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On Quaternions with Gaussian Oresme Coefficients

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Abstract. At this paper, we describe Gaussian Oresme numbers taking into account the Oresme numbers. Furthermore, we investigate their some basic characteristic properties such as Binet formula and Cassini identity, etc. Moreover, we define quaternions with Gaussian Oresme coefficients and obtain their some spectacular properties.

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1. Introduction

1.1. **A Brief Review on Quaternions.** The set of real quaternions can be represented as

$$H = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

and it is a 4-dimensional vector space on $\mathbb{R}$, for details see [9,27,32]. A general from of a real quaternion is represented as below:

$$q = \sum_{m=0}^{3} k_m e_m = k_0 e_0 + k_1 e_1 + k_2 e_2 + k_3 e_3,$$

where k_0, k_1, k_2, and k_3 are real coefficients and e_0, e_1, e_2, and e_3 are quaternion units, that satisfies

$$e_0^2 = 1,\ e_0 e_i = e_i e_0 = e_i,\ i = 1, 2, 3,\ e_1^2 = e_2^2 = e_3^2 = -1.$$

The multiplication of quaternion units is listed in Table 1:

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

Table 1. The multiplication of the quaternion units

For any quaternions $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$, the addition operation is defined as follows:

$$p + q = S_{(p+q)} + V_{(p+q)},$$

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the vector product is defined as follows:

$$p \times q = (p_2q_3 - p_3q_2)i + (p_3q_1 - p_1q_3)j + (p_1q_2 - p_2q_1)k,$$

and moreover, the quaternion product is defined as follows:

$$pq = S_pS_q - \langle V_p, V_q\rangle + S_qV_q + S_qV_p + V_p \times V_q,$$

where "$\langle\ ,\ \rangle$" denotes the inner product and "$\times$" denotes the vector product in $\mathbb{R}^3$. In this definition S_q and V_q are the scalar part and vector parts, respectively. For $\lambda \in \mathbb{R}$, the scalar product is defined as

$$\lambda q = (\lambda q_0) + (\lambda q_1)i + (\lambda q_2)j + (\lambda q_3)k.$$

The conjugate of q is given by

$$\bar{q} = S_q - V_q.$$

Also, the norm of q is defined as below:

$$\|q\| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

The inverse of a real quaternion q is

$$q^{-1} = \frac{\bar{q}}{\|q\|}, \ \ \|q\| \neq 0.$$

1.2. **On the Oresme Numbers.** The Oresme numbers are defined by the following recurrence relation:

$$O_n = O_{n-1} - \frac{1}{4}O_{n-2}$$

for $n \geq 2$ with $O_0 = 0, O_1 = \frac{1}{2}$. These numbers are obtained as a special case of Horadam numbers. Horadam numbers are defined as $W_n = W_n(W_0, W_1; p, q)$, for $n \geq 0$:

$$W_{n+2} = pW_{n+1} - qW_n,$$

where p, q, n are integers and $W_0 = a$, $W_1 = b$. It is a general form of some famous number sequences, please see e.g. [2, 5–8, 13–18, 21, 22, 28–30]. Moreover the author, in [16], presents some identities for Oresme numbers as follows:

$$O_n = \frac{n}{2^n} \text{ (Binet Formula)},$$

$$O_{n+1}O_{n-1} - O_n^2 = -\frac{1}{2^{2n}} \text{ (Cassini Formula)},$$

$$O_{n+3} = \frac{3}{4}O_{n+1} - \frac{1}{4}O_n,$$

$$O_{n+3} = \frac{3}{4}O_{n+2} - \frac{1}{16}O_n,$$

$$\sum_{j=0}^{n} O_j = 4\left(\frac{1}{2} - O_{n+2}\right).$$

In [3], the authors defined the quaternions with the Oresme coefficients as below:

$$QO_n = O_ne_0 + O_{n+1}e_1 + O_{n+2}e_2 + O_{n+3}e_3.$$

In Table 2, we have listed some quaternions with Oresme coefficients.

