= \alpha\left(\int_\Omega u^p \, dx\right)\left(\int_\Omega v \, dx\right) + \alpha\left(\int_\omega f v \, dx\right)\left(\int_\Omega v \, dx\right) \\ &\le \alpha^2\left(\int_\Omega u^p \, dx\right)^2 + \frac{1}{2}\left(\int_\Omega v \, dx\right)^2 + \alpha^2\left(\int_\omega f v \, dx\right)^2. \end{aligned} \quad (3.14)$$

Also, since $2 < q < \infty$ and $L^q(\omega) \hookrightarrow L^2(\omega)$, we have that $\|f\|_{L^2(\omega)} \le K_2\|f\|_{L^q(\omega)}$, with $K_2 := |\Omega|^{\frac{1}{2}-\frac{1}{q}} > 0$. Thus, from (3.14) we obtain

$$\begin{aligned} \frac{1}{4p}\left(\int_\Omega v \, dx\right)^2 < \frac{1}{2}\left(\int_\Omega v \, dx\right)^2 &\le \alpha^2\left(\int_\Omega u^p \, dx\right)^2 + \alpha^2\left(\int_\omega f v \, dx\right)^2 \\ &\le \alpha^2\left(\int_\Omega u^p \, dx\right)^2 + \alpha^2\|f\|^2_{L^2(\omega)}\|v\|^2 \\ &\le \alpha^2\|u\|^{2p}_{L^p} + \alpha^2 K_2^2\|f\|^2_{L^q(\omega)}\|v\|^2_{H^2}. \end{aligned} \quad (3.15)$$

In order to control the term $\|u\|_{L^p}^{2p}$ we will use the Gagliardo-Nirenberg interpolation inequality (2.6) (see Lemma 2.3 above). In fact, we observe that for $p \in (1,2)$, we have that $\frac{4(p-1)}{p} < 2$; thus, from inequality (2.6) we can obtain

$$\alpha^2\|u\|_{L^p}^{2p} = \alpha^2\|u^{p/2}\|^4 \le \alpha^2 C\|\nabla(u^{p/2})\|^{\frac{4(p-1)}{p}}\|u^{p/2}\|_{L^{2/p}}^{4/p} + \alpha^2 C\|u^{p/2}\|_{L^{2/p}}^4.$$

Moreover, applying the Young inequality and taking into account that $\|u^{p/2}\|_{L^{2/p}}^4 = \|u\|_{L^1}^{2p} = m_0^{2p}$ and that $\alpha \in (0,1]$; from the last inequality we deduce the following estimate

$$\begin{aligned}
\alpha^2\|u\|_{L^p}^{2p} &\le \frac{2\alpha}{p^2}\|\nabla(u^{p/2})\|^2 + C\|u^{p/2}\|_{L^{2/p}}^{\frac{4}{2-p}} + \alpha^2 C\|u\|_{L^1}^{2p} \\
&\le \frac{2\alpha}{p^2}\|\nabla(u^{p/2})\|^2 + C. \qquad (3.16)
\end{aligned}$$

Thus, from (3.15) and (3.16) we have

$$\frac{1}{4p}\left(\int_\Omega v\,dx\right)^2 \le \frac{2\alpha}{p^2}\|\nabla(u^{p/2})\|^2 + \alpha^2 K_2^2\|f\|_{L^q(\omega)}^2\|v\|_{H^2}^2 + C. \qquad (3.17)$$

Then, adding (3.13) and (3.17), and using the equivalent norm provided in (2.9) we obtain

$$\begin{aligned}
\frac{\alpha}{p(p-1)}\frac{d}{dt}\|u^{p/2}\|^2 + \frac{2\alpha}{p^2}\|\nabla(u^{p/2})\|^2 &+ \frac{1}{4p}\|v\|_{H^2}^2 + \frac{1}{p}\|\nabla v\|^2 \\
&\le K_1^2\|f\|_{L^q(\omega)}^2\|v\|_{H^1}^2 + \alpha^2 K_2^2\|f\|_{L^q(\omega)}^2\|v\|_{H^2}^2 + C \\
&\le (K_1^2 + K_2^2)\|f\|_{L^q(\omega)}^2\|v\|_{H^2}^2 + C,
\end{aligned}$$

which implies

$$\frac{\alpha}{p(p-1)}\frac{d}{dt}\|u^{p/2}\|^2 + \left(\frac{1 - 4p(K_1^2 + K_2^2)\|f\|_{L^q(\omega)}^2}{4p}\right)\|v\|_{H^2}^2 \le C. \qquad (3.18)$$

