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On Roman Domination in Middle and Splitting Graphs

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Abstract

For a graph $G=(V,E)$, a Roman dominating function(RDF) is a function $f:V\rightarrow\{0,1,2\}$ having the property that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$. The weight of an RDF $((w(f))$ is the sum of assignments for all vertices. The minimum weight of an Roman dominating function on graph G is the Roman domination number, denoted by $\gamma_R(G)$. In this paper, we study on this variant of the domination number for middle and splitting graphs of some special graphs.

Keywords: Graph vulnerability, domination, Roman domination.

1. Introduction

Let G be a simple and undirected graph with sets of vertex $V(G)$ and edge $E(G)$. For any vertex $v\in V(G)$, the *open neighbourhood* of v is $N(v)=\{u\in V(G): uv\in E(G)\}$ and *closed neighbourhood* of v is $N[v]=N(v)\cup\{v\}$. The *distance* $d(u,v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. The *eccentricity* of a vertex u, written as $\epsilon(u)$, the maximum value of all $d(u,v)$ values.

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The *radius* of a graph G, written as $rad\,G$, the minimum value of all $\epsilon(u)$ values [West 2001]. *The* number of the neighbour vertices of the vertex v is called *degree* of v and denoted by $deg_G(v)$, the minimum degree is denoted by $\delta=\delta(G)$ and the maximum degree is denoted by $\Delta=\Delta(G)$.

A vertex cover of a graph G is a set $Q\subseteq V(G)$ that contains at least one endpoint of every edge. The vertices in Q cover $E(G)$ [West 2001]. A vertex v is said to be pendant vertex if $deg_G(v)=1$. A vertex u is called support if u is adjacent to a pendant vertex [Harary 1969]. A vertex of degree zero is called an isolated vertex. The largest integer not greater than x is denoted by $\lfloor x\rfloor$ and the least integer not less than x is denoted by $\lceil x\rceil$. Let $v\in S\subseteq V.$ A vertex u is called a *private neighbour* of v with respect to S (denoted by u is an $S-pn$ of v) if $u\in N[v]-N[S-\{v\}]$. An $S-pn$ of v is external if it is a vertex of $V-S$. The set $pn(v,S)=N[v]-N[S-\{v\}]$ of all $S-pn's$ of v is called the *private neighbourhood set of v* with respect to S. The set S is said to be *irredundant* if for every $v\in S, pn(v,S)\neq\emptyset$ [Cockayne et al. 2004].

The graph with n vertices labeled $x_1,x_2,\dots,x_n$ and the edges $x_1x_2,x_2x_3,\dots,x_{n-1}x_n$ is called a path of length $n-1$, denoted by P_n. The cycle of length n, C_n is the graph with n vertices $x_1,x_2,\dots,x_n$ and the edges $x_1x_2,x_2x_3,\dots,x_nx_1$ [Hartsfield and Ringel 1990]. Paths are trees. A tree is a path if and only if its maximum degree is 2. The wheel with $n+1$ vertices, $W_{1,n}$, is the graph that consists of an $n-cycle$ and one additional vertex that is adjacent to all the vertices of the cycle. The complete graph K_n is the graph with n vertices and every vertex is adjacent to every other vertex [Hartsfield and Ringel 1990]. A star is a tree consisting of one vertex adjacent to all the others. The $n+1$ −vertex star is the biclique $S_{1,n}$ [West 2001]. The complement $\bar{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv\in E(\overline{G})$ if and only if $uv \notin E(G)$. A complementary prism of G, denoted by $G\bar{G}$, is the graph obtained by taking a copy of G and a copy of its complement $\bar{G}$ and then joining corresponding vertices by an edge. For arbitrary graphs G and H, we define the *Cartesian product* of G and H to be the graph GxH with vertices $\{(u,v)|u \in G, v \in H\}$. Two vertices (u_1,v_1) and (u_2,v_2) are adjacent in GxH if and only if one of the following conditions is true: $u_1 = u_2$ and v_1 is adjacent to v_2 in H; or $v_1 = v_2$ and u_1 is adjacent to u_2 in G. If $G = P_m$ and $H = P_n$, then the Cartesian product GxH is called the *mxn grid graph* is denoted $G_{m,n}$.

The domination in graph theory has an important role in many fields of study such as optimization, design and analysis of communication networks, social sciences and military surveillance. A *dominating set* in a graph G is a set of vertices of G such that every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [Haynes et all. 1998].

