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NOTES ON SURFACES WITH CONSTANT GAUSS CURVATURE ALONG A CURVE IN THE LIE GROUP

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Abstract

Original scientific paper

In this paper, the linear combination of the Lie Frenet frame of a given curve is used to create a parametric surface family. Using the coefficients of the surface's first and second fundamental forms, the Gauss curvature of this parametric surface is determined. Also, sufficient conditions for the surface when its Gauss curvature is constant along the given curve are derived. Moreover, sufficient conditions are found when the finding surface is a ruled surface, as a member of the family. Finally, a few examples to support our theory are created in the Lie group.

Keywords: Gauss curvature , Lie group, surfaces.

LİE GRUBUNDA BİR EĞRİ BOYUNCA SABİT GAUSS EĞRİLİKLİ YÜZEYLER ÜZERİNE NOTLAR

Özet

Orijinal bilimsel makale

Bu çalışmada, parametrik bir yüzey ailesi oluşturmak için verilen bir eğrinin Lie Frenet çatısının lineer kombinasyonu kullanılmıştır. Yüzeyin birinci ve ikinci temel form katsayıları kullanılarak bu parametrik yüzeyin Gauss eğriliği hesaplanmıştır. Ayrıca verilen eğri boyunca yüzeyin Gauss eğriliğinin sabit olması durumunda yeterli koşullar üretilmiştir. Dahası, ortaya çıkan yüzeyin, ailenin bir üyesi olarak açılabilir bir regle yüzey olduğu durumlardaki koşullar üretilmiştir. Son olarak, Lie Group'ta teorimizi destekleyecek bazı örnekler oluşturulmuştur.

Anahtar Kelimeler: *Gauss eğriliği, Lie grup, yüzeyler.*

1 Introduction

Differential geometry of curves and surfaces is an important and widely researched area in such fields as engineering and physics. Calculating the Gaussian curvature of a two-dimensional surface becomes necessary in many propagation problems in engineering [1]. In engineering, there exist correlations between a structure's stability and its Gaussian curvature at every point on its surface. Consequently, a useful formula for calculating the curvature is occasionally needed by engineers who work on the design of structures. A few other technical applications, such as computer vision and engineering, occasionally confront the tough task of determining a surface's Gaussian curvature in order to obtain three-dimensional depth data or range.

In general, curves in differential geometry, most studies on surfaces, and special curves on surfaces are examined. Moreover, the concept of the construction of a surface is an important issue in differential geometry. Until today, numerous studies have been focused on constructing surfaces with a common special curve such as a geodesic, an asymptotic, or a line of curvature [2-7]. To create this, they used the curve and its Frenet frame on the surface. On the other hand, the concept of curvature is also a widely used concept in differential geometry. Gauss's work was a start in this regard. If we express the Gauss curvature K as the product of the principal curvature, κ_1 and κ_2, then $K = 0$ if one of the principal curvatures is zero. It is also important to understand that Gauss curvature is an intrinsic property of surfaces. This leads to Minding's Theorem which states that two surfaces of the same constant Gauss curvature K are locally isometric. The results of Minding's Theorem lead to the fact that surfaces of positive constant Gauss curvature $K > 0$ are locally isometric to a sphere, and

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surfaces of negative constant Gauss curvature $K<0$ are locally isometric to a pseudosphere and surfaces of zero Gauss curvature $K=0$ are locally isometric to a plane [8,9]. These surfaces are known as developable surfaces because they can be created from a flat sheet of material without being stretched or torn as a result. Additionally, applications for real developable surfaces are widespread in the fields of engineering and manufacturing. For example, an aircraft designer utilizes them to create the wings of an airplane.

On the other hand, recently, Bayram [10,11] introduced the theory of obtaining surfaces with constant mean and Gauss curvature through a given curve in the Minkowski and Euclidean 3-spaces, respectively.

The major goal of this research is to investigate how to derive sufficient conditions for the parametric surface when its Gauss curvature is constant along the given curve and illustrate some examples to present our theory.

2 Prelimınaries

In this section, we will give a summary of the theory of the Lie Group. For more information, we may refer to [12-14].

