Analytical Study of Fractional Reaction-Diffusion Brusselator System

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Keywords Abstract


In this paper, a time fractional reaction-diffusion Brusselator system with the Caputo type fractional derivatives, is solved by Laplace Adomian decomposition method (LADM). Two numerical examples supported by graphics, tables, and discussion are provided, by comparing the numerical results obtained with the exact solution when \( \alpha = \beta = 1 \), it is observed that they are in perfect agreement, this confirms the accuracy and smoothness of the current method.

1. Introduction

Fractional differential equations have gained much attention recently due to be an important and useful tool to show the hidden aspects in many phenomena occurring from real world, including mathematics [1–3], continuum mechanics [4], magnetohydrodynamic [5], and other areas. One of the most important reaction and diffusion equations is called the Brusselator system, which is used to describe the mechanism of chemical reaction-diffusion with nonlinear oscillations [6, 7]. In recent years, many researchers have resorted to using modern and different methods to solve this system or find approximate solutions, such a second-order scheme by Twizell, Gumel and Cao in [8], authors of [9–11] employed the Adomian decomposition method, Why-Teong Ang in [12] used the dual-reciprocity boundary element method, in [13] by insert combination of collocation method using the radial basis functions (RBFs) with first order accurate forward difference approximation, by Laplace transform method and the new homotopy perturbation method in [14], authors of [15] resorted to a computational study with applications in chemical processes etc, the solutions to the system were also studied fractional order, such as q-homotopy analysis transform method in [16], by fractional reduced differential transform method in [17]. Our main concern in this work, is to apply Laplace Adomian decomposition method (LADM) [18–22], to find approximate solutions to the reaction-diffusion Brusselator system (RDBS), which has the following form [16]

\[
\psi_\alpha^{\alpha}(x, \eta, \zeta) = b - (a + 1)\psi + \psi^2\phi + \gamma(\psi_{xx} + \psi_{\eta\eta}), \quad 0 < \alpha \leq 1,
\]

\[
\phi_\beta^{\beta}(x, \eta, \zeta) = a\psi - \psi^2\phi + \gamma(\phi_{xx} + \phi_{\eta\eta}), \quad 0 < \beta \leq 1,
\]

with the following initial conditions

\[
\psi(x, \eta, 0) = h(x, \eta), \quad \phi(x, \eta, 0) = j(x, \eta),
\]

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both \((\psi, \phi)\) refer to the dimensionless concentrations of two reactants; \(a\) and \(b\) are positive real constants (constants concentrations), \(\gamma\) is a constant values for the diffusion coefficient and, \(0 < \alpha, \beta \leq 1\), are parameters representing the order of the time fractional derivatives in Caputo sense.

2. Preliminaries and notations

These are some basic definitions and properties of fractional calculus, for more details, see [1–3].

**Definition 1** ( [2, 23] Riemann-Liouville integral) Let \(\psi \in L^1(a, b), a, b \in \mathbb{R}\). The left sided Riemann–Liouville fractional integral operator of order \(\alpha \geq 0\), of a function \(\psi\) is defined by

\[
\begin{align*}
I_0^\alpha \psi(\zeta) &= \frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta - \tau)^{\alpha - 1} \psi(\tau) d\tau, \quad \alpha > 0, \\
I_0^0 \psi(\zeta) &= \psi(\zeta), \\
I_0^\alpha \psi(\zeta) &= \psi(\zeta), \quad \alpha = 0,
\end{align*}
\]

where ,

\[
\Gamma(\omega) = \int_0^\infty \zeta^{\omega-1} e^{-\zeta} d\zeta, \quad \text{Re}(\omega) > 0.
\]

The operator \(I_0^\alpha\) satisfy the following properties

1) \(I_0^\alpha I_0^\beta \psi(\zeta) = I_0^{\alpha + \beta} \psi(\zeta) = I_0^\beta I_0^\alpha \psi(\zeta),\)

2) \(I_0^\alpha \zeta^\rho = \frac{\Gamma(\rho + 1)}{\Gamma(\alpha + \rho + 1)} \zeta^{\alpha + \rho}, \rho > -1.\)

**Definition 2** ( [2, 23] Caputo derivative) Let \(\alpha \geq 0\), and \(n = [\alpha] + 1\). If \(\psi \in AC^n[a, b], a, b \in \mathbb{R}\), then the Caputo fractional derivative operator \(D_0^\alpha \psi(\zeta)\) exist almost everywhere on \([a, b]\), is defined as

\[
D_0^\alpha \psi(\zeta) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(n - \alpha)} \int_0^\zeta (\zeta - \tau)^{n - \alpha - 1} \psi^{(n)}(\tau) d\tau, & n - 1 < \alpha < n, \\
\psi^{(n)}(\zeta), & \alpha = n, n \in \mathbb{N}.
\end{array} \right.
\]

The operator \(D_0^\alpha\) satisfy the following properties

1) \(D_0^{\alpha + \beta} \psi(\zeta) = D_0^\beta D_0^\alpha \psi(\zeta),\)