Now, from assumption (2.10) we deduce that $4p(K_1^2 + K_2^2)\|f\|_{L^q(\omega)}^2 < 1$; thus, integrating (3.18) in $(0,T)$ we conclude that there exists a positive constant

$$C := C(m_0, T, \|u_0\|_{L^p}, K_1, K_2, \|f\|_{L^q(\omega)})$$

such that

$$\|v\|_{L^2(H^2)} \le C.$$

Therefore, v is bounded in $L^2(H^2(\Omega))$.

Step 3: $u^{\frac{q+1}{2}}$, is bounded in $L^\infty(L^2(\Omega)) \cap L^2(H^1(\Omega))$, for $2 < q < \infty$.

Testing $(3.10)_1$ by $(q+1)u^q$, integrating with respect to spatial variable, using the Young inequality, and considering the 2D interpolation inequality (2.7), we have

$$\begin{aligned}
\frac{d}{dt}\|u^{\frac{q+1}{2}}\|^2 + \frac{4q}{q+1}\|\nabla(u^{\frac{q+1}{2}})\|^2 &= q\|u^{q+1}\Delta v\|_{L^1} \le C\|u^{\frac{q+1}{2}}\|_{L^4}^2\|\Delta v\| \\
&\le C\|u^{\frac{q+1}{2}}\|\|u^{\frac{q+1}{2}}\|_{H^1}\|\Delta v\| \\
&\le \|u^{\frac{q+1}{2}}\|_{H^1}^2 + C\|u^{\frac{q+1}{2}}\|^2\|v\|_{H^2}^2.
\end{aligned}$$

Then, adding $\frac{4q}{q+1}\|u^{\frac{q+1}{2}}\|^2$ to both sides of the previous inequality and using the equivalent norm (2.8), we obtain

$$\frac{d}{dt}\|u^{\frac{q+1}{2}}\|^2 + \frac{4q}{q+1}\|u^{\frac{q+1}{2}}\|_{H^1}^2 \le C\left(1 + \|v\|_{H^2}^2\right)\|u^{\frac{q+1}{2}}\|^2. \qquad (3.19)$$

Therefore, applying the Gronwall lemma in (3.19) and using that $v \in L^2(W^{2,q}(\Omega))$, we deduce that

$$\|u^{\frac{q+1}{2}}\|_{L^\infty(L^2)\cap L^2(H^1)} \leq C(q, \|u_0\|_{L^q}, \|f\|_{L^q(\omega)}).$$

Thus, $u^{\frac{q+1}{2}}$ is bounded in $L^\infty(L^2(\Omega)) \cap L^2(H^1(\Omega))$.

Step 4: v is bounded in S_v.

From previous step and interpolating, we deduce that $u^{\frac{q+1}{2}} \in L^\infty(L^2(\Omega))\cap L^2(H^1(\Omega)) \hookrightarrow L^4(Q)$; hence $u^2 \in L^{q+1}(Q)$. Then, using that $p \in (1,2)$, we deduce that $u^p \in L^{q+1}(Q) \hookrightarrow L^q(Q)$. Now, taking into account that $f \in L^q(\omega)$, $u^p \in L^q(\Omega)$ and $v \in L^2(H^2(\Omega))$, we obtain that $u(t,\cdot) + fv(t,\cdot)\,1_\omega \in L^q(\Omega)$ for any $t \in (0,T)$. Then, applying Theorem 2.6 (for $s = q > 2$) to problem

