Journal of Universal Mathematics
Vol.5 No.2 pp.105-116 (2022)
ISSN-2618-5660
DOI: 10.33773/jum.1134983

NUMERICAL SIMULATION OF GENERALIZED OSKOLKOV EQUATION VIA THE SEPTIC B-SPLINE COLLOCATION METHOD

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Abstract. In this paper, one of the nonlinear evolution equation (NLEE) namely generalised Oskolkov equation which defines the dynamics of an incompressible visco-elastic Kelvin-Voigt fluid is investigated. We discuss numerical solutions of the equation for two test problems including shock wave and Gaussian initial condition, applying the collocation finite element method. The algorithm, based upon Crank Nicolson approach in time, is unconditionally stable. To demonstrate the efficiency and accuracy of the numerical algorithm, error norms L_2, L_∞ and invariant I are calculated and the obtained results are given both in tabular and graphical form. The obtained numerical results provide the method is more suitable and systematically handle the solution procedures of nonlinear equations arising in mathematical physics.

1. Introduction

Nonlinear evolution equations (NLEEs) are special classes of the category of partial differential equations (PDEs), which have been studied intensively in past several decades [1]. Various methods [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] have been devised to find the exact and approximate solutions of PDEs in order to provide more information for understanding physical phenomena arising in numerous scientific and engineering fields such as mathematics, physics, mechanics, biology, ecology, optical fiber, chemical reaction and so on [18].

The Oskolkov system which describes the nonlinear phenomena in incompressible viscoelastic Kelvin–Voigt fluid and fluid dynamics, based upon Oskolkov equation,

$$u_t - \lambda u_{xxt} - \alpha u_{xx} + u u_x = 0, \tag{1.1}$$

and modified Oskolkov equation

$$u_t - \lambda u_{xxt} - \alpha u_{xx} + u^2 u_x = 0, \tag{1.2}$$

Date: **Received:** 2022-06-23; **Accepted:** 2022-07-27.
2000 *Mathematics Subject Classification.* 65N30, 65D07.
Key words and phrases. Generalised Oskolkov equation, Shock wave, Collocation.

has the form

$$(1 - \lambda\triangledown^2)u_t = \alpha \triangledown^2 u - (u \bullet \triangledown)u - \triangledown^2 p + f = 0, \quad \triangledown \bullet u = 0. \tag{1.3}$$

Let's immediately note that α is viscosity and λ plays a vital role and should be negative. Its negative ness has physical meaning [19, 20]. Various kinds of Oskolkov equation are solved by several methods [21, 22, 23, 24, 25, 26, 27, 28] to construct exact solutions. Even so, numerical methods for initial-boundary value problem of the generalized Oskolkov equation have not been investigated considerable. Karakoc et al. [29] have applied collocation finite element method to find the numerical solutions of the equation.

The work recreation is organized as follows: In the next section, by giving information about septic B-spline functions, collocation method has been applied to the equation. In Section 3, stability analysis of the method is established. Some numerical examples are reported in Section 4 to validate performance of the method. Finally, we finish the paper with a brief conclusion.

2. Numerical applications

In this section, we consider the generalised Oskolkov equation as

$$u_t + \gamma(u^p)_x + \sigma u_{xx} + \eta u_{xxt} = 0, \tag{2.1}$$

the initial condition $u(x, 0) = f(x)$ $a \leq x \leq b$,and the boundary conditions

$$\begin{array}{ll} u_N(a,t) = 0, & u_N(b,t) = 0, \\ (u_N)_x(a,t) = 0, & (u_N)_x(b,t) = 0 \\ (u_N)_{xx}(a,t) = 0, & (u_N)_{xx}(b,t) = 0, \quad t > 0 \end{array} \tag{2.2}$$

where γ, σ and η are constants and x is the space coordinate and t symbolizes time differentiation.

For simplicity, let us assume that the grid distribution $a = x_0 < x_1 < ... < x_N = b$ of a finite interval $[a, b]$ is taken into consideration with mesh spacing $h = \frac{b-a}{N} = (x_{m+1} - x_m)$. First of all, we assume $\psi_m(x)$ to be the septic B-splines having nodal points x_m through which the uniformly distributed N nodal points are taken as $a = x_0 < x_1 < ... < x_N = b$ on the ordinary real axis. Thus, this assumption yields a set of B-splines consisting of $\{\psi_{-3},\psi_{-2},..., \psi_{N+2}, \psi_{N+3}\}$ and forming a basis for functions defined over $[a, b]$. The septic B-splines $\psi_m(x)$ are defined by the following relationships [30]:

$$\psi_m(x) = \frac{1}{h^7} \begin{cases} a, & [x_{m-4}, x_{m-3}], \\ a - 8b, & [x_{m-3}, x_{m-2}], \\ a - 8b + 28c, & [x_{m-2}, x_{m-1}], \\ a - 8b + 28c - 56d, & [x_{m-1}, x_m], \\ a - 8b + 28c - 56d, & [x_m, x_{m+1}], \\ e - 8f + 28g, & [x_{m+1}, x_{m+2}], \\ e - 8f, & [x_{m+2}, x_{m+3}], \\ e, & [x_{m+3}, x_{m+4}], \\ 0, & otherwise. \end{cases} \tag{2.3}$$