2) \(D_0^\alpha \zeta^\rho = \frac{\Gamma(1 + \rho)}{\Gamma(1 + \rho - \alpha)} \zeta^{\rho - \alpha}, \rho > -1,\)

3) \(D_0^\alpha I_0^\alpha \psi(\zeta) = \psi(\zeta),\)

4) \(I_0^\alpha D_0^\alpha \psi(\zeta) = \psi(\zeta) - \sum_{k=0}^{n-1} \psi^{(k)}(0^+) \frac{\omega^k}{k!}.\)

**Definition 3** [2, 23] The Mittag-Leffler type of two-parameters function \(E_{\lambda, \rho}(\cdot)\) is presented as

\[
E_{\lambda, \rho}(\omega) = \sum_{k=0}^{+\infty} \frac{\omega^k}{\Gamma(\lambda k + \rho)}, \quad \lambda, \rho > 0, \quad \omega \in \mathbb{C}.
\]

3. Laplace transform and important properties

**Definition 4** [24] Let \(\psi(\zeta)\) be defined for \(\zeta \in (0, +\infty)\). Then, when the improper integral exists, the Laplace transform \(\Psi(s)\) of \(\psi(\zeta)\) is defined by

\[
\mathcal{L}[\psi(\zeta)] = \Psi(s) = \int_0^{+\infty} e^{-s\zeta} \psi(\zeta) d\zeta.
\]

Note that \(\Psi\) is a function of the new variables \(s\), while the original function \(\psi\) is a function of the variable \(\zeta\).
3.1. Properties:

These are the basic properties of the Laplace transform, which are used in this work.

1. \( \mathcal{L}[\zeta^\rho] = \frac{\Gamma(\rho+1)}{s^{\rho+1}}, \quad \mathcal{L}^{-1}\left(\frac{1}{s^{\rho+1}}\right) = \frac{\zeta^\rho}{\Gamma(\rho+1)}, \quad \rho > -1. \) (\( \mathcal{L}^{-1} \) denotes the inverse of \( \mathcal{L} \)).

2. \( \mathcal{L}[s^{-\alpha} \psi(\zeta)] = s^{-\alpha} \Psi(s) \).

3. For \( m - 1 < \alpha \leq m, m \in \mathbb{N}^* \)

\[
\mathcal{L}[D_\zeta^\alpha \psi(\zeta)] = s^\alpha \Psi(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} \psi^{(k)}(0). \tag{8}
\]

4. When \( 0 < \alpha \leq 1 \)

\[
\mathcal{L}[D_\zeta^\alpha \psi(\zeta)] = s^\alpha \Psi(s) - s^{\alpha-1} \psi(0). \tag{9}
\]

4. Analysis of (LADM)

This section describes the implementation of (LADM) as shown below [18–22], we consider a general fractional partial differential equation with initial conditions of the form

\[
D_\zeta^\alpha \psi(x, \zeta) + R \psi(x, \zeta) + N \psi(x, \zeta) = w(x, \zeta), \quad m - 1 < \alpha \leq m, m \in \mathbb{N}^*. \tag{10}
\]

Operating the Laplace transform on both sides of Eq.(10), we then obtain

\[
\mathcal{L}[D_\zeta^\alpha \psi(x, \zeta)] + \mathcal{L}[R \psi(x, \zeta)] + \mathcal{L}[N \psi(x, \zeta)] = \mathcal{L}[w(x, \zeta)]. \tag{11}
\]

By using the formula (8) in the above Eq.(11), we have

\[
s^\alpha \mathcal{L}[\psi(x, \zeta)] - \sum_{k=0}^{m-1} s^{\alpha-k-1} \psi^{(k)}(0) + \mathcal{L}[R \psi(x, \zeta)] + \mathcal{L}[N \psi(x, \zeta)] = \mathcal{L}[w(x, \zeta)], \tag{12}
\]

on simplifying

\[
\mathcal{L}[\psi(x, \zeta)] = \sum_{k=0}^{m-1} \frac{\psi^{(k)}(0)}{s^{k+1}} + \frac{1}{s^\alpha} \mathcal{L}[w(x, \zeta)] - \frac{1}{s^\alpha} \mathcal{L}[R \psi(x, \zeta)] - \frac{1}{s^\alpha} \mathcal{L}[N \psi(x, \zeta)]. \tag{13}
\]

Application of \( \mathcal{L}^{-1} \), on the above equation Eq. (13), we then obtain

\[
\psi(x, \zeta) = W(x, \zeta) - \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha} \mathcal{L}[R \psi(x, \zeta)] + N \psi(x, \zeta)\right\}, \tag{14}
\]

where \( W(x, \zeta) \) is caused by \( w(x, \zeta) \) and the given initial conditions. Now, by (ADM) technique [25–28]

\[
\psi(x, \zeta) = \sum_{n=0}^{+\infty} \psi_n(x, \zeta) \tag{15}
\]

The nonlinear term in the problem can be decomposed as

\[
N \psi(x, \zeta) = \sum_{n=0}^{+\infty} \mathcal{L}^{-1}[\psi_0, \psi_1, \cdots, \psi_n]. \tag{16}
\]
and \( \mathcal{A}_n \) is the Adomian polynomials shown below see [25, 26]

\[
\mathcal{A}_n(\psi_0, \psi_1, \cdots, \psi_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[ N \left( \sum_{i=0}^{\infty} (\lambda^i \psi_i) \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, 3, \cdots.
\] (17)