$$\left\{\begin{array}{rcl} -\Delta v + v &=& \alpha u^p + \alpha f v\, 1_\omega \;\text{ in }\; \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} &=& 0 \;\text{ on }\; \partial\Omega, \end{array}\right.$$

we conclude that $v(t,\cdot) \in W^{2,q}(\Omega)$ for each $t \in (0,T)$ and

$$\begin{aligned} \|v\|_{W^{2,q}} &\leq \alpha C\left(\|u^p\|_{L^q} + \|f\|_{L^q(\omega)}\|v\|_{L^\infty}\right) \\ &\leq C\left(\|u^p\|_{L^q} + K_3\|f\|_{L^q(\omega)}\|v\|_{W^{2,q}}\right); \end{aligned}$$

thus,

$$\left(1 - K_3\|f\|_{L^q(\omega)}\right)\|v\|_{W^{2,q}} \leq C\|u^p\|_{L^q}, \tag{3.20}$$

where $K_3 := K_3(\Omega) > 0$ is a fixed constant given by the embedding $W^{2,q}(\Omega) \hookrightarrow L^\infty(\Omega)$. Notice that from (2.10) we deduce that $K_3\|f\|_{L^q(\omega)} < 1$; hence, using that $u^p \in L^q(\Omega)$, from (3.20) we conclude that

$$\begin{aligned} \|v\|^q_{L^q(W^{2,q})} &\leq \frac{C}{\left(1 - K_3\|f\|_{L^q(\omega)}\right)^q}\|u^p\|^q_{L^q} \\ &\leq C\left(m_0, T, \|f\|_{L^q(\omega)}, \|u_0\|_{H^1}, \beta\right). \end{aligned} \tag{3.21}$$

Consequently, v is bounded in the space S_v.

Step 5: u is bounded in S_u.

Testing $(3.10)_1$ by $-\Delta u$ we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\Delta u\|^2 = -(u\Delta v + \nabla u \cdot \nabla v, \Delta u). \tag{3.22}$$

Then, applying the Hölder and Young inequalities we obtain

$$\begin{aligned} -(u\Delta v, \Delta u) &\leq \|u\|_{L^s}\|\Delta v\|_{L^q}\|\Delta u\| \leq \widehat{K}\|u\|_{H^1}\|\Delta v\|_{L^q}\|\Delta u\| \\ &\leq \widehat{K}^2\|u\|^2_{H^1}\|\Delta v\|^2_{L^q} + \frac{1}{4}\|\Delta u\|^2, \end{aligned} \tag{3.23}$$

where $\frac{1}{s} + \frac{1}{q} = \frac{1}{2}$ and $\widehat{K} := \widehat{K}(\Omega) > 0$ is a constant given by the embedding $L^s(\Omega) \hookrightarrow H^1(\Omega)$.

Now, using again the Hölder and Young inequalities we deduce

$$-(\nabla u \cdot \nabla v, \Delta v) \leq \|\nabla u\|\|\nabla v\|_{L^\infty}\|\Delta u\| \leq \frac{1}{4}\|\Delta u\|^2 + \|\nabla u\|^2\|\nabla v\|^2_{L^\infty}. \tag{3.24}$$

Then, from (3.22)-(3.24) we have

$$\frac{d}{dt}\|\nabla u\|^2 + \|\Delta u\|^2 \leq \widehat{K}^2\|u\|^2_{H^1}\|\Delta v\|^2_{L^q} + \|\nabla u\|^2\|\nabla v\|^2_{L^\infty};$$

which together with the fact that $(\int_\Omega u(t)\,dx)^2 = m_0^2$ and the equivalent norms (2.8) and (2.9) imply that

$$\frac{d}{dt}\|u\|_{H^1}^2 + \|u\|_{H^2}^2 \le 2\widehat{K}^2\|u\|_{H^1}^2\|\Delta v\|_{L^q}^2 + 2\|\nabla u\|^2\|\nabla v\|_{L^\infty}^2 + 2m_0^2. \qquad (3.25)$$

Thus, considering that $\Delta v \in L^q(\Omega)$, for $2 < q < \infty$, and $(\nabla u, \nabla v) \in L^2(Q) \times L^q(L^\infty(\Omega))$, from (3.25) and the Gronwall lemma we deduce that

$$\begin{aligned} \|u\|_{L^\infty(H^1)\cap L^2(H^2)} &\le C(m_0, T, \|f\|_{L^q(\omega)}, \|u_0\|_{H^1}, \beta) \\ &\le C. \end{aligned} \qquad (3.26)$$

Then, u is bounded in $L^\infty(H^1(\Omega)) \cap L^2(H^2(\Omega))$.