The septic B-spline functions are employed to overcome the higher order derivatives in the equation and when the bases are chosen at a high degree, generally better numerical results are obtained [31]. To obtain the numerical solutions of the equation (2.1) using septic B- splines as approximation functions by the collocation method, we assume that the approximate solution $u_N(x,t)$ of Eq. (2.1) is

$$u_N(x,t) = \sum_{j=-3}^{N+3} \psi_j(x)\delta_j(t), \tag{2.4}$$

where $\delta_j(t)$ are unknown time dependent quantities to be determined. Using Eqs. (2.3) and (2.4), nodal values of u_m; u'_m; u''_m; u'''_m and u^{iv}_m in terms of the element parameters δ_m in the following form

(2.5)

$$\begin{aligned}
u_N(x_m,t) &= u_m = \delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3},\\
u'_m &= \tfrac{7}{h}(-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1} + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}),\\
u''_m &= \tfrac{42}{h^2}(\delta_{m-3} + 24\delta_{m-2} + 15\delta_{m-1} - 80\delta_m + 15\delta_{m+1} + 24\delta_{m+2} + \delta_{m+3}),\\
u'''_m &= \tfrac{210}{h^3}(-\delta_{m-3} - 8\delta_{m-2} + 19\delta_{m-1} - 19\delta_{m+1} + 8\delta_{m+2} + \delta_{m+3}),\\
u^{iv}_m &= \tfrac{840}{h^4}(\delta_{m-3} - 9\delta_{m-1} + 16\delta_m - 9\delta_{m+1} + \delta_{m+3})
\end{aligned}$$

where the symbols $'$, $''$, $'''$ and ^{iv} indicate first, second, third and fourth differentiation with respect to x, respectively. Putting the node values of u_m and its derivatives given by Eqs. (2.5) into Eq. (2.1) yields the following set of ordinary differential equations of the form:

(2.6)

$$\begin{aligned}
&\left(\dot\delta_{m-3} + 120\dot\delta_{m-2} + 1191\dot\delta_{m-1} + 2416\dot\delta_m + 1191\dot\delta_{m+1} + 120\dot\delta_{m+2} + \dot\delta_{m+3}\right)\\
&+\tfrac{14\gamma z_m}{h}\left(-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1} + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}\right)\\
&+\tfrac{42\sigma}{h^2}(\delta_{m-3} + 24\delta_{m-2} + 15\delta_{m-1} - 80\delta_m + 15\delta_{m+1} + 24\delta_{m+2} + \delta_{m+3})\\
&+\tfrac{42\eta}{h^2}(\dot\delta_{m-3} + 24\dot\delta_{m-2} + 15\dot\delta_{m-1} - 80\dot\delta_m + 15\dot\delta_{m+1} + 24\dot\delta_{m+2} + \dot\delta_{m+3}) = 0,
\end{aligned}$$

where

(2.7)

$$z_m = pu_m^{p-1} = p\left(\delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3}\right)^{p-1},$$

and $\cdot$ states derivative with respect to t. The term pu^{p-1} in non-linear term $pu^{p-1}u_x$, is taken as Eq. (2.6) considering that the quantity pu^{p-1} is locally constant, with the linearization form presented by Rubin and Graves [32]. Both the finite difference approach and the Crank-Nicolson diagrams described below can be applied to the Eq.(2.6):

$$\delta_i = \frac{\delta_i^{n+1} + \delta_i^n}{2}, \quad \dot\delta_i = \frac{\delta_i^{n+1} - \delta_i^n}{\Delta t}. \tag{2.8}$$