Roman domination is a variant of the domination and was introduced by Cockayne et al. in 2004 [Cockayne et al.2004]. A *Roman dominating function* on a graph $G = (V,E)$ is a function $f: V \to \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The idea is that colours 1 and 2 represent either one or two Roman legions stationed at a given location (vertex v). A nearby location (an adjacent vertex (u) is considered to be *unsecured* if no legions are stationed there (i.e. $f(u) = 0$). An unsecured location (u) can be secured by sending a legion to u from an adjacent location (v). But Emperor Constantine the Great, in the fourth century A.D., decreed that a legion cannot be sent from a location v if doing so leaves that location unsecured (i.e. if $f(v) = 1$). Thus, two legions must be stationed at a location ($f(v) = 2$) before one of the legions can be sent to an adjacent location. Thus, Roman domination appears to be a new variety of both historical and mathematical interest [Stewart 1999]. A function $f = (V_0, V_1, V_2)$ is a *Roman dominating function (RDF)* if $V_2 \succ V_0$, where $\succ$ means that the set V_2 dominates the set V_0, i.e. $V_0 \subseteq N[V_2]$. For a graph $G = (V,E)$, let $f: V \to \{0,1,2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V| \in f(v) = i\}$ and $|V_i| = n_i$, for $i = 0,1,2$. Note that there exists a 1-1 correspondence between the functions $f: V \to \{0,1,2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$. The weight of f is $w_f = f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1$. The minimum weight of w_f for every Roman dominating function f on G is called Roman domination number of G. We denote this number with $\gamma_R(G)$. A Roman dominating function of G with weight $\gamma_R(G)$ is called a γ_R −function of G [Cockayne et al.2004, Mojdeh et al. 2019].

A graph G is a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$. Graphs of the form $G = K_1 + H$, where $\gamma(G) = 1$ and $\gamma_R(G) = 2$ are Roman graphs. Equivalently, any graph G of order n having a vertex of degree $n-1$ is a Roman graph. Complete bipartite graphs are Roman, i.e. $K_{m,n}$ where $\min\{m,n\} \neq 2$, in which case either $\gamma(G) = 1$ and $\gamma_R(G) = 2$, or $\gamma(G) = 2$ and $\gamma_R(G) = 4$. A graph G is Roman if and only if it has a $\gamma_R - function\ f = (V_0, V_1, V_2)$ with $n_1 = |V_1| = 0$.

The following figure shows the $f(v)$ values for each vertex $v \in V(G)$. Since $w(f) = \sum_{v \in V} f(v) = 2n_2 + n_1$, we have $\gamma_R(G) = 3$.

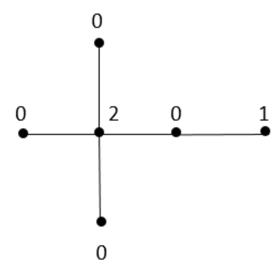

Figure 1.1. The graph G

To date, many articles have been published on the topic of domination as total domination, strong weak domination [Aytaç and Turacı 2015] , exponential domination, semitotal domination [Kartal and Aytaç 2020] etc. and Roman domination. Cockayne et al. introduced the properties of Roman dominating functions. Blidia et al (2020), Chambers et al. (2009), Liu and Chang (2012) researched the bounds on Roman dominating functions and Bermudo et al. (2014) and Xing et al. (2006) discovered the relationships with some domination parameters. Cockayne et al. introduced a linear-time algorithm for computing Roman domination problem on trees [Cockayne et al.2004]. Shirkol et al. (2121) researched the middle Roman domination number of path, cycle, star, double star, wheel, friendship and corona graph. Kazemi (2012) studied on Roman domination for Mycieleski’s structure. McRae showed that the decision problem corresponding to Roman dominating functions (DECIDE-RDF) was NP-complete for bipartite graphs, split graphs and planar graphs. Liedloff et al. (2008) discovered that there were linear-time algorithms for computing the Roman domination number on cographs and interval graphs.

In this paper, our aim is to present the Roman domination number of middle and splitting graphs for $P_n, C_n, S_{1,n}, W_{1,n}$ and K_n. With this work, we have knowledge about the Roman domination number of some special graphs. These results can later be used in larger structures that are their combination. This paper is organized as follows: Section 2 is devoted to some known results about the Roman domination number. Sections 3 and 4 are about the Roman domination number of Middle and Splitting graphs, respectively.