n	-1	0	1	2
QO_n	$-2 + \frac{1}{2}e_2 + \frac{2}{4}e_3$	$\frac{1}{2}e_1 + \frac{2}{4}e_2 + \frac{3}{8}e_3$	$\frac{1}{2} + \frac{2}{4}e_1 + \frac{3}{8}e_2 + \frac{4}{16}e_3$	$\frac{2}{4} + \frac{3}{8}e_1 + \frac{4}{16}e_2 + \frac{5}{32}e_3$

Table 2. Some Oresme Quaternions

In [12], Halici investigated the complex Fibonacci quaternions and give the generating function and Binet formula for these quaternions. In [4], Arslan introduced the Gaussian Pell quaternion and Gaussian Pell-Lucas quaternion.

Then, the author obtained some interesting identities of them. Any complex quaternion λ is defined in the following form:

$$\lambda = \lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$$

where each coefficient λ_i is a complex number and e_0, e_1, e_2, e_3 are quaternionic units. The set of all complex quaternions is denoted by H_c. The complex quaternion λ can be written as

$$\lambda = k + ik', \ i^2 = -1,$$

where k and k' are real quaternions.

In literature, there are many amazing papers that are interested in Oresme numbers, Gaussian-type numbers, Horadam sequence, quaternions and their spectacular properties. For example; in [26], the authors studied Oresme hybrid numbers and hybrationals based on the known Oresme sequence and gave some properties of Oresme hybrid numbers. In [10], the authors defined the Horadam hybrid quaternions and gave some of their properties. Moreover, they investigated the relations between the Fibonacci hybrid quaternions and the Lucas hybrid quaternions which connected the Fibonacci quaternions and Lucas quaternions. In [11], the authors considered determinants for some families of Toeplitz-Hessenberg matrices whose entries are Oresme numbers. In particular, they established a connection between the Oresme and the Fibonacci and Pell sequences via Toeplitz-Hessenberg determinants. In [24], the authors defined the generalization of the matrix form of the Oresme sequence, extending it to the field of integers. In [23], using the Leonardo Pisano numbers and hybrid numbers, the authors investigated Leonardo Pisano polynomials and hybrinomials. In [1], the authors provided De Moivre's formula for the light-like Pauli quaternions. In [31], Yılmaz and Özkan took into account the generalized Gaussian Fibonacci numbers. In [20], Yılmaz and Karaca constructed new number systems, called the harmonic complex Fibonacci sequences (HCF) and the harmonic hybrid Fibonacci (HHF) sequences. Moreover, they examined some algebraic properties such as Binet-like-formula, partial sums related to these sequences. In [19], the authors present, in a unified manner, results which are valid on both split quaternions with quaternion coefficients and quaternions with dual coefficients, simultaneously, calling the attention to the main differences between these two quaternions.

At this paper, initially, we consider Gaussian Oresme numbers and examine some spectacular properties of them. Then, we define quaternions with Gaussian Oresme coefficients and obtain some of their characteristic properties.

2. Gaussian Oresme Numbers

In [25], the authors defined generalized Gaussian Fibonacci sequence, denoted by Gf_n, as below:

$$Gf_{n+1} = pGf_n + qGf_{n-1}$$

where $Gf_0 = a$, $Gf_1 = b$ are initial values. For $p = 1$, $q = -\frac{1}{4}$, $a = -2i$, $b = \frac{1}{2}$, we get the Gaussian Oresme sequence. In other words, the Gaussian Oresme sequence, denoted by GO_n, defined by the following recurrence relation

$$GO_n = GO_{n-1} - \frac{1}{4} GO_{n-2}, \quad \text{for n} \geq 2 \tag{2.1}$$

with initial conditions $GO_0 = -2i$, $GO_1 = \frac{1}{2}$. We remind that the Gaussian Oresme sequence can be rewritten as below:

$$GO_n = O_n + iO_{n-1}.$$

We have listed some values of the Gaussian Oresme numbers in the following table.