Therefore, the above process allows us to derive a recursion relationship between the two time levels based on the parameters δ_m^{n+1}, δ_m^n as follows:

$$\begin{aligned}
&\lambda_1\delta_{m-3}^{n+1} + \lambda_2\delta_{m-2}^{n+1} + +\lambda_3\delta_{m-1}^{n+1} + \lambda_4\delta_m^{n+1} + \lambda_5\delta_{m+1}^{n+1} + \lambda_6\delta_{m+2}^{n+1} + \lambda_7\delta_{m+3}^{n+1}\\
&= \lambda_7\delta_{m-3}^{n} + \lambda_6\delta_{m-2}^{n} + \lambda_5\delta_{m-1}^{n} + \lambda_4\delta_m^{n} + \lambda_3\delta_{m+1}^{n} + \lambda_2\delta_{m+2}^{n} + \lambda_1\delta_{m+3}^{n},
\end{aligned} \tag{2.9}$$

where

$$
\begin{aligned}
\lambda_1 &= [1 - Ez_m + T + M]\,, \\
\lambda_2 &= [120 - 56Ez_m + 24T + 24M]\,, \\
\lambda_3 &= [1191 - 245Ez_m + 15T + 15M]\,, \\
\lambda_4 &= [2416 - 80T - 80M]\,, \\
\lambda_5 &= [1191 + 245Ez_m + 15T + 15M], \\
\lambda_6 &= [120 + 56Ez_m + 24T + 24M]\,, \\
\lambda_7 &= [1 + Ez_m + T + M]\,, \\
E &= \tfrac{7}{h}\gamma\Delta t,\ T = \tfrac{21}{h^2}\sigma\Delta t,\ M = \tfrac{42}{h^2}\eta.
\end{aligned}
\tag{2.10}
$$

To assure a unique solution; we eliminate the parameters $\{\delta_{-3}, \delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}, \delta_{N+3}\}$ from the system (2.9) using the boundary conditions (2.2). In this case, the system, becomes a matrix equation for the $N+1$ unknowns $d = (\delta_0, \delta_1, \ldots, \delta_N)^T$, then last system can be written in the matrix form

$$
Pd^{n+1} = Qd^n. \tag{2.11}
$$

Using the following initial and boundary conditions,

$$
\begin{aligned}
&u_N(x,0) = u\,(x_m,0)\,, && m = 0,1,2,...,N \\
&(u_N)_x\,(a,0) = 0, && (u_N)_x\,(b,0) = 0, \\
&(u_N)_{xx}\,(a,0) = 0, && (u_N)_{xx}\,(b,0) = 0.
\end{aligned}
\tag{2.12}
$$

the values of the initial parameters δ_m^0 at the initial time are obtained. So, initial vector d^0 can be determined in the following system of algebraic equations in matrix form:

$$
Wd^0 = R, \tag{2.13}
$$

where W is

$$
W =
$$

$$
\begin{bmatrix}
1536 & 2712 & 768 & 24 & & & & & & \\
\frac{82731}{81} & \frac{210568.5}{81} & \frac{104796}{81} & \frac{10063.5}{81} & 1 & & & & & \\
\frac{9600}{81} & \frac{96597}{81} & \frac{195768}{81} & \frac{96474}{81} & 120 & 1 & & & & \\
 & & \ddots & & & & & & & \\
 & & & & & & & & & \\
 & & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\
 & & & & 1 & 120 & \frac{96474}{81} & \frac{195768}{81} & \frac{96597}{81} & \frac{9600}{81} \\
 & & & & & 1 & \frac{10063.5}{81} & \frac{104796}{81} & \frac{210568.5}{81} & \frac{82731}{81} \\
 & & & & & & 24 & 768 & 2712 & 1536
\end{bmatrix},
$$

$d^0 = (\delta_0, \delta_1, \delta_2, ..., \delta_{N-2}, \delta_{N-1}, \delta_N)^T$ and $R = (u(x_0,0), u(x_1,0), ..., u(x_{N-1},0), u(x_N,0))^T$.

2.1. **Stability of the scheme.** To implement the von Neumann stability analysis, generalised Oskolkov equation is linearized by considering quantity u^{p-1} in nonlinear term $u^{p-1}u_x$ is locally fixed. Writing Fourier formula

$$
\delta_m^n = \xi^n e^{imkh} \tag{2.14}
$$

where k is the mode number and h is the space step size, into the system of Eq. (2.9) we get the following system:

(2.15)

$$\omega_1\xi^{n+1}e^{i(m-3)kh}+\omega_2\xi^{n+1}e^{i(m-2)kh}+\omega_3\xi^{n+1}e^{i(m-1)kh}+\omega_4\xi^{n+1}e^{imkh}+\omega_5\xi^{n+1}e^{i(m+1)kh}+ \omega_6\xi^{n+1}e^{i(m+2)kh}+\omega_7\xi^{n+1}e^{i(m+3)kh}=\omega_7\xi^{n}e^{i(m-3)kh}+\omega_6\xi^{n}e^{i(m-2)kh}+\omega_5\xi^{n}e^{i(m-1)kh}+ \omega_4\xi^{n}e^{imkh}+\omega_3\xi^{n}e^{i(m+1)kh}+ \omega_2\xi^{n}e^{i(m+2)kh}+\omega_1\xi^{n}e^{i(m+3)kh}$$

