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# Certain Finite Sums Pertaining to Leibnitz, Harmonic and Other Special Numbers

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**Abstract**

The present manuscript deals with some certain finite sums and identities pertaining to some special numbers. Using generating functions methods, some relations and identities involving the Apostol type Euler and combinatorial numbers, and also the Fubini type numbers and polynomials, are given. Then, by using some certain classes of special finite sums involving the following rational sum which is defined by Simsek (2021b):

$$\boldsymbol{y}(r,\vartheta)=\sum_{b=0}^{r}\frac{(-1)^r}{(1+b)\vartheta^{b+1}(\vartheta-1)^{r-b+1}},$$

many new certain finite sums and formulas related to the Leibnitz, Harmonic, Changhee, and Daehee numbers are obtained. Moreover, some applications of these results are presented.

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## 1. INTRODUCTION

It is well known that certain finite sums including special numbers and special functions have taken their place among the main subjects of the studies of many researchers in recent years. Because these sums contain formulas and relations that are frequently used in mathematics, engineering and other branches of science due to their properties. For this reason, they are frequently used in modeling design and other situations involving many real-world problems.

In present manuscript, we give some certain finite sums and relations covering both the Leibnitz, Harmonic, Changhee, Daehee, and Apostol type Euler numbers, and also the Fubini type numbers and polynomials. In order to obtain these results, let us introduce the following notations and definitions that we will use throughout this manuscript:

Let $\mathbb{N}=\{1,2,3,\ldots\}$, $\mathbb{C}$ denotes by complex numbers and $\mathbb{N}_0=\{0,1,2,3,\ldots\}$. Let

$$\binom{\omega}{k}k!=(\omega)_k=\omega(\omega-1)\ldots(\omega-k+1)$$

with $(\omega)_0=1, k\in\mathbb{N}, \omega\in\mathbb{C}$ and

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$$0^s = \begin{cases} 1, & s = 0 \\ 0, & s \in \mathbb{N} \end{cases}$$

(Gould, 1972;-;Srivastava & Kızılateş, 2019).

The numbers $E_r^{*(-k)}(\vartheta)$ are defined by

$$\mathcal{Z}_E(t,k,\vartheta) = \left(\frac{\vartheta e^t + \vartheta^{-1}e^{-t}}{2}\right)^k = \sum_{r=0}^{\infty} E_r^{*(-k)}(\vartheta)\frac{t^r}{r!}, \tag{1}$$

where $k \in \mathbb{N}$ and $\vartheta \in \mathbb{C}$. The numbers $E_r^{*(-k)}(\vartheta)$ are called the second kind Apostol type Euler numbers of order $-k$ (Simsek, 2017; 2018; 2022a).

The numbers $W_r^{(-k)}(\vartheta)$ are defined by

$$\mathcal{Z}_W(t,k,\vartheta) = (\vartheta e^t + \vartheta^{-1}e^{-t} + 2)^k = \sum_{r=0}^{\infty} W_r^{(-k)}(\vartheta)\frac{t^r}{r!}, \tag{2}$$

where $k \in \mathbb{N}$ and $\vartheta \in \mathbb{C}$ (Simsek, 2017; 2018; 2022a; see also Kucukoglu & Simsek, 2018; Kucukoglu et al., 2019).

By using (1) and (2), one has

$$W_r^{(-k)}(\vartheta) = 2^k \sum_{j=0}^{k} \binom{k}{j} E_r^{*(-j)}(\vartheta) \tag{3}$$

(Simsek, 2017; 2018; 2022a).

The Daehee numbers are defined by

$$D_r = (-1)^r \frac{r!}{r+1} \tag{4}$$

(Kim & Kim, 2013; see also Simsek, 2019).

The Changhee numbers are defined by

$$Ch_r = (-1)^r \frac{r!}{2^r} \tag{5}$$

(Kim et al., 2013; see also Simsek, 2019).

The Leibnitz numbers are defined by

$$\boldsymbol{l}(r,b) = \frac{\binom{r}{b}^{-1}}{(r+1)}, \tag{6}$$

where $b = 0,1,2,\ldots,r$ and $r \in \mathbb{N}_0$ (for detail, see Simsek, 2021a). Recently, Simsek (2021a) has given some results associated with the Leibnitz, Daehee, Changhee and other well-known special numbers.