2. Known Results

Theorem 2.1. [Cockayne et al. 2004] For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R −function, and let S be a $\gamma - set$ of G. Then, $V_1 \cup V_2$ is a dominating set of G and $(\emptyset, \emptyset, S)$ is a Roman dominating function. Hence, $\gamma(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_R(G)$. But, $\gamma_R(G) \leq 2|S| = 2\gamma(G)$.

Theorem 2.2. [Cockayne et al. 2004] For any graph G of order $n, \gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K_n}$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R −function. The equality $\gamma(G)$= $\gamma_R(G)$ implies that we have equality in $\gamma(G) \leq |V_1| + |V_2| = |V_1| + 2|V_2| = \gamma_R(G)$. Hence, $|V_2| = 0$, which implies that $V_0 = \emptyset$. Therefore, $\gamma_R(G) = |V_1| = |V| = n$. This implies that $\gamma(G) = n$, which, in turn, implies that $G = \overline{K_n}$.

Theorem 2.3. [Cockayne et al.2004] Let $f = (V_0, V_1, V_2)$ be a γ_R −function. Then

(a) $G[V_1]$, the subgraph induced by V_1 has maximum degree 1.
(b) No edge of G joins V_1 and V_2.
(c) Each vertex of V_0 is adjacent to at most two vertices of V_1.
(d) V_2 is a $\gamma - set$ of $G[V_0 \cup V_2]$.
(e) Let $H = G[V_0 \cup V_2]$. Then each vertex $v \in V_2$ has at least two *H-pn's (*i.e. private neighbours relative to V_2 in the graph *H).*
(f) If v is isolated in $G[V_2]$ and has precisely one external *H-pn*, say $w \epsilon V_0$, then $N(w) \cap V_1 = \emptyset$.
(g) Let k_1 equal the number of non-isolated vertices in $G[V_2]$, let $C = \{v \in V_0 : |N(v) \cap V_2| \geq 2\}$, and let $|C| = c$. Then $n_0 \geq n_2 + k_1 + c$.

Theorem 2.4. [Cockayne et al. 2004] Let $f = (V_0, V_1, V_2)$ be a γ_R-function of an isolate-free graph G, such that n_1 is a minimum. Then,

(a) V_1 is independent and $V_0 \cup V_2$ is a vertex cover.
(b) $V_0 \succ V_1$.
(c) Each vertex of V_0 is adjacent to at most one vertex of V_1, i.e. V_1 is a 2-packing.
(d) Let $v \in G[V_2]$ have exactly two external *H-pn's* w_1 and w_2 in V_0. Then there do not exist vertices $y_1, y_2 \in V_1$ such that (y_1, w_1, v, w_2, y_2) is the vertex sequence of a path P_5.
(e) $n_0 \geq 3n/7$ and this bound is sharp even for trees.

Theorem 2.5. [Mojdeh et al. 2019] If G is a connected graph of order n, then $\gamma_R(G) \leq \frac{4n}{5}$.

Theorem 2.6. [Mojdeh et al. 2019] For any graph G of order n, we have $\gamma_R(G\bar{G}) \leq n + \gamma(G)$.

Theorem 2.7. [Mojdeh et al. 2019] If G is a graph with no isolated vertices, then $\gamma_R(G\bar{G}) \leq \frac{3n}{2}$.

Theorem 2.8. [Cockayne et al. 2004] For any graph G of order n and maximum degree Δ, $\frac{2n}{\Delta+1} \leq \gamma_R(G)$.

Theorem 2.9. [Cockayne et al. 2004] For a graph G on n vertices, $\gamma_R(G) \leq n\frac{2+\ln((1+\delta(G)/2)}{1+\delta(G)}$.

Theorem 2.10. [Cockayne et al. 2004] For the classes of paths P_n and cycles C_n, $\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.

Theorem 2.11. [Cockayne et al. 2004] Let $G = K_{m_1,\dots,m_n}$ be the complete *n-partite* graph with $m_1 \leq m_2 \leq \dots \leq m_n$.

(a) If $m_1 \geq 3$ then $\gamma_R(G) = 4$.
(b) If $m_1 = 2$ then $\gamma_R(G) = 3$.
(c) If $m_1 = 1$ then $\gamma_R(G) = 2$.

Theorem 2.12. [Cockayne et al. 2004] If G is a graph of order n which contains a vertex of degree $n - 1$, then $\gamma(G) = 1$ and $\gamma_R(G) = 2$.