If Eq.(2.15) is simplified, we obtain the following growth factor

$$\xi=\frac{a-b-ic}{a+b+ic}, \tag{2.16}$$

which gives

$$\begin{aligned}&a=A(2\cos(3kh)+240\cos(2kh)+2382\cos(kh)+2416)+\\&B(2\cos(3kh)+48\cos(2kh)+30\cos(kh)-80,\\&b=C(2\cos(3kh)+48\cos(2kh)+30\cos(kh)-80,\\&c=D(2\sin(3kh)+112\sin(2kh)+490\sin(kh),\end{aligned}$$

where

$$A=1,\quad B=\frac{42}{h^2}\eta,\ C=\frac{21}{h^2}\sigma\Delta t,\ D=\frac{7}{h}\gamma\Delta tZ_m,\qquad m=0,1,\ldots,N.$$

$|\xi|<1$, is found when we take the modulus of Eq.(2.16). Thus, the linearized algorithm is unconditionally stable.

3. Computer applications and discussions

In this section, the two well-known test problems are investigated namely motion of shock wave and evolution of solitary waves with Gaussian initial condition. The accuracy of the present numerical method is controlled using the following error norms, L_2 and L_∞, respectively [33]:

$$L_2=\left\|u^{exact}-u_N\right\|_2\simeq\sqrt{h\sum_{j=1}^{N}\left|u_j^{exact}-(u_N)_j\right|^2}, \tag{3.1}$$

$$L_\infty=\left\|u^{exact}-u_N\right\|_\infty\simeq\max_j\left|u_j^{exact}-(u_N)_j\right|,\quad j=1,2,...,N. \tag{3.2}$$

In addition to these error norms, the lowest invariant, of which formulae is given below, is computed

$$I=\int_a^b u\,dx\simeq h\sum\nolimits_{j=1}^{N}u_j^n, \tag{3.3}$$

which corresponds to mass.

3.1. **Propagation of shock wave.**

Among others, the first test problem has been considered as the shock wave solution of the Eq. (2.1) which exact solution is given in the form

$$u(x,t)=\left[A+BD(\frac{\mu}{2}-\frac{a\mu}{a+\cosh\left[\mu(x-\kappa t)\right]-\sinh\left[\mu(x-\kappa t)\right]})\right]^{2/(p-1)}, \tag{3.4}$$

where

$$
\begin{aligned}
A &= -\tfrac{1}{2}\frac{\sigma(p-1)\sqrt{\frac{\gamma\sigma\left(\sigma^2-\frac{8\sigma^2(p+1)}{(p+3)^2}\right)\sqrt{2\eta(p+1)}}{\gamma(p+3)}}}{\gamma(p+3)\left(\sigma^2-\frac{8\sigma^2(p+1)}{(p+3)^2}\right)},\\
B &= \frac{1}{\gamma(p+3)\left(\sigma^2-\frac{8\sigma^2(p+1)}{(p+3)^2}\right)},\\
D &= \sigma\sqrt{2\eta(p+1)}\sqrt{\frac{\gamma\sigma\left(\sigma^2-\frac{8\sigma^2(p+1)}{(p+3)^2}\right)\sqrt{2\eta(p+1)}}{\eta(p+3)}},\\
\mu &= \tfrac{\sqrt{8}}{4}\sqrt{\tfrac{p^2-2p+1}{\eta(p+1)}},\\
\kappa &= \tfrac{\sigma\sqrt{2\eta(p+1)}}{\eta(p+3)}
\end{aligned}
$$

and a, κ, μ, p are arbitrary constants [34]. We take Eq. (3.4) as initial condition at $t=0$ of the form

$$
u(x,0)=\left[A+BD\left(\frac{\mu}{2}-\frac{a\mu}{a+\cosh(\mu x)-\sinh(\mu x)}\right)\right]^{2/(p-1)}. \tag{3.5}
$$

We have selected values of parameters as $\sigma=0.1$; $\eta=7$; $h=\Delta t=0.1$; $\gamma=0.5, 0.33$; $p=2,3$ and $a=0.3, 0.5$ through the region $x\ \epsilon\ [-50,50]$ for the computational work.