Now, from $(3.10)_1$, (3.21) and (3.26) we obtain the following estimate

$$\begin{aligned} \|\partial_t u\|_{L^2(Q)} &\le \|u\|_{L^2(Q)} + \|u\|_{L^2(L^{\bar{s}})}\|\Delta v\|_{L^2(L^q)} + \|\nabla u\|_{L^{\bar{s}}(L^2)}\|\nabla v\|_{L^q(L^2)} \\ &\le C(m_0, T, \|f\|_{L^q(\omega)}, \|u_0\|_{H^1}, \beta), \end{aligned} \qquad (3.27)$$

where $\frac{1}{\bar{s}} + \frac{1}{q} = 1$. Therefore, from (3.26) and (3.27) we conclude that $u \in S_u$.

Finally, from (3.21), (3.26) and (3.27) we deduce that all elements of the set $\mathcal{L}_\alpha$ are bounded in $S_u \times S_v$. Moreover, the estimate (3.9) follows from (3.21) and (3.26)-(3.27). □

Consequently, in virtue of Lemmas 3.2 and 3.3 we deduce that the operator $\mathcal{L}$ and the set $\mathcal{L}_\alpha$ satisfy the conditions of the Leray-Schauder fixed-point theorem. Therefore, we deduce that $\mathcal{L}(\cdot,\cdot)$ has a fixed-point $(u, v) \in S_u \times S_v$; namely, $\mathcal{L}(u, v) = (u, v)$. This fixed-point is a strong solution of system (1.2)-(1.4). Furthermore, the estimate (2.11) follows from (3.21) and (3.26)-(3.27).

3.2. Uniqueness

Following a classical comparison argument, let $(u_1, v_1), (u_2, v_2) \in S_u \times S_v$ two possible solutions of system (1.2)-(1.4). Then, subtracting equations (1.2)-(1.4) for (u_1, v_1) and (u_2, v_2), and then denoting $u := u_1 - u_2$ and $v := v_1 - v_2$, we can obtain the following system:

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta u &=& \nabla\cdot(u_1\nabla v + u\nabla v_2) \quad \text{in} \quad Q, \\ -\Delta v + v &=& u_1^p - u_2^p + fv\,1_\omega \quad \text{in} \quad Q, \\ u(0,x) &=& 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &=& \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega. \end{array} \right. \qquad (3.28)$$

Testing $(3.28)_1$ by u and $(3.28)_2$ by $-\Delta v$, then integrating by parts in Ω we have

$$\begin{aligned} \frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + \|\Delta v\|^2 + \|\nabla v\|^2 &= (u_1\Delta v, u) + (\nabla u_1\cdot\nabla v, u) - (u\nabla v_2, \nabla u) \\ &\quad -(u_1^p, \Delta v) + (u_2^p, \Delta v) - (fv\,1_\omega, \Delta v). \end{aligned} \qquad (3.29)$$