n	0	1	2	3	4	5	6
GO_n	$-2i$	$1/2$	$(1+i)/2$	$(3+4i)/8$	$(2+3i)/8$	$(5+8i)/32$	$(3+5i)/8$

Table 3. Some Gaussian Oresme numbers

Theorem 2.1 (Generating function)**.** *The generating function for the Gaussian Oresme numbers is*

$$f(x) = \sum_{n=0}^{\infty} (GO_n)\, x^n = \frac{-2i + (\frac{1}{2} + 2i)x}{1 - x + \frac{x^2}{4}}.$$

Proof. By exploiting the definition of the generating function, we have:

$$f(x) = GO_0 + GO_1 x + GO_2 x^2 + GO_3 x^3 + ... + GO_n x^n + \cdots ,$$

$$-xf(x) = -GO_0 x - GO_1 x^2 - GO_2 x^3 - ... - GO_{n-1} x^n - \cdots ,$$
$$\frac{1}{4}x^2 f(x) = \frac{1}{4}GO_0 x^2 + \frac{1}{4}GO_1 x^3 + \frac{1}{4}GO_2 x^4 + ... + \frac{1}{4}GO_{n-2} x^n + \cdots .$$

From here;

$$\begin{aligned}(1 - x + \frac{1}{4}x^2)f(x) &= GO_0 + (GO_1 - GO_0)\,x \\ &\quad + \left(GO_2 - GO_1 + \frac{1}{4}GO_0\right)x^2 \\ &\quad + \left(GO_3 - GO_2 + \frac{1}{4}GO_1\right)x^3 \\ &\quad \vdots \\ &\quad + \left(GO_n - GO_{n-1} + \frac{1}{4}GO_{n-2}\right)x^n + \cdots\end{aligned}$$

and as a result;

$$\begin{aligned}f(x) &= \frac{GO_0 + (GO_1 - GO_0)\,x}{1 - x + \frac{x^2}{4}} \\ &= \frac{-2i + (\frac{1}{2} + 2i)x}{1 - x + \frac{x^2}{4}}.\end{aligned}$$

So, the proof is completed. □

Note that the generating function helps us to obtain the Binet formula which is an explicit closed-form formula for the coefficients of these generating functions.

Theorem 2.2 (Binet formula)**.** *For $n \geq 0$:*

$$GO_n = 2^{-n}(-A + B(n + 1)),$$

where $A = (4i + 1)\,,\ B = (2i + 1)\,.$

Proof. By using the generating function and the definition of the Gaussian Oresme numbers, we have

$$\begin{aligned}f(x) &= \frac{GO_0 + (GO_1 - GO_0)\,x}{1 - x + \frac{x^2}{4}} \\ &= \frac{-2i + (\frac{1}{2} + 2i)x}{1 - x + \frac{x^2}{4}} \\ &= \frac{-2i + (\frac{1}{2} + 2i)x}{(\frac{x}{2} - 1)^2} \\ &= \frac{A}{(\frac{x}{2} - 1)} + \frac{B}{(\frac{x}{2} - 1)^2},\end{aligned}$$

where $A = (4i+1)$ and $B = (2i+1)$. It can be rewritten

$$\begin{aligned}
\frac{-2i + (\frac{1}{2} + 2i)x}{1 - x + \frac{x^2}{4}} &= -\frac{(4i+1)}{(1-\frac{x}{2})} + \frac{(2i+1)}{(1-\frac{x}{2})^2} \\
&= \left(\sum_{n=0}^{\infty} -A2^{-n}x^n\right) + \left(\sum_{n=0}^{\infty} B2^{-n}(n+1)x^n\right) \\
&= \sum_{n=0}^{\infty} \left(-A2^{-n} + B2^{-n}(n+1)\right) x^n,
\end{aligned}$$

where $f(x) = \sum_{n=0}^{\infty} \left(-A2^{-n} + B2^{-n}(n+1)\right) x^n$. In other words, the Binet formula is obtained as below:

$$\begin{aligned}
GO_n &= -A2^{-n} + B2^{-n}(n+1) \\
&= 2^{-n}(-A + B(n+1)).
\end{aligned}$$