Case I. For the first case, we choose $p=2$ with the parameters as $\gamma=0.5$, $\sigma=0.1$, $\eta=7$, $h=0.1$, $a=0.3$ and $\Delta t=0.1$ over the interval $-10\leq x\leq 10$. Amplitude of the wave is found as $A=-0.096$. Numerical values of the invariants and error norms have been reported at some predefined times up to $t=5$ in Table (1). It is observed from table that the errors are noticeably small and invariant of solutions are almost unchanged as time grows. We have drawn graphs of the numerical solution of a shock wave in Fig. (1).

TABLE 1. Invariants and error norms for Case I

t	I	L_2-Norm	$L_\infty-Norm$
0.0	17.0705095646	0.0000000000	0.0000000000
1.0	17.4126759927	0.0006036666	0.0001831731
2.0	17.1657255366	0.0010375625	0.0003252532
3.0	16.9222773872	0.0015305140	0.0004885243
4.0	16.6822818739	0.0020426984	0.0006586780
5.0	16.4456900304	0.0025654551	0.0008324359

Case II. For the present case, to provide the simulation through the region $-10\leq x\leq 10$, the parameters $\gamma=0.33$, $\sigma=0.1$, $\eta=7$, $a=0.5$, $h=0.1$ and $\Delta t=0.1$ are used for $p=3$. It is computed that the amplitude of shock wave has -0.116. In Table (2), we list values of the invariant and error norms for different time levels. The table shows that invariant is almost constant as the time increases. The behaviours of solutions have been shown in Fig. (1) for $x\in[-10,10]$ and $0\leq t\leq 5$. Distribution of errors at $t=10$ are depicted in Fig. (2) for $p=2$ and 3, respectively.

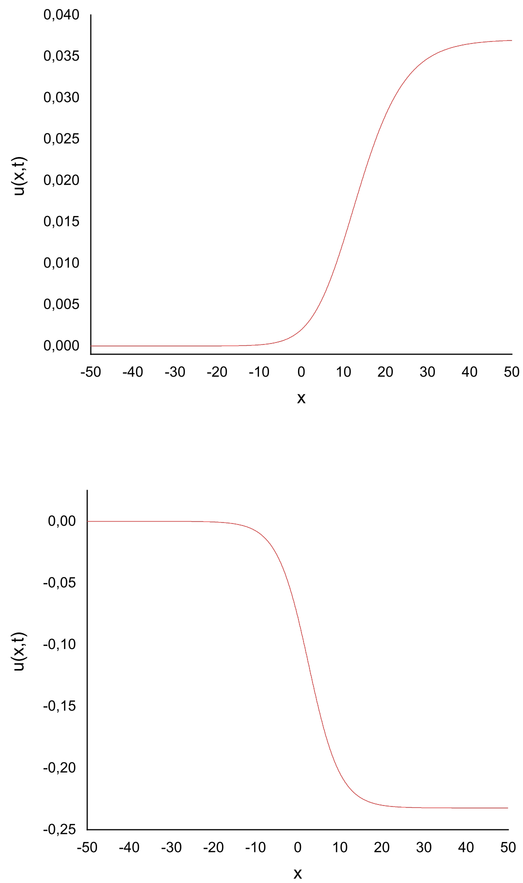

FIGURE 1. Shock wave profiles for a)$p = 2$, $\gamma = 0.5$, $\sigma = 0.1$, $\eta = 7$, $h = 0.1$, $a = 0.3$ and $\Delta t = 0.1$ b)$p = 3$, $\gamma = 0.33$, $\sigma = 0.1$, $\eta = 7$, $h = 0.1$, $a = 0.5$ and $\Delta t = 0.1$.

TABLE 2. Invariants and error norms for Case II

t	I	$L_2 - Norm$	$L_\infty - Norm$
0.0	-1.8145676980	0.0000000000	0.0000000000
1.0	-1.7917168932	0.0251287079	0.0078733542
2.0	-1.7848595276	0.0273149732	0.0083127545
3.0	-1.7782317207	0.0318511153	0.0096966411
4.0	-1.7718229053	0.0380046052	0.0117056417
5.0	-1.7656224690	0.0452176097	0.0141149714

3.2. **Evolution of Waves.** For the last application, Gaussian initial condition to display the evolution of waves are considered.