The Frenet formulas for a unit speed curve $\alpha(s)$ in the Lie group are expressed as

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & (\tau-\tau_G) \\ 0 & -(\tau-\tau_G) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},$$

where κ and τ are the curvature functions of $\alpha(s)$ and τ_G is called Lie torsion which is defined by $\tau_G=\frac{1}{2}<T,[N,B]>$.

Definition 2.1: $h=\frac{\tau-\tau_G}{\kappa}$ is the harmonic curvature function [13].

Definition 2.3: Let $P(s,t)$ be a surface in the 3-dimesional Lie Group, then the Gauss curvature of this surface, such that the unit surface normal η, is defined by

$$K=\frac{ln-m^2}{EG-F^2}, \tag{1}$$

where $l=\left\langle\eta, \frac{\partial^2 P}{\partial s^2}\right\rangle$, $m=\left\langle\eta, \frac{\partial^2 P}{\partial t\partial s}\right\rangle$, $n=\left\langle\eta, \frac{\partial^2 P}{\partial t^2}\right\rangle$, $E=\left\langle\frac{\partial P}{\partial s}, \frac{\partial P}{\partial s}\right\rangle$, $F=\left\langle\frac{\partial P}{\partial s}, \frac{\partial P}{\partial t}\right\rangle$, $G=\left\langle\frac{\partial P}{\partial t}, \frac{\partial P}{\partial t}\right\rangle$ [9].

3 Surfaces with Constant Gauss Curvature

In this section, we introduce surfaces whose has the constant Gauss curvature in the three dimensional Lie group. Furthermore, some examples of the surface are obtained in the study and are given visualized.

Let $\alpha(s)$ be an arc length parametrized curve on a surface $P(s,t)$ in G. Then the parametric form of a surface $P(s,t)$ along given curve $\alpha(s)$ and its Frenet frame is defined as follows

$$P(s,t)=\alpha(s)+x(s,t)T(s)+y(s,t)N(s)+z(s,t)B(s) \tag{2}$$

$$L_1\le s\le L_2 \quad \text{and} \quad T_1\le t\le T_2,$$

where $x(s,t)$, $y(s,t)$, $z(s,t)$ are all C^1 functions. These functions are called the marching-scale functions.
According to the definition of an isoparametric curve on a surface, there exists a parameter $t_0\in[T_1,T_2]$ such that $\alpha(s)=P(s,t_0)$, $L_1\le s\le L_2$, that is,

$$x(s,t_0)=y(s,t_0)=z(s,t_0)=0, \tag{3}$$

$$L_1\le s\le L_2 \quad \text{ve} \quad T_1\le t_0\le T_2$$

Now we calculate the Gauss curvature of $P(s,t)$ given by (2) using the equation (1).

First, we will find the unit surface normal using the following equation

$$\eta(s,t)=\frac{\frac{\partial P(s,t)}{\partial s}\times\frac{\partial P(s,t)}{\partial t}}{\left\|\frac{\partial P(s,t)}{\partial s}\times\frac{\partial P(s,t)}{\partial t}\right\|}.$$

So we get

$$\frac{\partial P}{\partial s}=\left(1+\frac{\partial x}{\partial s}(s,t)-y(s,t)\kappa(s)\right)T(s)+\left(x(s,t)\kappa(s)+\frac{\partial y}{\partial s}(s,t)-\left(\tau(s)-\tau_G(s)\right)z(s,t)\right)N(s)+\left(\left(\tau(s)-\tau_G(s)\right)y(s,t)+\frac{\partial z}{\partial s}(s,t)\right)B(s),$$

$$\frac{\partial P}{\partial t}=\frac{\partial x}{\partial t}(s,t)T(s)+\frac{\partial y}{\partial t}(s,t)N(s)+\frac{\partial z}{\partial t}(s,t)B(s),$$

$$\left\|\frac{\partial P(s,t)}{\partial s}\times\frac{\partial P(s,t)}{\partial t}\right\|=\left(\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\right)^{\frac{1}{2}}.$$

We obtain

$$\eta(s,t_0)=\left(\frac{\partial y}{\partial t}B(s)-\frac{\partial z}{\partial t}N(s)\right)\left(\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\right)^{-\frac{1}{2}}(s,t_0).$$