Applying the Hölder and Young inequalities and taking into account the 2D interpolation inequality (2.7) we obtain

$$\begin{aligned} (u_1\Delta v, u) \le \|u_1\|_{L^4}\|\Delta v\|\|u\|_{L^4} &\le \frac{1}{12}\|\Delta v\|^2 + C\|u_1\|_{L^4}^2\|u\|\|u\|_{H^1} \\ &\le \frac{1}{12}\|\Delta v\|^2 + \frac{1}{10}\|u\|_{H^1}^2 + C\|u_1\|_{L^4}^4\|u\|^2, \end{aligned} \qquad (3.30)$$

$$\begin{aligned} (\nabla u_1\cdot\nabla v, u) \le \|\nabla u_1\|\|\nabla v\|_{L^4}\|u\|_{L^4} &\le C\|\nabla u_1\|\|v\|_{H^2}\|u\|^{1/2}\|u\|_{H^1}^{1/2} \\ &\le \frac{1}{12}\|v\|_{H^2}^2 + \frac{1}{10}\|u\|_{H^1}^2 + C\|\nabla u_1\|_{L^4}^4\|u\|^2, \end{aligned} \qquad (3.31)$$

$$-(u\nabla v_2, \nabla u) \le \|u\|\|\nabla v_2\|_{L^\infty}\|\nabla u\| \le \frac{1}{10}\|u\|_{H^1}^2 + C\|u\|^2\|\nabla v_2\|_{L^\infty}^2, \qquad (3.32)$$

$$
\begin{aligned}
-(fv\,1_\omega,\Delta v) \le \|f\|_{L^q(\omega)}\|v\|_{L^s}\|\Delta v\| &\le K_1\|f\|_{L^q(\omega)}\|v\|_{H^1}\|\Delta v\| \\
&\le \frac{1}{4}\|\Delta v\|^2 + K_1^2\|f\|^2_{L^q(\omega)}\|v\|^2_{H^2}, \qquad (3.33)
\end{aligned}
$$

where $\frac{1}{q}+\frac{1}{s}=\frac{1}{2}$ and $K_1 := K_1(\Omega)$ is the constant given by the embedding $H^1(\Omega)\hookrightarrow L^s(\Omega)$.

Moreover, using again the Hölder and Young inequalities, considering the 2D interpolation inequality (2.7) and taking into account the pointwise inequality $|a^p - b^p| \le C|a-b|(|a|^{p-1}+|b|^{p-1})$, we can obtain

$$
\begin{aligned}
-(u_1^p,\Delta v)+(u_2^p,\Delta v) &= -(u_1^p-u_2^p,\Delta v) \\
&\le C\|u\|_{L^4}\|u_1^{p-1}\|_{L^4}\|\Delta v\| + C\|u\|_{L^4}\|u_2^{p-1}\|_{L^4}\|\Delta v\| \\
&\le \frac{1}{12}\|\Delta v\|^2 + C\|u\|\|u\|_{H^1}\|u_1^{p-1}\|^2_{L^4} + C\|u\|\|u\|_{H^1}\|u_2^{p-1}\|^2_{L^4} \\
&\le \frac{1}{12}\|\Delta v\|^2 + \frac{1}{10}\|u\|^2_{H^1} \\
&\quad + C\left(\|u_1\|^{4(p-1)}_{L^{4(p-1)}} + \|u_2\|^{4(p-1)}_{L^{4(p-1)}}\right)\|u\|^2. \qquad (3.34)
\end{aligned}
$$

On the other hand, integrating $(3.28)_2$ in Ω we have

$$
\int_\Omega v\,dx = \int_\Omega (u_1^p - u_2^p)\,dx + \int_\omega fv\,dx;
$$

then,

$$
\left(\int_\Omega v\,dx\right)^2 = \left(\int_\Omega (u_1^p-u_2^p)\,dx\right)\left(\int_\Omega v\,dx\right) + \left(\int_\omega fv\,dx\right)\left(\int_\Omega v\,dx\right). \qquad (3.35)
$$

Thus, from (3.35) and arguing as in estimates (3.15) and (3.34) we can obtain

$$
\frac{7}{12}\left(\int_\Omega v\,dx\right)^2 \le \frac{1}{10}\|u\|^2_{H^1} + C\left(\|u_1\|^{4(p-1)}_{L^{4(p-1)}} + \|u_2\|^{4(p-1)}_{L^{4(p-1)}}\right)\|u\|^2 + K_2^2\|f\|^2_{L^q(\omega)}\|v\|^2_{H^2}, \qquad (3.36)
$$

where $K_2 := |\Omega|^{\frac{1}{2}-\frac{1}{q}} > 0$.