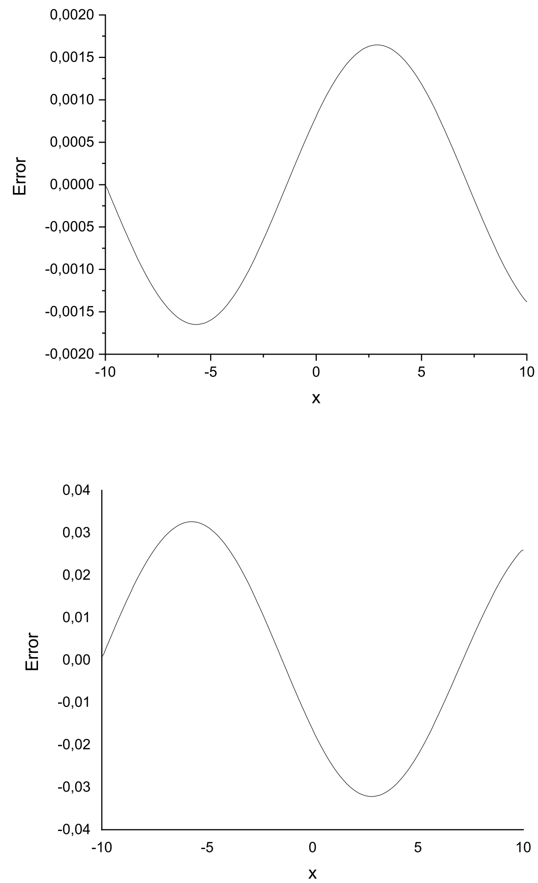

FIGURE 2. Error distributions at $t = 10$ for the parameters a)$p = 2, \gamma = 0.5$, $\sigma = 0.1$, $\eta = 7$ and $a = 0.3$ b)$p = 3$, $\gamma = 0.33$, $\sigma = 0.1$, $\eta = 7$ and $a = 0.5$

3.2.1. *Gaussian Initial Condition.* For the equation under consideration, the evolution of waves is now investigated using the Gaussian initial condition

$$u(x,0) = \exp\left(-x^2\right), \tag{3.6}$$

and boundary condition

$$u(-10,t) = u(10,t) = 0 \quad , \quad t > 0 \tag{3.7}$$

for different values of h and Δt [34]. For the numerical simulations, two sets of parameters $\gamma = 0.5, 0.33$, $\sigma = 0.1$, $\eta = 7$, $a = 0.3$, $h = \Delta t = 0.1$ and $h = \Delta t = 0.01$ in the region $[-10, 10]$ are selected for $p = 2, 3$. Results are reported in Table(3). Evolution of a train of waves with Gaussian initial condition is plotted in Fig. (3) for $p = 2, 3$.

TABLE 3. Invariant and error norms for Gaussian initial condition

t	$p = 2$		p=3	
	h=Δt=0.1	h=Δt=0.01	h=Δt=0.1	h=Δt=0.01
	I	I	I	I
0.0	1.7724537283	1.7724549574	1.7724537283	1.7724549574
1.0	1.8037525365	1.8028647197	1.8030392033	1.8025903847
2.0	1.8344852561	1.8328243023	1.8328702827	1.8320166101
3.0	1.8640699962	1.8617827701	1.8618301065	1.8606202391
4.0	1.8916727204	1.8889600314	1.8897528713	1.8882439336
5.0	1.9162436998	1.9133811566	1.9164297119	1.9146920566

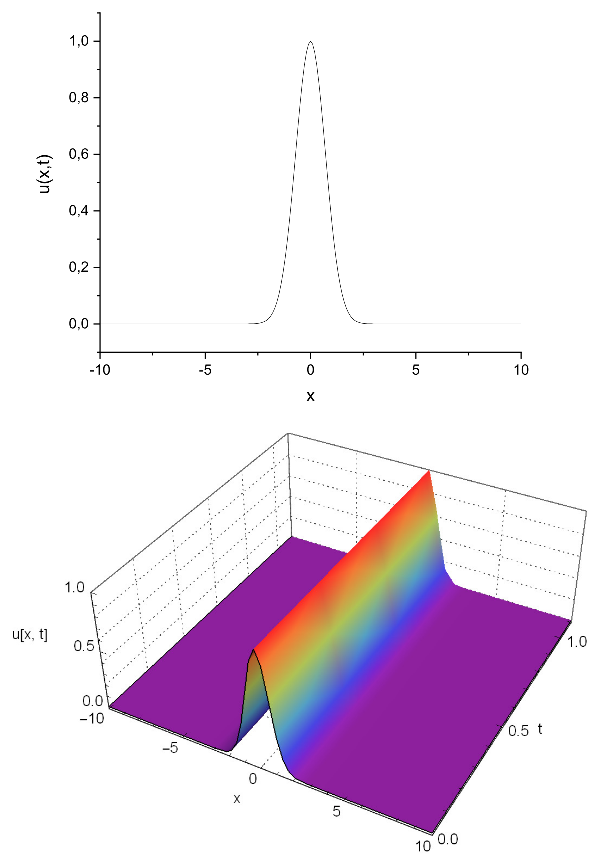

FIGURE 3. Generated waves profiles with a)$p = 2, \gamma = 0.5$, $\sigma = 0.1$, $\eta = 7$ and $a = 0.3$ b)$p = 3$, $\gamma = 0.33$, $\sigma = 0.1$, $\eta = 7$ and $a = 0.5$ with various values of h and Δt.