Theorem 2.13. [Cockayne et al. 2004] For the 2 x n grid graph $G_{2,n}, \gamma_R(G_{2,n}) = n + 1$.

Theorem 2.14. [Cockayne et al. 2004] If G is any isolate-free graph of order n, then $\gamma_R(G) = n$ if and only if n is even and $G = \binom{n}{2} K_2$.

Theorem 2.15. [Cockayne et al. 2004] If G is a connected graph of order n, then $\gamma_R(G) = \gamma(G) + 1$ if and only if there is a vertex $v\epsilon V$ of degree $n - \gamma(G)$.

Theorem 2.16. [Cockayne et al. 2004] If T is a tree on two or more vertices, then $\gamma_R(T) = \gamma(T) + 1$ if and only if T is a wounded spider.

Theorem 2.17. [Cockayne et al. 2004] If G is a connected graph of order n, then $\gamma_R(G) = \gamma(G) + 2$ if and only if:

(a) G does not have a vertex of degree $n - \gamma(G)$.
(b) either G has a vertex of degree $n - \gamma(G) - 1$ or G has two vertices v and w such that $|N[v] \cup N[w]| = n - \gamma(G) + 2$.

Theorem 2.18. [Cockayne et al. 2004] If G is a connected graph and $\gamma_R(G) = \gamma(G) + 2$, then $2 \leq rad(G) \leq 4$ and $3 \leq diam(G) \leq 8$.

Theorem 2.19. [Cockayne et al. 2004] If T is a tree of order $n \geq 2$, then $\gamma_R(G) = \gamma(G) + 2$ if and only if either (i) T is a healthy spider or (ii) T is a pair of wounded spiders T_1 and T_2, with a single edge joining $v \in V(T_1)$ and $w \in V(T_2)$, subject to the following conditions:

(1) if either tree is a P_2, then neither vertex in P_2 are joined to the head vertex of the other tree.
(2) v and w are not both foot vertices.

3. *Roman Domination for Middle Graphs*

In this section, we gived some results about the Roman domination number of middle graphs for path graph P_n, cycle graph C_n, star graph $S_{1,n}$, wheel graph $W_{1,n}$ and complete graph K_n.

Definition 3.1. [Ramakrishnan 1988] The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$, and two vertices of $M(G)$ are adjacent if either they are adjacent edges of G or one is a vertex and the other is an edge of G, incident with it.

The star graph $S_{1,4}$ and the middle graph $M(S_{1,4})$ can be depicted as in the following figures:

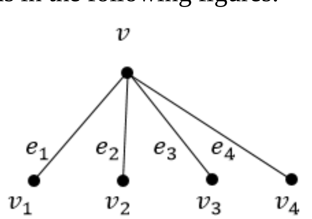

Figure 3.1. Star graph $S_{1,4}$

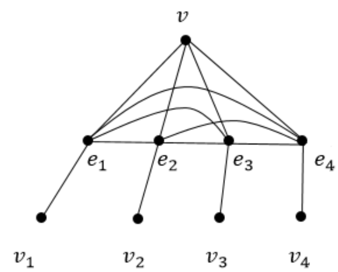

Figure 3.2. Middle graph $M(S_{1,4})$

Theorem 3.1. **[**Shirkol et al. 2121**]** Let $M(P_n)$ be a middle graph of the path graph of order $2n - 1$ for $n \geq 2$. Then,

$$\gamma_R\big(M(P_n)\big) = n + 1.$$

Theorem 3.2. **[**Shirkol et al. 2121**]** Let $M(C_n)$ be a middle graph of the cycle graph of order $2n$. Then,

$$\gamma_R(M(C_n) = n.$$

Theorem 3.3. **[**Shirkol et al. 2121**]** Let $M(W_{1,n})$ be a middle graph of the wheel graph of order $3n + 1$. Then,

$$\gamma_R(M\big(W_{1,n}\big) = n - 1.$$

Theorem 3.4. **[**Shirkol et al. 2121**]** Let $M(S_{1,n})$ be a middle graph of the star graph of order $2n + 1$. Then,

$$\gamma_R(M\big(S_{1,n}\big) = n + 1.$$

Proof. The vertex v and vertices $e_1, e_2, \dots, e_n$ are make up the complete graph K_{n+1}. We know $\gamma_R(K_n) = 2$. Since the remaining vertices $v_1, v_2, \dots, v_n$ are independent, $V_2 \succ V_0$ and V_1 and V_2 have no edges between them, $e_i \in V_2$, where each i is related to exactly one $1,2, \dots, n$. So, $|V_2| = 1$ and $v_i \in V_0$ $i = \overline{1,n}$ due to $d(v_i, e_i) = 1$. Hence, $|V_1| = n - 1$ and we have $\gamma_R(M\big(S_{1,n}\big) = f(V) = \sum_{v\in V} f(v) = 2n_2 + n_1 = 2 + n - 1 = n + 1$.