Simsek (2021a (Theorem 2.9. and Theorem 2.10.)) gave the following formulas

$$2q!\sum_{b=0}^{q}\boldsymbol{l}(q,b)-q!\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)=2(-1)^{q}D_{q} \tag{7}$$

and

$$2(1+q)\sum_{b=0}^{q}\boldsymbol{l}(q,b)-(1+q)\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)=2. \tag{8}$$

The Harmonic numbers, $H_r$, are defined by

$$H_r=\sum_{b=1}^{r}\frac{1}{b}$$

(for detail, see Simsek, 2021b, 2021c, 2021d; 2022b, 2022c). From the above equation, we have

$$H_{r+1}-H_r=\frac{1}{r+1} \tag{9}$$

(Simsek, 2021b, 2021c; 2022b, 2022c).

Kilar and Simsek (2017) defined the following the Fubini type numbers $a_r^{(n)}$:

$$\frac{2^n}{(2-e^t)^{2n}}=\sum_{r=0}^{\infty}a_r^{(n)}\frac{t^r}{r!}, \tag{10}$$

where $n\in\mathbb{N}_0$ and $|t|<\ln(2)$ (for detail, see Kilar & Simsek, 2019a, 2019b; 2021a, 2021b; Kilar, 2023a, 2023b).

When $n=1$ in (10), we get

$$a_r^{(1)}=a_r.$$

Kilar and Simsek (2017) defined the following polynomials $a_r^{(n)}(x)$:

$$\frac{2^n}{(2-e^t)^{2n}}e^{xt}=\sum_{r=0}^{\infty}a_r^{(n)}(x)\frac{t^r}{r!}. \tag{11}$$

From (10) and (11), we have

$$a_r^{(n)}(x)=\sum_{b=0}^{r}\binom{r}{b}x^{r-b}a_b^{(n)}$$

(Kilar & Simsek, 2017; 2019a, 2019b; 2021a, 2021b; Kilar, 2023a, 2023b).

Two parametric polynomials $a_r^{(C,n)}(x,y)$ and $a_r^{(S,n)}(x,y)$ are defined, respectively, by

$$\mathcal{Z}_{ac}(t,n,x,y) = \frac{2^n e^{xt}}{(2-e^t)^{2n}}\cos(yt) = \sum_{r=0}^{\infty} a_r^{(C,n)}(x,y)\frac{t^r}{r!} \tag{12}$$

and

$$\mathcal{Z}_{as}(t,n,x,y) = \frac{2^n e^{xt}}{(2-e^t)^{2n}}\sin(yt) = \sum_{r=0}^{\infty} a_r^{(S,n)}(x,y)\frac{t^r}{r!} \tag{13}$$

(Srivastava & Kızılateş, 2019). When $y = 0$ in (12), one can see that

$$a_r^{(C,n)}(x,0) = a_r^{(n)}(x).$$

Simsek (2021b) defined the following finite sum which is called the numbers $\boldsymbol{y}(r,\vartheta)$:

$$\boldsymbol{y}(r,\vartheta) = \sum_{b=0}^{r} \frac{(-1)^r}{(1+b)\vartheta^{b+1}(\vartheta-1)^{r+1-b}} \tag{14}$$

(see also Simsek, 2021c, 2021d; 2022b, 2022c).

Simsek (2022b (Equation (59))) gave the following formula

$$\boldsymbol{y}(r-1,\vartheta) + (\vartheta-1)\boldsymbol{y}(r,\vartheta) = \frac{(-1)^r}{(r+1)\vartheta^{r+1}}. \tag{15}$$

When $\vartheta = \frac{1}{2}$ in the above equation, we get

$$2\boldsymbol{y}\left(r-1,\frac{1}{2}\right) - \boldsymbol{y}\left(r,\frac{1}{2}\right) = \frac{(-1)^r 2^{r+2}}{r+1} \tag{16}$$

(Simsek, 2022b (Equation (61))).

## 2. MAIN RESULTS

Using the generating functions which are introduced previous section, many miscellaneous identities involving the Fubini type numbers, the numbers $E_m^{*(-k)}(\vartheta)$ and the numbers $W_m^{(-k)}(\vartheta)$, are given. Using derivative and integrate operators, some certain finite sums and formulas pertaining to the Leibnitz, Harmonic, Changhee and Daehee numbers and also the numbers $\boldsymbol{y}(m,\vartheta)$, are presented. Moreover, some applications of the obtained results are given.