To calculate Gauss curvature, we easily get

$$E(s,t_0)=\left\langle\frac{\partial P}{\partial s}(s,t_0), \frac{\partial P}{\partial s}(s,t_0)\right\rangle=\left\|\frac{\partial P}{\partial s}(s,t_0)\right\|^2=\|T(s)\|^2=1,$$

$$F(s,t_0)=\left\langle\frac{\partial P}{\partial s}(s,t_0), \frac{\partial P}{\partial t}(s,t_0)\right\rangle=\frac{\partial x}{\partial t}(s,t_0)$$

$$G(s,t_0)=\left\langle\frac{\partial P}{\partial t}(s,t_0), \frac{\partial P}{\partial t}(s,t_0)\right\rangle=\left(\frac{\partial x}{\partial t}(s,t_0)\right)^2+\left(\frac{\partial y}{\partial t}(s,t_0)\right)^2+\left(\frac{\partial z}{\partial t}(s,t_0)\right)^2,$$

$$l(s,t_0)=\left\langle\eta, \frac{\partial^2 P}{\partial s\partial s}\right\rangle=\frac{-\kappa(s)\frac{\partial z}{\partial t}}{\sqrt{\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2}}(s,t_0),$$

$$m(s,t_0)=\left\langle\eta, \frac{\partial^2 P}{\partial t\partial s}\right\rangle=\frac{\left(\frac{\partial z}{\partial t}\right)^2\left(\tau(s)-\tau_G(s)\right)-\frac{\partial z\partial x}{\partial t\partial t}\kappa(s)-\frac{\partial^2 y}{\partial t\partial s}\frac{\partial z}{\partial t}+\left(\frac{\partial y}{\partial t}\right)^2\left(\tau(s)-\tau_G(s)\right)+\frac{\partial^2 z}{\partial t\partial s}\frac{\partial y}{\partial t}}{\sqrt{\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2}}(s,t_0),$$

and

$$n(s,t_0)=\left\langle \eta, \frac{\partial^2 P}{\partial t \partial t}\right\rangle = \frac{\frac{\partial^2 z}{\partial t \partial t}\frac{\partial y}{\partial t}-\frac{\partial^2 y}{\partial t \partial t}\frac{\partial z}{\partial t}}{\sqrt{\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2}}(s,t_0).$$

Then substituting these values into equation (1), Gauss curvature of the given surface is obtained as

$$K = \frac{A-B}{\left(\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\right)^2}(s,t_0), \tag{4}$$

where

$$A=\kappa(s)\left(\frac{\partial^2 y}{\partial t^2}\left(\frac{\partial z}{\partial t}\right)^2-\frac{\partial z}{\partial t}\frac{\partial y}{\partial t}\frac{\partial^2 z}{\partial t^2}\right),$$

$$B=\left(\left(\left(\frac{\partial z}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2\right)(h(s)\kappa(s))-\frac{\partial z}{\partial t}\frac{\partial x}{\partial t}\kappa(s)-\frac{\partial^2 y}{\partial t \partial s}\frac{\partial z}{\partial t}+\frac{\partial^2 z}{\partial t \partial s}\frac{\partial y}{\partial t}\right)^2 \text{ and } h(s)=\frac{\tau(s)-\tau_G(s)}{2}.$$

In light of these results, we can state the following two theorems when the Gauss curvature of the surface is constant:

Theorem 3.1: Let $P(s,t)$ be the surface given by Equation (2). If the Gauss curvature of $P(s,t)$ in equation (4) along the isoparametric curve $\alpha(s)$ is a constant, then one of the following two conditions is satisfied:

$$1.\begin{cases} x(s,t_0)=y(s,t_0)=z(s,t_0)=\frac{\partial z}{\partial t}(s,t_0)\equiv 0, \\ \frac{\partial y}{\partial t}(s,t_0)\neq 0,\ h(s)=const.,\ \kappa(s)=const. \end{cases}$$

$$2.\begin{cases} x(s,t_0)=y(s,t_0)=z(s,t_0)=\frac{\partial x}{\partial t}(s,t_0)\equiv 0, \\ \frac{\partial z}{\partial t}(s,t_0)\neq 0\equiv\frac{\partial y}{\partial t}(s,t_0)\equiv\frac{\partial^2 y}{\partial t^2}(s,t_0), \\ h(s)=const.,\ \kappa(s)=const. \end{cases}$$