Therefore, from (3.29) and estimates (3.30)-(3.36) we arrive at

$$
\begin{aligned}
\frac{d}{dt}\|u\|^2 &+ \|\nabla u\|^2 + \frac{7}{12}\|\Delta v\|^2 \\
&\le \frac{2}{5}\|u\|^2_{H^1} + \frac{1}{12}\|v\|^2_{H^2} + C\left(\|u_1\|^{4(p-1)}_{L^{4(p-1)}} + \|u_2\|^{4(p-1)}_{L^{4(p-1)}}\right)\|u\|^2 \\
&\quad + C\left(\|u_1\|^4_{L^4} + \|\nabla u_1\|^4_{L^4} + \|\nabla v_2\|^2_{L^\infty}\right)\|u\|^2 + K_1^2\|f\|^2_{L^q(\omega)}\|v\|^2_{H^2}. \qquad (3.37)
\end{aligned}
$$

Now, using that $\int_\Omega u(t)\,dx = 0$ and then adding (3.36) and (3.37), and considering the equivalent norms (2.8) and (2.9), we deduce that

$$
\begin{aligned}
\frac{d}{dt}\|u\|^2 + \frac{1}{2}\|u\|^2_{H^1} + \frac{1}{2}\|v\|^2_{H^2} &\le C\left(\|u_1\|^{4(p-1)}_{L^{4(p-1)}} + \|u_2\|^{4(p-1)}_{L^{4(p-1)}}\right)\|u\|^2 \\
&\quad + (K_1^2+K_2^2)\|f\|^2_{L^q(\omega)}\|v\|^2_{H^2} \\
&\quad + C\left(\|u_1\|^4_{L^4} + \|\nabla u_1\|^4_{L^4} + \|\nabla v_2\|^2_{L^\infty}\right)\|u\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{d}{dt}\|u\|^2 + \frac{1}{2}\|u\|^2_{H^1} &+ \left(\frac{1-2(K_1^2+K_2^2)\|f\|^2_{L^q(\omega)}}{2}\right)\|v\|^2_{H^2} \\
&\le C\left(\|u_1\|^{4(p-1)}_{L^{4(p-1)}} + \|u_2\|^{4(p-1)}_{L^{4(p-1)}}\right)\|u\|^2 \\
&\quad + C\left(\|u_1\|^4_{L^4} + \|\nabla u_1\|^4_{L^4} + \|\nabla v_2\|^2_{L^\infty}\right)\|u\|^2. \qquad (3.38)
\end{aligned}
$$

Since $p \in (1,2)$, we deduce that $\frac{1}{2[p(K_1^2+K_2^2)]^{1/2}} < \frac{1}{[2(K_1^2+K_2^2)]^{1/2}}$; thus, if $\|f\|_{L^q(\omega)}$ is small enough such that $\|f\|_{L^q(\omega)} < \frac{1}{2[p(K_1^2+K_2^2)]^{1/2}}$; from (2.10) we deduce that $1 - 2(K_1^2 + K_2^2)\|f\|^2_{L^q(\omega)} > 0$. Therefore, considering that $(u_1, \nabla u_1) \in L^4(Q)\times L^4(Q)$, $\nabla v_2 \in L^2(L^\infty(\Omega))$, $(u_1, u_2) \in L^{4(p-1)}(Q) \times L^{4(p-1)}(Q)$ for $p \in (1,2)$ and $u_0 = 0$; then, from (3.38) and Gronwall lemma we deduce that $u = v = 0$. Consequently, $u_1 = u_2$ and $v_1 = v_2$, and the uniqueness follows.