**Theorem 2.1.** For $u \in \mathbb{N}_0$ and $d \in \mathbb{N}$ yields

$$\sum_{b=0}^{\left[\frac{u}{2}\right]} \binom{u}{2b}(-1)^b y^{2b}(x-d)^{u-2b} + (-1)^{d+1}\sum_{b=0}^{u}\binom{u}{b} W_{u-b}^{(-d)}\left(-\frac{1}{2}\right) a_b^{(C,d)}(x,y) = 0. \tag{17}$$

*Proof*. By using (2) and (12), we have

$$e^{(x-d)t}\cos(yt) = (-1)^d \mathcal{Z}_W\left(t,d,-\frac{1}{2}\right)\mathcal{Z}_{ac}(t,d,x,y).$$

From above equation, we get

$$\sum_{u=0}^{\infty}(x-d)^u\frac{t^u}{u!}\sum_{u=0}^{\infty}\frac{(-1)^u(yt)^{2u}}{(2u)!} = (-1)^d\sum_{u=0}^{\infty}W_u^{(-d)}\left(-\frac{1}{2}\right)\frac{t^u}{u!}\sum_{u=0}^{\infty}a_u^{(C,d)}(x,y)\frac{t^u}{u!}.$$

Thus,

$$\sum_{u=0}^{\infty}\sum_{b=0}^{\left[\frac{u}{2}\right]}(-1)^b\binom{u}{2b}(x-d)^{u-2b}y^{2b}\frac{t^u}{u!} = \sum_{u=0}^{\infty}\sum_{b=0}^{u}\binom{u}{b}(-1)^d W_{u-b}^{(-d)}\left(-\frac{1}{2}\right)a_b^{(C,d)}(x,y)\frac{t^u}{u!}.$$

Therefore, Equation (17) is derived.

When $y = 0$ in (17), we get the Corollary 2.2:

**Corollary 2.2.**

$$(x-d)^u = \sum_{b=0}^{u}\binom{u}{b}(-1)^d W_{u-b}^{(-d)}\left(-\frac{1}{2}\right)a_b^{(d)}(x). \tag{18}$$

Applying derivative operator $\frac{\partial^u}{\partial x^u}$ to the Equation (18), we get

$$\frac{\partial^u}{\partial x^u}\{(x-d)^u\} = \sum_{b=0}^{u}(-1)^d\binom{u}{b}W_{u-b}^{(-d)}\left(-\frac{1}{2}\right)\frac{\partial^u}{\partial x^u}\left\{a_b^{(d)}(x)\right\}. \tag{19}$$

Since

$$\frac{\partial^u}{\partial x^u}\left\{a_b^{(d)}(x)\right\} = (u)_b\, a_{b-u}^{(d)}(x)$$

(Kilar & Simsek, 2017), after some elementary calculations, Equation (19) is derived as follows:

$$u! = (-1)^d u!\, W_0^{(-d)}\left(-\frac{1}{2}\right)\, a_0^{(d)}(x). \tag{20}$$

Since

$$a_0^{(d)}(x) = 2^d,$$

Equation (20) reduced to the following special result:

$$W_0^{(-d)}\left(-\frac{1}{2}\right)=\frac{(-1)^d}{2^d}.$$

**Theorem 2.3.** For $u,d\in\mathbb{N}$ yields

$$\sum_{b=0}^{\left[\frac{u-1}{2}\right]}\binom{u}{2b+1}(-1)^b(x-d)^{u-2b-1}y^{2b+1}+\sum_{b=0}^{u}(-1)^{d+1}\binom{u}{b}W_{u-b}^{(-d)}\left(-\frac{1}{2}\right)a_b^{(S,d)}(x,y)=0. \tag{21}$$

*Proof*. By (2) and (13), we have

$$e^{(x-d)t}\sin(yt)=(-1)^d\mathcal{Z}_W\left(t,d,-\frac{1}{2}\right)\mathcal{Z}_{as}(t,d,x,y).$$

Using the above functional equation, we have

$$\sum_{u=0}^{\infty}\sum_{b=0}^{\left[\frac{u-1}{2}\right]}\binom{u}{1+2b}(-1)^b y^{1+2b}(x-d)^{u-2b-1}\frac{t^u}{u!}$$
$$=\sum_{u=0}^{\infty}\sum_{b=0}^{u}(-1)^d\binom{u}{b}W_{u-b}^{(-d)}\left(-\frac{1}{2}\right)a_b^{(S,d)}(x,y)\frac{t^u}{u!}.$$

Thus, Equation (21) is obtained.