Theorem 3.5. Let $M(K_n)$ be a middle graph of the complete graph of order $n + \binom{n}{2}$. Then,

$$\gamma_R(M(K_n) = n.$$

Proof. We can split the vertex set of the $M(K_n)$ as $V(M(P_n) = V\big(V(K_n)\big) \cup V(E(K_n)), v_i \in V(K_n), i = \overline{1,n}$ and $e_{i,j} \in V\big(E(K_n)\big), i = \overline{1,n-1}, j = \overline{\imath+1,n}$. $\big|V\big(V(K_n)\big)\big| = n$ and $|V(E(K_n))\,| = \binom{n}{2}$. Also, $\deg\big(e_{i,j}\big) = 2n - 4$ for $\forall e_{i,j} \in V(E(K_n))$ and $\deg(v_i) = n - 1$ for $\forall v_i \in V(K_n)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R - function$ and S be a $\gamma - set$ of $M(K_n)$. Then, $V_1 \cup V_2$ is a dominating set of $M(K_n)$

and $(\emptyset, \emptyset, S)$ is a Roman dominating function. We have two cases:

Case 1. $n \equiv 0(mod\ 2)$

In this case, $\frac{n}{2}$ vertices $e_{i,i+1}, i = 1,3,5, \ldots, n-1$ make up V_2. So, all vertices are dominated. $\gamma_R(M(K_n) = 2\gamma(M(K_n))$. $f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1 = 2\left(\frac{n}{2}\right) = n$.

Case 2. $n \equiv 1(mod\ 2)$

In this case, $\left\lfloor\frac{n}{2}\right\rfloor$ vertices $e_{i,i+1}, i = 1,3,5, \ldots, n-2$ make up V_2. So, $\forall e_{i,j}$ and $v_1, v_2, \ldots, v_{n-1}$ are dominated. The vertex v_n make up V_1. Hence, $f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1 = 2\left\lfloor\frac{n}{2}\right\rfloor + 1 = n$.

So, we have $\gamma_R(M(K_n) = n$ from Case 1 and Case 2.

4. Roman Domination for Splitting Graphs

In this section, we investigated the Roman domination number of splitting graphs for path graph P_n, cycle graph C_n, star graph $S_{1,n}$, wheel graph $W_{1,n}$ and complete graph K_n.

Definition 4.1.**[**Sampathkumar, E. and Walikar 1980**]** *Splitting graph* $S(G)$ of a graph G is obtained by taking a copy of G, for each vertex v of a graph G, take a new vertex v' and join v' to all the vertices of G adjacent to v.

The path graph P_5 and the splitting graph $S(P_5)$ can be depicted as in the following figures:

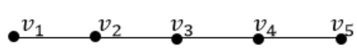

Figure 4.1. Path graph P_5

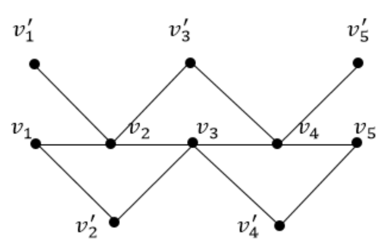

Figure 4.2. Splitting graph $S(P_5)$

Theorem 4.1. Let $S(P_n)$ be a splitting graph of the path graph of order $2n$. Then,

$$\gamma_R(S(P_n)) = \begin{cases} n, & n \equiv 0(mod\ 3) \\ n+1, & n \equiv 1,2(mod\ 3) \end{cases}$$

Proof. Let $f = (V_0, V_1, V_2)$ be the RDF of $S(P_n)$. The value of the function for the corresponding vertices v_i', $i = \overline{1,n}$ is $f(v_i') = 0$ if $f(v_i) = 0$; $f(v_i') = 1$ if $f(v_i) = 1\ or\ f(v_i) = 2$, respectively. We know $\gamma_R(P_n) = \left\lceil\frac{2n}{3}\right\rceil$ from the Theorem 1.9. So, we have three cases:

Case 1. $n \equiv 0(mod\ 3)$

In this case, $f(v_i) = 0, f(v_{i+1}) = 2, f(v_{i+2}) = 0$ and $f(v_i') = 0, f(v_{i+1}') = 1, f(v_{i+2}') = 0$ for $i = 1,4,7, \ldots, n-2$. Hence, $f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1 = \frac{2n}{3} + \frac{n}{3} = n$.