4. CONCLUSION

In this study, septic B-spline collocation method has been employed to obtain the numerical solutions of the generalised Oskolkov equation. The presented method has been shown to be unconditionally stable. To demonstrate the performance of the algorithm, two test problems including shock wave and evolution of waves with Gaussian initial condition have been examined. For shock wave L_2 and L_∞ error norms and for the Gaussian initial condition the invariant I_1 has been calculated. The obtained numerical results indicate that the error norms are satisfactorily small and the conservation law is marginally constant in all computer program run.

5. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The author(s) declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author(s) declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

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References

[1] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, in: London Mathematical Society Lecture Notes, Cambridge University Press, Cambridge (1991).
[2] T. Ak, S. B. G. Karakoc, A. Biswas, Numerical scheme to dispersive shallow water waves, Journal of Computational and Theoretical Nanoscience, Vol.13,N.10, pp.7084–7092 (2016).
[3] S. K. Bhowmik, S. B. G. Karakoc, Numerical approximation of the generalized regularized long wave equation using Petrov Galerkin finite element method, Numerical Methods for Partial Differential Equation, Vol.35,N.6, pp.2236–2257 (2019).
[4] T. Ak, S. B. G. Karakoc, T. Houria, Numerical simulation for treatment of dispersive shallow water waves with Rosenau KdV equation, The European Physical Journal Plus, Vol.131, pp.1–15 (2016).
[5] S. B. G. Karakoc, T. Geyikli, A. Bashan, A numerical solution of the Modified Regularized Long Wave MRLW equation using quartic B splines, TWMS J. App. Eng. Math., Vol.3,N.2, pp.231-244 (2013).
[6] T. Ak, S. B. G. Karakoc, A numerical technique based on collocation method for solving modified Kawahara equation, Journal of Ocean Engineering and Science, Vol.3,N.1, pp.67–75 (2018).
[7] H. Zeybek and S. B. G. Karakoc, Application of the Collocation Method With B-Splines to the GEW Equation, Electronic Transactions on Numerical Analysis, Vol.46, pp.71–88 (2017).

[8] K. S. Nisar, A. Ciancio, K. K. Ali, M. S. Osman, C. Cattani, D. Baleanu,A. Zafar, M.Raheel, M.Azeem, On beta-time fractional biological population model with abundant solitary wave structures, Alexandria Engineering Journal, Vol.61,N.3, pp.1996-2008 (2022).

[9] M. S. Osman, K. K. Ali, Optical soliton solutions of perturbing time-fractional nonlinear Schrödinger equations, Optik, Vol.209, 164589, (2020).

[10] K. S. Nisar, K. K. Ali, M. Inc, M. S. Mehanna, H. Rezazadeh, L. Akinyemi, New solutions for the generalized resonant nonlinear Schrödinger equation, Results in Physics, Vol.33, 105153, (2022).

[11] R. I. Nuruddeen, K. S.u Aboodh, K. K. Ali, Analytical Investigation of Soliton Solutions to Three Quantum Zakharov-Kuznetsov Equations, Commun. Theor. Phys., Vol.70, pp.405–412 (2018).

[12] K. Raslan, K. K. Ali, Numerical Study of MHD-Duct Flow Using the Two-Dimensional Finite Difference Method, Applied Mathematics and Information Sciences, Vol.14,N.4, pp.693-697 (2020).

[13] N. M. Yagmurlu, A. S. Karakaş, Numerical solutions of the equal width equation by trigonometric cubic B-spline collocation method based on Rubin–Graves type linearization, Numerical Methods for Partial Differential Equations, Vol.36,N.5, pp.1170-1183, (2020).

[14] S. Ozer, N. M. Yagmurlu, Numerical solutions of nonhomogeneous Rosenau type equations by quintic B-spline collocation method, Math Meth Appl Sci., Vol.45, pp.5545-5558 (2022).

[15] A. Bashan, N. M. Yagmurlu, A mixed method approach to the solitary wave, undular bore and boundary-forced solutions of the Regularized Long Wave equation, Computational and Applied Mathematics, Vol.41,N.169, pp.1-20 (2022).