So, the proof is completed. □

Example 2.3. GO_4 can be obtained by using Binet's formula. For $n = 4$,

$$\begin{aligned}
GO_4 &= 2^{-4}(-A + B(4+1)) \\
&= 2^{-4}(-(4i+1) + (2i+1)5) \\
&= \frac{3i+2}{8}.
\end{aligned}$$

Theorem 2.4 (Cassini identity)**.** *For $n > 0$, the following identity holds*

$$GO_{n+1}GO_{n-1} - GO_n^2 = -2^{-2n}B^2.$$

Proof. From the Binet formula for the Gaussian Oresme numbers;

$$\begin{aligned}
GO_{n+1}GO_{n-1} - GO_n^2 &= \left(2^{-n-1}(-A + B(n+2))\right)\left(2^{-n+1}(-A + Bn)\right) \\
&- \left((2^{-n}(-A + B(n+1))\right)^2 \\
&= 2^{-2n}(A^2 - ABn - AB(n+2) + B^2(n+2)n) \\
&- 2^{-2n}(A^2 - 2AB(n+1) + B^2(n+1)^2) \\
&= 2^{-2n}(A^2 - ABn - AB(n+2) + B^2(n+2)n \\
&- A^2 + 2AB(n+1) - B^2(n+1)^2) \\
&= -2^{-2n}B^2,
\end{aligned}$$

where $A = (4i+1)$ and $B = (2i+1)$. □

Example 2.5. For $n = 5$, Cassini identity can be observed as follows:

$$\begin{aligned}
GO_6GO_4 - GO_5^2 &= -2^{-10}B^2 \\
&= -2^{-10}(2i+1)^2 \\
&= 2^{-10}(3-4i) \\
&= \frac{3-4i}{2^{10}}.
\end{aligned}$$

Theorem 2.6 (Catalan identity)**.** *For $n, r \geq 0$,*

$$GO_{n+r}GO_{n-r} - GO_n^2 = -2^{-2n}B^2r^2.$$

Proof. From the Binet formula for the Gaussian Oresme numbers;

$$\begin{aligned}
GO_{n+r}GO_{n-r} - GO_n^2 &= \left(2^{-n-r}(-A + B(n+r+1))\right)\left(2^{-n+r}(-A + B(n-r+1))\right)\\
&- \left(2^{-n}(-A + B(n+1))\right)^2\\
&= 2^{-2n}(A^2 - 2ABn - 2AB + B^2(n^2 + 2n - r^2 + 1))\\
&- 2^{-2n}(A^2 - 2ABn - 2AB + B^2n^2 + 2B^2n + B^2))\\
&= -2^{-2n}B^2r^2,
\end{aligned}$$

where $A = (4i + 1)$ and $B = (2i + 1)$. So, the proof is completed. □

Example 2.7. For $n = 4$, $r = 1$,

$$\begin{aligned}
GO_5GO_3 - GO_4^2 &= -2^{-2n}B^2r^2\\
&= -2^{-8}(2i + 1)^2\\
&= 2^{-8}(3 - 4i)\\
&= \frac{3 - 4i}{2^8}.
\end{aligned}$$

Theorem 2.8 (d'Ocagne's identity)**.** *For $n, m \geq 0$,*

$$GO_{m+1}GO_n - GO_mGO_{n+1} = 2^{-m-n-1}B^2(n - m).$$

Proof. From the Binet formula for the Gaussian Oresme numbers;