Theorem 3.2: Let $P(s,t)$ be the surface given by Equation (2). If the Gauss curvature of the ruled surfaces $P(s,t)$ along the isoparametric curve $\alpha(s)$ is a constant, then the following condition is satisfied:

$$\begin{cases} x(s,t_0)=y(s,t_0)=z(s,t_0)=t-t_0, \\ 2h(s)\kappa(s)-\kappa(s)=const. \end{cases}$$

Corollary 3.3: If we take $2h(s)\kappa(s)-\kappa(s)=0$, then $K=0$. So $P(s,t)$ surfaces become developable ruled surfaces.

Example 3.4: Suppose that $\alpha(s)$ given by

$\alpha(s)=(\sin(s),\cos(s),0)$.

By straightforward calculations, we get the T, N *and* B vectors in the three dimensional Lie Group as follows

$T(s)=(\cos(s),\sin(s),0)$,

$N(s)=(-\sin(s),\cos(s),0)$,

$B(s)=(0,0,-1)$,

where $\kappa=1, \tau=0, \tau_G=\frac{1}{2}$ and $h(s)=-\frac{1}{2}$.

Case 1: Choosing marching-scale functions as $x(s,t)=st$, $y(s,t)=t$, $z(s,t)=st^2$ $\left(\frac{\partial y}{\partial t}\neq 0, \frac{\partial z}{\partial t}=0\right)$ and $t_0=0$.

Then, the first condition of Theorem 3.1 is satisfied and the surface $P_1(s,t)$ given by (2) in the Lie group G is obtained as

$P_1(s,t)=((1-t)\sin(s)+st\cos(s),(1-t)\cos(s)-st\sin(s),-st^2)$.

In Figure 1, the surface $P_1(s,t)$ with constant Gauss curvature $K(s,t_0)=-\frac{1}{4}$ along the curve $\alpha(s)$ can be seen as follows

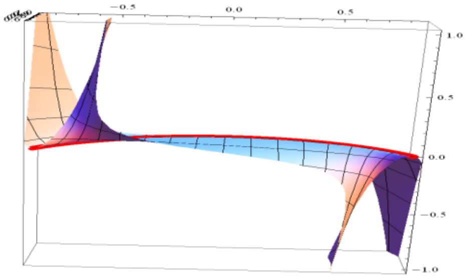

Figure 1. The surface $P_1(s,t)$ with constant Gauss curvature along the curve $\alpha(s)$.

Case 2: Choosing $x(s,t)=0$, $y(s,t)=sin(t)$, $z(s,t)=t^3$ $\left(\frac{\partial y}{\partial t}\neq 0, \frac{\partial z}{\partial t}=0\right)$ and $t_0=0$. Then, the first condition of Theorem 3.1 is satisfied and the surface $P_2(s,t)$ given by (2) in the Lie group G is obtained as

$P_2(s,t)=((1-sin(t))\sin(s),(1-sin(t))\cos(s),-t^3)$

In Figure 2, the surface $P_2(s,t)$ with $K(s,t_0)=-\frac{1}{4}$ along the curve $\alpha(s)$ can be seen as follows

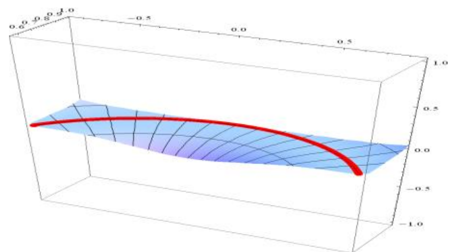

Figure 2. The surface $P_2(s,t)$ with constant Gauss curvature along the curve $\alpha(s)$.