Remark 3.4. If the initial data u_0 belongs to $W^{3/2,4}(\Omega)$ we can obtain more regularity for (u,v) and conclude that (u,v) does not blow-up at finite time. Indeed, arguing as in the proof of Theorem 2.7 we can obtain that $(u,v) \in S_u \times S_v$. Moreover, since $u \in L^\infty(H^1(\Omega)) \cap L^2(H^2(\Omega))$; hence, in particular, $u^p \in L^\infty(L^q(\Omega))$, for $2 < q < \infty$. Thus, using that $f \in L^q(\omega)$ from (3.20) we have

$$\|v\|_{W^{2,q}} \le \frac{C}{1 - K_3\|f\|_{L^q(\omega)}}\|u^p\|_{L^q}. \tag{3.39}$$

Then, considering (2.10), from (3.39) we deduce that

$$\begin{aligned}\|v\|_{L^\infty(W^{2,q})} &\le C(m_0, T, \|f\|_{L^q(\omega)}, \beta, \|u_0\|_{H^1})\|u\|_{L^\infty(L^q)} \\ &\le C.\end{aligned}$$

Consequently, $v \in L^\infty(W^{2,q}(\Omega)) \hookrightarrow L^\infty(Q)$.

Now, using again that $u \in L^\infty(H^1(\Omega)) \cap L^2(H^2(\Omega))$, in particular, $u \in L^{\frac{4q}{4+q}}(Q)$, for $2 < q < \infty$. Then, taking into account that $\nabla u \in L^\infty(L^2(\Omega)) \cap L^2(H^1(\Omega)) \hookrightarrow L^4(Q)$, $\nabla v \in L^\infty(W^{1,q}(\Omega)) \hookrightarrow L^\infty(Q)$ and $\Delta v \in L^\infty(L^q(\Omega))$, we have $\nabla\cdot(u\nabla v) = u\Delta v + \nabla u \cdot \nabla v \in L^4(Q)$. Therefore, applying Theorem 2.4 to $(1.2)_1$ (for $s = 4$) we deduce that $u \in L^\infty(W^{3/2,4}(\Omega)) \cap L^4(W^{2,4}(\Omega))$ and $\partial_t u \in L^4(Q)$. Then, from Sobolev embeddings we have $W^{3/2,4}(\Omega) \hookrightarrow L^\infty(\Omega)$. Consequently, $u \in L^\infty(Q)$ and we deduce that the pair (u,v) does not blow-up at finite time.

Remark 3.5. The case $p = 2$ is not clear. Indeed, integrating $(3.10)_2$ in space; then multiplying the respective equality by $\int_\Omega v\,dx$ and applying the Hölder and Young inequalities, we can obtain

$$\frac{1}{2}\left(\int_\Omega v\,dx\right)^2 \le \varepsilon\alpha^2\|u\|^4 + \frac{C}{\varepsilon}\left(\int_\Omega v\,dx\right)^2 + \alpha^2K_2^2\|f\|^2_{L^q(\omega)}\|v\|^2_{H^2}, \tag{3.40}$$

where $\varepsilon > 0$ is arbitrary and $K_2 := |\Omega|^{\frac{1}{2}-\frac{1}{q}}$.

Now, from the Gagliardo-Nirenberg inequality (see Lemma 2.3), using that $\|u\|_{L^1} = m_0$ and considering ε small enough we have

$$\varepsilon\alpha^2\|u\|^4 \le \varepsilon\|\nabla u\|^2\|u\|^2_{L^1} + \varepsilon C\|u\|^4_{L^1} \le \frac{\alpha}{2}\|\nabla u\|^2 + C. \tag{3.41}$$

Thus, arguing as in (3.18) and taking into account (3.40) and (3.41) we can deduce the following estimate

$$\frac{\alpha}{2}\frac{d}{dt}\|u\|^2 + \left(\frac{1 - 8(K_1^2 + K_2^2)\|f\|^2_{L^q(w)}}{8}\right)\|v\|^2_{H^2} \le C + \frac{C}{\varepsilon}\left(\int_\Omega v\,dx\right)^2.$$

The main difficulty here is to control the integral $\frac{C}{\varepsilon}\left(\int_\Omega v\,dx\right)^2$, since in (3.40) was considered ε small enough, which makes $\frac{C}{\varepsilon}$ very large and difficult to control. For this reason, the case that considers the quadratic chemical signal production term u^2 ($p = 2$) is not clear.

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