Combining (17) with (3), we get the Theorem 2.4:

**Theorem 2.4.** For $u\in\mathbb{N}_0$ and $d\in\mathbb{N}$ yields

$$\sum_{b=0}^{\left[\frac{u}{2}\right]}(-1)^b\binom{u}{2b}y^{2b}(x-d)^{u-2b}=(-2)^d\sum_{b=0}^{u}\binom{u}{b}\sum_{k=0}^{d}\binom{d}{k}a_b^{(C,d)}(x,y)E_{u-b}^{*(-k)}\left(-\frac{1}{2}\right).$$

By combining (21) with (3), we derive the Theorem 2.5:

**Theorem 2.5.** For $u,d\in\mathbb{N}$ yields

$$\sum_{b=0}^{\left[\frac{u-1}{2}\right]}(-1)^b\binom{u}{2b+1}y^{2b+1}(x-d)^{u-2b-1}=(-2)^d\sum_{b=0}^{u}\binom{u}{b}\sum_{k=0}^{d}\binom{d}{k}a_b^{(S,d)}(x,y)E_{u-b}^{*(-k)}\left(-\frac{1}{2}\right).$$

**Theorem 2.6.** For $u\in\mathbb{N}_0$ and $d\in\mathbb{N}$ yields

$$\frac{(1-d)^{u+1}+d(-d)^u}{u+1}=\sum_{b=0}^{u}\sum_{p=0}^{b}\binom{u}{b}\binom{b}{p}W_{u-b}^{(-d)}\left(-\frac{1}{2}\right)\frac{(-1)^d a_p^{(d)}}{b-p+1}. \tag{22}$$

*Proof*. Integrating both sides with respect to $x$ of (18), we obtain

$$\int_0^1 (x-d)^u dx = (-1)^d \sum_{b=0}^{u} \binom{u}{b} W_{u-b}^{(-d)}\left(-\frac{1}{2}\right) \int_0^1 a_b^{(d)}(x)dx.$$

Hence

$$\frac{(-1)^{u+2}d^{u+1} + (1-d)^{u+1}}{1+u} + \sum_{b=0}^{u}\sum_{p=0}^{b} \binom{b}{p}\binom{u}{b} W_{u-b}^{(-d)}\left(-\frac{1}{2}\right)\frac{(-1)^{d+1}a_p^{(d)}}{b+1-p} = 0.$$

From the above equation, we get the Equation (22).

Gould (1972 (p. 5, Equation (1.37)) gave the combinatorial finite sum:

$$\frac{(x+1)^{v+1}-1}{x(1+v)} = \sum_{s=0}^{v}\binom{v}{s}\frac{x^s}{1+s}. \tag{23}$$

Using (23), Equation (22) is reduced to as follows:

**Corollary 2.7.**

$$\sum_{b=0}^{u}\binom{u}{b}\frac{(-d)^u(-1)^b}{(b+1)d^b} = \sum_{b=0}^{u}\sum_{p=0}^{b}\binom{u}{b}\binom{b}{p} W_{u-b}^{(-d)}\left(-\frac{1}{2}\right)\frac{(-1)^d a_p^{(d)}}{b-p+1}. \tag{24}$$

Replacing $\vartheta$ by $\frac{1}{d}$ in (15), we have

$$\boldsymbol{y}(q-1, d^{-1}) + (d^{-1}-1)\boldsymbol{y}(q, d^{-1}) = -\frac{(-d)^{q+1}}{1+q}. \tag{25}$$

Replacing $\vartheta$ by $\frac{1}{d-1}$ in (15), we also have

$$\boldsymbol{y}(q-1, (d-1)^{-1}) + ((d-1)^{-1}-1)\boldsymbol{y}(q, (d-1)^{-1}) = -\frac{(1-d)^{q+1}}{q+1}. \tag{26}$$