$$\begin{aligned}
GO_{m+1}GO_n - GO_mGO_{n+1} &= \left(2^{-m-1}(-A + B(m+2))\right)\left(2^{-n}(-A + B(n+1))\right)\\
&- \left(2^{-m}(-A + B(m+1))\right)\left(2^{-n-1}(-A + B(n+2))\right)\\
&= 2^{-m-n-1}(A^2 - AB(n+1) - AB(m+2)\\
&+ B^2(m+2)(n+1)) - 2^{-m-n-1}(A^2 - AB(n+2)\\
&- AB(m+1) + B^2(m+1)(n+2))\\
&= 2^{-m-n-1}(A^2 - ABn - AB - ABm - 2AB\\
&+ (B^2m + 2B^2)(n+1)) - 2^{-m-n-1}\\
&\times (A^2 - ABn - 2AB - ABm - AB\\
&+ (B^2m + B^2)(n+2))\\
&= 2^{-m-n-1}(-B^2m + B^2n)\\
&= 2^{-m-n-1}B^2(n - m),
\end{aligned}$$

where $A = (4i + 1)$ and $B = (2i + 1)$. So, the proof is completed. □

Example 2.9. For $n = 4$, $m = 1$,

$$\begin{aligned}
GO_2GO_4 - GO_1GO_5 &= 2^{-1-4-1}B^2(4 - 1)\\
&= 2^{-6}3(2i + 1)^2\\
&= 2^{-6}(12i - 9).
\end{aligned}$$

3. Quaternions with Gaussian Oresme Numbers

In this section, we define quaternions with Gaussian Oresme coefficients and investigate some of their properties. Let us define the quaternions with Gaussian Oresme coefficients as below:

$$QGO_n = GO_ne_0 + GO_{n+1}e_1 + GO_{n+2}e_2 + GO_{n+3}e_3.$$

Note that, it verifies the equation $QGO_n = QO_n + iQO_{n-1}$. We have listed some values of the quaternions with Gaussian Oresme coefficients in Table 4.

n	QGO_n
0	$-2i + \frac{e_1}{2} + (\frac{1+i}{2})e_2 + (\frac{3+4i}{8})e_3$
1	$\frac{1}{2} + (\frac{1+i}{2})e_1 + (\frac{3+4i}{8})e_2 + (\frac{2+3i}{8})e_3$
2	$\frac{i+1}{2} + (\frac{3+4i}{8})e_1 + (\frac{2+3i}{8})e_2 + (\frac{8i+5}{32})e_3$
3	$\frac{3+4i}{8} + (\frac{2+3i}{8})e_1 + (\frac{5+8i}{32})e_2 + (\frac{3+5i}{32})e_3$

Table 4. Some Gaussian Oresme Quaternions

The conjugate, complex conjugate, norm of the quaternions with Gaussian Oresme coefficients are defined as below, respectively:

$$\begin{aligned}
QGO_n^* &= GO_n e_0 - GO_{n+1}e_1 - GO_{n+2}e_2 - GO_{n+3}e_3,\\
\overline{QGO_n} &= \overline{GO_n}e_0 + \overline{GO_{n+1}}e_1 + \overline{GO_{n+2}}e_2 + \overline{GO_{n+3}}e_3,\\
N_{QGO_n} &= QGO_n QGO_n^* = GO_n^2 + GO_{n+1}^2 + GO_{n+2}^2 + GO_{n+3}^2.
\end{aligned}$$

Theorem 3.1. *For $n > 1$, the quaternions with Gaussian Oresme coefficients verifies:*

i) $QGO_{n+1} = QGO_n - \frac{1}{4}QGO_{n-1}$.
ii) $QGO_n + QGO_n^* = 2GO_n$.

Proof.

i) Using the equation (2.1) and the definition of quaternions with Gaussian Oresme coefficients, we get

$$\begin{aligned}
QGO_n - \frac{1}{4}QGO_{n-1} &= (GO_n + GO_{n+1}e_1 + GO_{n+2}e_2 + GO_{n+3}e_3)\\
&\quad - \frac{1}{4}(GO_{n-1} + GO_n e_1 + GO_{n+1}e_2 + GO_{n+2}e_3)\\
&= \left(GO_n - \frac{1}{4}GO_{n-1}\right) + \left(GO_{n+1} - \frac{1}{4}GO_n\right)e_1\\
&\quad + \left(GO_{n+2} - \frac{1}{4}GO_{n+1}\right)e_2 + \left(GO_{n+3} - \frac{1}{4}GO_{n+2}\right)e_3\\
&= GO_{n+1} + GO_{n+2}e_1 + GO_{n+3}e_2 + GO_{n+4}e_3\\
&= QGO_{n+1}.
\end{aligned}$$