[16] N. M. Yagmurlu, B. Karaagaç, A. Esen, A Lumped Galerkin finite element method for the generalized Hirota-Satsuma coupled KdV and coupled MKdV equations, Tbilisi Math. J. Vol.12,N.3, pp.159-173 (2019).

[17] N. M. Yagmurlu, A. Gagir, Numerical Simulation of Two Dimensional Coupled Burgers Equations by Rubin-Graves Type Linearization, Mathematical Sciences and Applications E-Notes, Vol.9,N.4, pp.158-169 (2021).

[18] E. Pindza, E. Maré, Solving the Generalized Regularized Long Wave Equation Using a Distributed Approximating Functional Method, International Journal of Computational Mathematics, Vol.2014, 12 pages (2014).

[19] M. M. Roshid, T. Bairagi, Harun-Or-Roshid, M.M. Rahman, Lump, interaction of lump and kink and solitonic solution of nonlinear evolution equation which describe incompressible viscoelastic Kelvin–Voigt fluid, Partial Differential Equations in Applied Mathematics, Vol.5 100354, (2022).

[20] A. O. Kondyukov, T. G. Sukacheva, Phase space of the initial-boundary value problem for the Oskolkov system of nonzero order, Computational Mathematics and Mathematical Physics, Vol.55,N.5, pp.823-828 (2015).

[21] M. Alquran,Bright and dark soliton solutions to the Ostrovsky-Benjamin-Bona-Mahony (OS-BBM) equation, J. Math. Comput. Sci., Vol.2,N.1, pp.15-22 (2012).

[22] Khan, K., Akbar, M. A., Alam, M. N.: Traveling wave solutions of the nonlinear Drinfel'd–Sokolov–Wilson equation and modified Benjamin–Bona–Mahony equations, Journal of the Egyptian Mathematical Society, Vol.21,N.3, pp.233-240 (2013).

[23] G. A. Sviridyuk, A. S. Shipilov, On the stability of solutions of the Oskolkov equations on a graph, Differential Equations, Vol.46,N.5, pp.742-747 (2010).

[24] S. Akcagil, T. Aydemir, O. F. Gozukizil, Exact travelling wave solutions of nonlinear pseudoparabolic equations by using the G'/G Expansion Method, New Trends in Mathematical Sciences, Vol.4,N.4, pp.51-66 (2016).

[25] M. M. Roshid, O. H. Roshid, Exact and explicit traveling wave solutions to two nonlinear evolution equations which describe incompressible viscoelastic Kelvin-Voigt fluid, Heliyon, Vol.4, pp.1-21 (2018).

[26] O. F. Gozukizil, S. Akcagil, The tanh-cothmethod for some nonlinear pseudoparabolic equations with exact solutions, Advances in Difference Equations, Vol.143, pp.1-19 (2013).

[27] M. Roshid, H. Bashar, Breather wave and kinky periodic wave solutions of one-dimensional Oskolkov equation. Mathematical Modelling of Engineering Problems, Vol.6,N.3, pp.460-466 (2019).

[28] M. N. Alam and C. Tunc, An analytical method for solving exact solutions of the nonlinear Bogoyavlenskii equation and the nonlinear diffusive predator-prey system, Alexandria Engineering Journal, Vol.55, pp.1855-1865 (2016).

[29] S. B. G. Karakoc, S.K. Bhowmik, D. Y. Sucu, A Novel Scheme Based on Collocation Finite Element Methodto Generalised Oskolkov Equation, journal of Science and Arts, Vol.4,N.47, pp.895–908 (2021).

[30] M. P. Prenter, Splines and variational methods, Courier Corporation, New York, (2008).

[31] A. Esen, A lumped Galerkin method for the numerical solution of the modified equal-width wave equation using quadratic B-splines, International Journal of Computer Mathematics, Vol.83, pp.449-459 (2011).

[32] G. S. Rubin, A. Randolph, J. Graves, A cubic spline approximation for problems in fluid mechanics, NASA STI/Recon Technical Report N., Vol.75, 33345, (1975).

[33] S. B. G. Karakoc, A Quartic Subdomain Finite Element Method for the Modified KdV Equation, Statistic, Optimization and Information Computing, Vol.6,N.4, pp.609–618 (2018).

[34] T. Ak, T. Aydemir, A. Saha, A. Kara, Propagation of nonlinear shock waves for the generalised Oskolkov equation and its dynamic motions in thepresence of an external periodic perturbation, Pramana. J. Phys, Vol.90,N.78, pp.1-16 (2018).

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