Case 3: Choosing $x(s,t)=0$, $y(s,t)=st^3$, $z(s,t)=ssin(t)$ and $t_0=0$. Then, the second condition of Theorem 3.1 is satisfied and the surface $P_3(s,t)$ given by (2) in the Lie group G is obtained as

$P_3(s,t)=((1-st^3)\sin(s),(1-st^3)\cos(s),-ssin(t))$

In Figure 3, the surface $P_3(s,t)$ with $K(s,t_0)=-\frac{1}{4}$ along the curve $\alpha(s)$ can be seen as follows

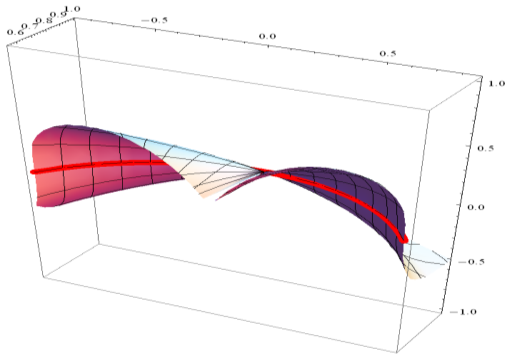

Figure 3: The surface $P_3(s,t)$ with constant Gauss curvature along the curve $\alpha(s)$.

Case 4: Choosing $x(s,t)=t^2, y(s,t)=0, z(s,t)=st$ and $t_0=0$. Then, the second condition of Theorem 3.1 is satisfied and the surface $P_4(s,t)$ given by (2) in the Lie group G is obtained as

$$P_4(s,t)=(\sin(s)-t^2\cos(s), \cos(s)-t^2\sin(s), -st)$$

In Figure 4, the surface $P_4(s,t)$ with $K(s,t_0)=-\frac{1}{4}$ along the curve $\alpha(s)$ can be seen as follows

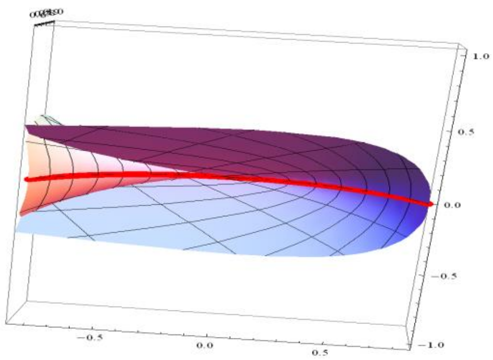

Figure 4: The surface $P_4(s,t)$ with constant Gauss curvature along the curve $\alpha(s)$.

Case 5: Choosing marching-scale functions as $x(s,t)$= $y(s,t)$=$z(s,t)$= $t-t_0$ and $t_0=0$. Then, Theorem 3.2 is satisfied and the ruled surface $P_5(s,t)$ given by (2) in the Lie group G is obtained as

$$P_5(s,t)=((1-t)\sin(s)+t\cos(s), (1-t)\cos(s)-t\sin(s), -t)$$

In Figure 5, the surface $P_5(s,t)$ with $K(s,t_0)=-1$ along the curve $\alpha(s)$ can be seen as follows

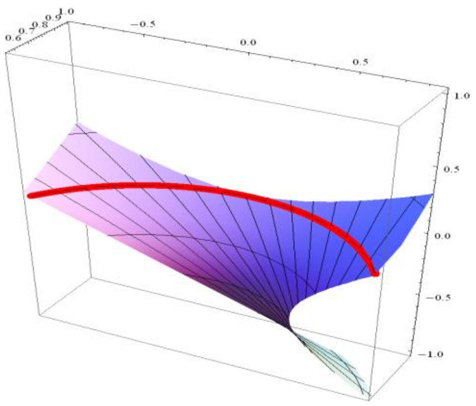

Figure 5. The surface $P_5(s,t)$ with constant Gauss curvature along the curve $\alpha(s)$.

4 Conclusion

In this paper, we defined sufficient conditions to find the surfaces which have constant Gauss curvature along a given curve in the Lie group. Moreover, we derived sufficient conditions when the finding surface is a ruled surface, which is commonly utilized in mechanical engineering. Finally, using the same base curve $\alpha(s)$ and various marching-scale functions, we created the surfaces $P_i(s,t), for\ 1\le i\le 5$ with constant Gauss curvature, and illustrated them in Figures 1-5 for the parameters $-1\le s\le 1$ and $-2\le t\le 2$, respectively.

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Declaration

Ethics committee approval is not required.

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