ii) From the definition of quaternions with Gaussian Oresme coefficients and its quaternion conjugate, we obtain

$$\begin{aligned}
QGO_n + QGO_n^* &= (GO_n + GO_{n+1}e_1 + GO_{n+2}e_2 + GO_{n+3}e_3)\\
&\quad + ((GO_n - GO_{n+1}e_1 - GO_{n+2}e_2 - GO_{n+3}e_3)\\
&= 2GO_n.
\end{aligned}$$

□

It is known that, the generating function is a way of coding an infinite sequence by treating them as the coefficients of a formal power series. Let us compute the generating function for the quaternions with Gaussian Oresme coefficients.

Theorem 3.2 (Generating function)**.** *The generating function for the quaternions with Gaussian Oresme coefficients is given by*

$$g(x) = \sum_{n=0}^{\infty} (QGO_n)\, x^n = \frac{\left[-2i + \frac{e_1}{2} + (\frac{1+i}{2})e_2 + (\frac{3+4i}{8})e_3\right] + x\left[(\frac{1+4i}{2}) + (\frac{i}{2})e_1 - (\frac{1}{8})e_2 - (\frac{1+i}{8})e_3\right]}{1 - x + \frac{x^2}{4}}.$$

Proof. By exploiting the definition of the generating function, we have:

$$\begin{aligned} g(x) &= QGO_0 + QGO_1 x + QGO_2 x^2 + QGO_3 x^3 + ... + QGO_n x^n + \cdots, \\ -xg(x) &= -QGO_0 x - QGO_1 x^2 - QGO_2 x^3 ... - QGO_{n-1} x^n - \cdots, \\ \frac{1}{4}x^2 g(x) &= \frac{1}{4}QGO_0 x^2 + \frac{1}{4}QGO_1 x^3 + \frac{1}{4}QGO_2 x^4 + ... + \frac{1}{4}QGO_{n-2} x^n + \cdots. \end{aligned}$$

From here;

$$\begin{aligned} (1 - x + \frac{1}{4}x^2)g(x) = {} & QGO_0 + (QGO_1 - QGO_0)\, x \\ & + \left(QGO_2 - QGO_1 + \frac{1}{4}QGO_0\right) x^2 \\ & + \left(QGO_3 - QGO_2 + \frac{1}{4}QGO_1\right) x^3 \\ & \vdots \\ & + \left(QGO_n - QGO_{n-1} + \frac{1}{4}QGO_{n-2}\right) x^n + \cdots \end{aligned}$$

and as a result;

$$\begin{aligned} g(x) &= \frac{QGO_0 + (QGO_1 - QGO_0)\, x}{1 - x + \frac{x^2}{4}} \\ &= \frac{\left[-2i + \frac{e_1}{2} + (\frac{1+i}{2})e_2 + (\frac{3+4i}{8})e_3\right] + x\left[(\frac{1+4i}{2}) + (\frac{i}{2})e_1 - (\frac{1}{8})e_2 - (\frac{1+i}{8})e_3\right]}{1 - x + \frac{x^2}{4}}. \end{aligned}$$

So, the proof is completed. □

Theorem 3.3 (Binet formula)**.** *For $n \geq 0$*

$$QGO_n = 2^{-n}(-A + B(n+1)),$$

where

$$A = \left[1 + 4i + ie_1 - \frac{1}{4}e_2 - \left(\frac{1+i}{4}\right)e_3\right]$$

and

$$B = \left[1 + 2i + \left(\frac{1+2i}{2}\right)e_1 + \left(\frac{1+2i}{4}\right)e_2 + \left(\frac{1+2i}{8}\right)e_3\right].$$