Combining (26) and (25) with (22), we get

$$\begin{aligned}\boldsymbol{y}(q-1, d^{-1}) + (d^{-1}-1)\boldsymbol{y}(q, d^{-1}) - \boldsymbol{y}(q-1, (d-1)^{-1}) \\ -((d-1)^{-1}-1)\boldsymbol{y}(q, (d-1)^{-1}) \\ = \sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p} W_{q-b}^{(-d)}\left(-\frac{1}{2}\right)\frac{(-1)^d a_p^{(d)}}{b-p+1}.\end{aligned}$$

From here, we get the Theorem 2.8:

**Theorem 2.8.** For $d \in \mathbb{N}\backslash\{1,2\}$ and $q \in \mathbb{N}$ yields

$$\boldsymbol{y}(q-1, d^{-1}) - \boldsymbol{y}(q-1, (d-1)^{-1}) + \left(\frac{1-d}{d}\right)\boldsymbol{y}(q, d^{-1}) + \left(\frac{2-d}{1-d}\right)\boldsymbol{y}(q, (d-1)^{-1}) = \sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p}W_{q-b}^{(-d)}\left(-\frac{1}{2}\right)\frac{(-1)^d a_p^{(d)}}{b-p+1}. \tag{27}$$

Now it is time to give some interesting applications of Equation (24).

Substituting $x = -1$ into (23), we get

$$\sum_{b=0}^{q}\frac{\binom{q}{b}(-1)^b}{(b+1)} = \frac{1}{q+1}. \tag{28}$$

When $d = 1$ in (24), and using (28), we get the Corollary 2.9:

**Corollary 2.9.**

$$1 = (q+1)\sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p}W_{q-b}^{(-1)}\left(-\frac{1}{2}\right)\frac{(-1)^{1-q}a_p}{b-p+1}. \tag{29}$$

Moreover, substituting $d = 2$ into (22) we obtain

$$(2^{q+1}-1)(-1)^q = (q+1)\sum_{b=0}^{q}\sum_{p=0}^{b}\binom{b}{p}\binom{q}{b}W_{q-b}^{(-2)}\left(-\frac{1}{2}\right)\frac{a_p^{(2)}}{b-p+1}.$$

When $x = 1$ in (23), and using the previous equation, we also obtain

$$\sum_{b=0}^{q}\binom{q}{b}\frac{(-1)^q}{b+1} = \sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p}W_{q-b}^{(-2)}\left(-\frac{1}{2}\right)\frac{a_p^{(2)}}{b+1-p}. \tag{30}$$

Also, substituting $d = 2$ into (24), we obtain

$$\sum_{b=0}^{q}\binom{q}{b}\frac{(-1)^{q+b}}{(1+b)2^{b-q}} = \sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p}W_{q-b}^{(-2)}\left(-\frac{1}{2}\right)\frac{a_p^{(2)}}{b-p+1}. \tag{31}$$

Combining (30) with (31), we derive the following presumably known result:

**Corollary 2.10.**

$$\sum_{b=0}^{\varphi}\frac{\binom{\varphi}{b}}{1+b}=\sum_{b=0}^{\varphi}\binom{\varphi}{b}\frac{(-1)^b}{(1+b)2^{b-\varphi}}.$$

By using (22) and (4), we have

$$\frac{(-1)^{q}}{q!}\mathrm{D}_{q}-\frac{(d-1)^{q+1}}{d^{q+1}(q+1)}=\frac{1}{d^{q+1}}\sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p}W_{q-b}^{(-d)}\left(-\frac{1}{2}\right)\frac{(-1)^{d+q}a_p^{(d)}}{b-p+1}.$$

Using (7) and the above equation, we have the Theorem 2.11:

**Theorem 2.11.** For $q,d\in\mathbb{N}$ yields

$$\begin{aligned}\sum_{b=0}^{q}\boldsymbol{l}(q,b)-\frac{1}{2}\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)&\\&=\frac{(1-d^{-1})^{q+1}}{q+1}+\frac{1}{d^{1+q}}\sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p}W_{q-b}^{(-d)}\left(-\frac{1}{2}\right)\frac{(-1)^{d+q}a_p^{(d)}}{b-p+1}.\end{aligned}\tag{32}$$