Case 2. $n \equiv 1(mod\ 3)$

In this case, $f(v_i) = 0, f(v_{i+1}) = 2, f(v_{i+2}) = 0$ and $f(v_i') = 0, f(v_{i+1}') = 1, f(v_{i+2}') = 0$ for $i = 1,4,7, \ldots, n-3$. Also, $f(v_n) = f(v_n') = 1$. Hence, $f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1 = \left\lceil\frac{2n}{3}\right\rceil + \frac{n-1}{3} + 1 = \left\lceil\frac{2n}{3}\right\rceil + \frac{n+2}{3} = n+1$.

Case 3. $n \equiv 2(mod\ 3)$

In this case, $f(v_i) = 0, f(v_{i+1}) = 2, f(v_{i+2}) = 0$ and $f(v_i') = 0, f(v_{i+1}') = 1, f(v_{i+2}') = 0$ for $i = 1,4,7, \ldots, n-4$. Also, $f(v_n) = 2$ and $f(v_n') = 1$. Hence, $f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1 = \left\lceil\frac{2n}{3}\right\rceil + \frac{n-2}{3} + 1 = \left\lceil\frac{2n}{3}\right\rceil + \frac{n+1}{3} = n+1$.

So, from Case 1, Case 2 and Case 3, we have

$$\gamma_R(S(P_n)) = \begin{cases} n, & n \equiv 0(mod\ 3) \\ n+1, & n \equiv 1,2(mod\ 3) \end{cases}$$

Corollary 4.1. Let P_n be a path graph of order n and $S(P_n)$ be a splitting graph of the path graph of order $2n$. Then,

$$\gamma_R(S(P_n)) = \begin{cases} \gamma_R(P_n) + \dfrac{n}{3}, & n \equiv 0(mod\ 3) \\ \gamma_R(P_n) + \dfrac{n+2}{3}, & n \equiv 1(mod\ 3) \\ \gamma_R(P_n) + \dfrac{n+1}{3}, & n \equiv 2(mod\ 3) \end{cases}$$

Theorem 4.2. Let $S(C_n)$ be a splitting graph of the cycle graph of order $2n$. Then,

$$\gamma_R(S(C_n)) = \begin{cases} n, & n \equiv 0(mod\ 3) \\ n+1, & n \equiv 1,2(mod\ 3) \end{cases}$$

Proof. The proof is similar to the proof of the Theorem 4.1.

Theorem 4.3. Let $S(S_{1,n})$ of order $2n+2$, $S(W_{1,n})$ of order $2n+2$ be splitting graphs of a star graph and a wheel graph respectively and $S(K_n)$ of order $2n$ be a splitting graph of a complete graph. Then, $\gamma_R\left(S(S_{1,n})\right) = \gamma_R(S_{1,n}) + 1, \gamma_R\left(S(W_{1,n})\right) = \gamma_R(W_{1,n}) + 1$ and $\gamma_R(S(K_n)) = \gamma_R(K_n) + 1$.

Proof. We denote $S_{1,n}$, $W_{1,n}$ and K_n by G. $\exists u \in G|$ $\deg(u) = n$ in G. Let $f = (V_0, V_1, V_2)$ be the RDF of $S(G)$. We denote the corresponding vertices of v_i by $v_i', i = \overline{1,n}$ in $S(G)$. If $f(u) = 2$, then $f(v_i) = f(v_i') = 0$, since $d(u) = d(v_i) = d(v_i') = 1$. So, $\{u'\} = V_1$. We have $\gamma_R(G) = 2\gamma(G)$ and $\gamma_R(S(G)) = \gamma_R(G) + 1$.

Corollary 4.2. Let G be a graph of order n. If $\exists v \in V(G)|\deg(v) = n - 1$, then $\gamma_R(S(G)) = \gamma_R(G) + 1$.

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