Proof. By exploiting the generating function and the definition of the quaternions with Gaussian Oresme coefficients, we have

$$\begin{aligned} g(x) &= \frac{QGO_0 + (QGO_1 - QGO_0)\, x}{1 - x + \frac{x^2}{4}} \\ &= \frac{\left[-2i + \frac{e_1}{2} + (\frac{1+i}{2})e_2 + (\frac{3+4i}{8})e_3\right] + x\left[(\frac{1+4i}{2}) + (\frac{i}{2})e_1 - (\frac{1}{8})e_2 - (\frac{1+i}{8})e_3\right]}{1 - x + \frac{x^2}{4}} \\ &= \frac{A}{(\frac{x}{2} - 1)} + \frac{B}{(\frac{x}{2} - 1)^2}, \end{aligned}$$

where

$$A = \left[1 + 4i + ie_1 - \frac{1}{4}e_2 - \left(\frac{1+i}{4}\right)e_3\right]$$

and

$$B = \left[1 + 2i + \left(\frac{1+2i}{2}\right)e_1 + \left(\frac{1+2i}{4}\right)e_2 + \left(\frac{1+2i}{8}\right)e_3\right].$$

It can be rewritten

$$\begin{aligned}\frac{\left[-2i + \frac{e_1}{2} + (\frac{1+i}{2})e_2 + (\frac{3+4i}{8})e_3\right] + x\left[(\frac{1+4i}{2}) + (\frac{i}{2})e_1 - (\frac{1}{8})e_2 - (\frac{1+i}{8})e_3\right]}{1 - x + \frac{x^2}{4}} &= -\frac{A}{(1-\frac{x}{2})} + \frac{B}{(1-\frac{x}{2})^2} \\ &= \left(\sum_{n=0}^{\infty} -A2^{-n}x^n\right) + \left(\sum_{n=0}^{\infty} B2^{-n}(n+1)x^n\right) \\ &= \sum_{n=0}^{\infty} 2^{-n}\left(-A + B(n+1)\right)x^n,\end{aligned}$$

where

$$f(x) = \sum_{n=0}^{\infty}\left(-A2^{-n} + B2^{-n}(n+1)\right)x^n, \text{ i.e.,}$$

$$\begin{aligned}QGO_n &= -A2^{-n} + B2^{-n}(n+1) \\ &= 2^{-n}(-A + B(n+1)).\end{aligned}$$

So, the proof is completed. □

Example 3.4. For $n = 1$,

$$\begin{aligned}QGO_1 &= 2^{-1}(-A + B(1+1)) \\ &= 2^{-1}\left[-\left(1 + 4i + ie_1 - \frac{1}{4}e_2 - \left(\frac{1+i}{4}\right)e_3\right)\right] \\ &+ \left[1 + 2i + \left(\frac{1+2i}{2}\right)e_1 + \left(\frac{1+2i}{4}\right)e_2 + \left(\frac{1+2i}{8}\right)e_3\right] \\ &= -\frac{1}{2} - 2i - \frac{ie_1}{2} + \frac{1}{8}e_2 + \left(\frac{1+i}{8}\right)e_3 \\ &+ 1 + 2i + \left(\frac{1+2i}{2}\right)e_1 + \left(\frac{1+2i}{4}\right)e_2 + \left(\frac{1+2i}{8}\right)e_3 \\ &= \frac{1}{2}\left(\frac{1+i}{2}\right)e_1 + \left(\frac{3+4i}{8}\right)e_2 + \left(\frac{2+3i}{8}\right)e_3.\end{aligned}$$