Combining (9) with (8), we derive

$$\sum_{b=0}^{q}\boldsymbol{l}(q,b)-\frac{1}{2}\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)=H_{q+1}-H_{q}.$$

Using the above equation and (32), we get the Corollary 2.12:

**Corollary 2.12.**

$$H_{q+1}-H_{q}-\frac{(1-d^{-1})^{q+1}}{q+1}=\sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p}W_{q-b}^{(-d)}\left(-\frac{1}{2}\right)\frac{(-1)^{d+q}a_p^{(d)}}{d^{q+1}(b-p+1)}.$$

Combining (32) with (27), we have the Corollary 2.13:

**Corollary 2.13.** For $d\in\mathbb{N}\backslash\{1,2\}$ and $q\in\mathbb{N}$ yields

$$\begin{aligned}d(d-1)\big(\boldsymbol{y}(q-1,d^{-1})-\boldsymbol{y}(q-1,(d-1)^{-1})\big)-(d-1)^2\boldsymbol{y}(q,d^{-1})+d(d-2)\boldsymbol{y}(q,(d-1)^{-1})&\\=(d-1)d^{q+2}(-1)^{q}\left(\sum_{b=0}^{q}\boldsymbol{l}(q,b)-\frac{1}{2}\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)\right)-\frac{d(1-d)^{q+2}}{q+1}.&\end{aligned}$$

When $d=3$ into Corollary 2.13, and using (16), we get the Corollary 2.14:

**Corollary 2.14.**

$$3\boldsymbol{y}\left(q-1,\frac{1}{3}\right)-2\boldsymbol{y}\left(q,\frac{1}{3}\right)=(-3)^{q+2}\left(\sum_{b=0}^{q}\boldsymbol{l}(q,b)-\frac{1}{2}\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)\right)$$

or, equivalently,

$$3\boldsymbol{y}\left(q-1,\frac{1}{3}\right)-2\boldsymbol{y}\left(q,\frac{1}{3}\right)=(-3)^{q+2}\left(H_{q+1}-H_{q}\right).$$

When $d=2$ in (32), we get

$$2\sum_{b=0}^{q}\boldsymbol{l}(q,b)-\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)$$
$$=\frac{2^{-q}}{(q+1)}+2^{-q}(-1)^{q}\sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p}W_{q-b}^{(-2)}\left(-\frac{1}{2}\right)\frac{a_p^{(2)}}{b+1-p}.$$

Using (4), (5) and the above equation, we have

$$2\sum_{b=0}^{q}\boldsymbol{l}(q,b)-\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)$$
$$=\frac{2Ch_{q+1}}{(2+q)(1+q)D_{q+1}}\left(1+(1+q)(-1)^{q}\sum_{b=0}^{q}\sum_{p=0}^{b}\binom{q}{b}\binom{b}{p}W_{q-b}^{(-2)}\left(-\frac{1}{2}\right)\frac{a_p^{(2)}}{b-p+1}\right).$$

Using the previous equation and (30) (or (31)), the following result is derived:

**Corollary 2.15.**

$$2\sum_{b=0}^{q}\boldsymbol{l}(q,b)-\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)=\frac{2Ch_{q+1}}{(1+q)(2+q)D_{q+1}}\left(1+(q+1)\sum_{b=0}^{q}\frac{\binom{q}{b}}{b+1}\right)$$

or, equivalently,

$$2\sum_{b=0}^{q}\boldsymbol{l}(q,b)-\sum_{b=0}^{q-1}\boldsymbol{l}(q-1,b)=\frac{Ch_{q+1}}{(1+q)(2+q)D_{q+1}}\left(2+\sum_{b=0}^{q}\binom{q}{b}\frac{(-1)^{b}(q+1)}{2^{b-q-1}(b+1)}\right).$$

## 3. CONCLUSION

In this manuscript, some certain finite sums and identities included some special numbers were studied. Using certain special polynomials and numbers with generating functions, and integrating some results, various novel formulas, combinatorial finite sums and identities were given. These various results pertaining to the Leibnitz, Harmonic, Apostol type Euler, Changhee, Daehee, combinatorial and Fubini type numbers, and also the Fubini type polynomials. As a result, the results obtained in present manuscript may be usefulness in related sciences especially engineering and mathematics.

## CONFLICT OF INTEREST

The author declares no conflict of interest.

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