Theorem 3.5 (Cassini identity)**.** *For $n > 0$, the following identity holds*

$$QGO_{n+1}QGO_{n-1} - QGO_n^2 = -2^{-2n}B^2.$$

Proof. From the Binet formula;

$$\begin{aligned}QGO_{n+1}QGO_{n-1} - QGO_n^2 &= \left(2^{-n-1}(-A + B(n+2))\right)\left(2^{-n+1}(-A + Bn)\right) \\ &- \left((2^{-n}(-A + B(n+1))\right)^2 \\ &= 2^{-2n}(A^2 - ABn - AB(n+2) + B^2(n+2)n) \\ &- 2^{-2n}(A^2 - 2AB(n+1) + B^2(n+1)^2) \\ &= 2^{-2n}(A^2 - ABn - AB(n+2) + B^2(n+2)n \\ &- A^2 + 2AB(n+1) - B^2(n+1)^2) \\ &= -2^{-2n}B^2,\end{aligned}$$

where $A = \left[1 + 4i + ie_1 - \frac{1}{4}e_2 - \left(\frac{1+i}{4}\right)e_3\right]$ and $B = \left[1 + 2i + \left(\frac{1+2i}{2}\right)e_1 + \left(\frac{1+2i}{4}\right)e_2 + \left(\frac{1+2i}{8}\right)e_3\right]$.

So, the proof is completed. □

Example 3.6. For $n = 3$,

$$\begin{aligned} QGO_4QGO_2 - QGO_3^2 &= -2^{-6}\left[1 + 2i + \left(\frac{1+2i}{2}\right)e_1 + \left(\frac{1+2i}{4}\right)e_2 + \left(\frac{1+2i}{8}\right)e_3\right]^2 \\ &= -2^{-6}\left(\frac{1+2i}{8}\right)^2 [8 + 4e_1 + 2e_2 + e_3]^2 \\ &= -2^{-12}(4i - 3)(43 + 68e_1 + 24e_2 + 32e_3). \end{aligned}$$

Theorem 3.7 (Catalan identity)**.** *For $n, r \geq 0$, the following property holds*

$$QGO_{n+r}QGO_{n-r} - QGO_n^2 = -2^{-2n}B^2r^2.$$

Proof. By considering the Binet formula;

$$\begin{aligned} QGO_{n+r}QGO_{n-r} - QGO_n^2 &= \left(2^{-n-r}(-A + B(n + r + 1))\right)\left(2^{-n+r}(-A + B(n - r + 1))\right) \\ &\quad - \left(2^{-n}(-A + B(n + 1))\right)^2 \\ &= 2^{-2n}(A^2 - 2ABn - 2AB + B^2(n^2 + 2n - r^2 + 1)) \\ &\quad - 2^{-2n}(A^2 - 2ABn - 2AB + B^2n^2 + 2B^2n + B^2)) \\ &= -2^{-2n}B^2r^2, \end{aligned}$$

where $A = \left[1 + 4i + ie_1 - \frac{1}{4}e_2 - \left(\frac{1+i}{4}\right)e_3\right]$ and $B = \left[1 + 2i + \left(\frac{1+2i}{2}\right)e_1 + \left(\frac{1+2i}{4}\right)e_2 + \left(\frac{1+2i}{8}\right)e_3\right]$.
So, the proof is completed. □

Example 3.8. For $n = 2$, $r = 1$,

$$\begin{aligned} QGO_3QGO_1 - QGO_2^2 &= -2^{-2n}B^2r^2 \\ &= -2^{-4}\left[1 + 2i + \left(\frac{1+2i}{2}\right)e_1 + \left(\frac{1+2i}{4}\right)e_2 + \left(\frac{1+2i}{8}\right)e_3\right]^2 \\ &= -2^{-4}\left(\frac{1+2i}{8}\right)^2 [8 + 4e_1 + 2e_2 + e_3]^2 \\ &= -2^{-10}(4i - 3)(43 + 68e_1 + 24e_2 + 32e_3). \end{aligned}$$

4. Conclusion

In this paper, we consider the Oresme numbers, defined by Horadam in [16], and describe the Gaussian Oresme numbers. Then, we investigate some of their characteristic properties, such as the Binet formula, generating function, Cassini identity, etc. In the following section, we define the quaternions with Gaussian Oresme coefficients and obtain some properties for them. Finally, we illustrate the results with some examples.

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Conflicts of Interest

The author/authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

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