Substituting Eq. (15) and Eq. (16) in Eq. (14) we get

\[
\sum_{n=0}^{\infty} \psi_n(x, \zeta) = W(x, \zeta) - \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left( R \left( \sum_{n=0}^{\infty} \psi_n(x, \zeta) + \sum_{n=0}^{\infty} s^\alpha \mathcal{A}_n \right) \right) \right\}.
\] (18)

We obtain \( \psi_0(x, \zeta) = W(x, \zeta) \), and a recurrence relation as shown below

\[
\psi_{n+1}(x, \zeta) = -\mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[ R \psi_n(x, \zeta) + \mathcal{A}_n \right] \right\}, \quad n \geq 0.
\] (19)

5. Implementation (LADM) to (RDBS)

[16] Consider the (RDBS) (1).

\[
\begin{align*}
\psi^\alpha \psi(x, \eta, \zeta) &= b - (a + 1) \psi + \psi^2 \phi + \gamma (\psi_{xx} + \psi_{\eta \eta}), \quad 0 < \alpha \leq 1, \\
\phi^\beta \phi(x, \eta, \zeta) &= a \psi - \psi^2 \phi + \gamma (\phi_{xx} + \phi_{\eta \eta}), \quad 0 < \beta \leq 1,
\end{align*}
\] (20)

with the following initial conditions

\[
\psi(x, \eta, 0) = h(x, \eta), \quad \phi(x, \eta, 0) = j(x, \eta).
\] (21)

Operating \( \mathcal{L} \) on both sides of eqs. (20), we get

\[
\begin{align*}
\mathcal{L}[D^\alpha \psi(x, \eta, \zeta)] &= \mathcal{L}[b - (a + 1) \psi + \psi^2 \phi + \gamma (\psi_{xx} + \psi_{\eta \eta})], \\
\mathcal{L}[D^\beta \phi(x, \eta, \zeta)] &= \mathcal{L}[a \psi - \psi^2 \phi + \gamma (\phi_{xx} + \phi_{\eta \eta})].
\end{align*}
\] (22)

Now, by using the formula (9), in the above eqs. (22), we have

\[
\begin{align*}
s^\alpha \mathcal{L} \psi(x, \eta, \zeta) - s^{\alpha-1} \psi(x, \eta, 0) &= \mathcal{L}[b - (a + 1) \psi + \psi^2 \phi + \gamma (\psi_{xx} + \psi_{\eta \eta})], \\
s^\beta \mathcal{L} \phi(x, \eta, \zeta) - s^{\beta-1} \phi(x, \eta, 0) &= \mathcal{L}[a \psi - \psi^2 \phi + \gamma (\phi_{xx} + \phi_{\eta \eta})].
\end{align*}
\] (23)

Or

\[
\begin{align*}
\mathcal{L} \psi(x, \eta, \zeta) &= \left[ \psi(x, \eta, 0) \right]_s + \frac{b}{s^{\alpha+1}} + \frac{1}{s^\alpha} \mathcal{L} \left[ -(a + 1) \psi + \psi^2 \phi + \gamma (\psi_{xx} + \psi_{\eta \eta}) \right], \\
\mathcal{L} \phi(x, \eta, \zeta) &= \left[ \phi(x, \eta, 0) \right]_s + \frac{1}{s^\beta} \mathcal{L} \left[ a \psi - \psi^2 \phi + \gamma (\phi_{xx} + \phi_{\eta \eta}) \right].
\end{align*}
\] (24)

Thus, we apply \( \mathcal{L}^{-1} \), to the above equation (24) we get

\[
\begin{align*}
u(x, \eta, \zeta) &= \psi(x, \eta, 0) + \frac{b \tau^\alpha}{\Gamma(\alpha + 1)} + \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[ -(a + 1) \psi + \psi^2 \phi + \gamma (\psi_{xx} + \psi_{\eta \eta}) \right] \right\}, \\
\phi(x, \eta, \zeta) &= \phi(x, \eta, 0) + \mathcal{L}^{-1} \left\{ \frac{1}{s^\beta} \mathcal{L} \left[ a \psi - \psi^2 \phi + \gamma (\phi_{xx} + \phi_{\eta \eta}) \right] \right\}.
\end{align*}
\] (25)

Then, we apply the Adomian decomposition method in (25) we obtain

\[
\begin{align*}
\sum_{n=0}^{\infty} \psi_n &= h(x, \eta) + \frac{b \zeta^\alpha}{\Gamma(\alpha + 1)} + \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[ -(a + 1) \sum_{n=0}^{\infty} \psi_n + \gamma (\nabla^2 \sum_{n=0}^{\infty} \psi_n) + \sum_{n=0}^{\infty} \mathcal{A}_n (\psi, \phi) \right] \right\}, \\
\sum_{n=0}^{\infty} \phi_n &= j(x, \eta) + \mathcal{L}^{-1} \left\{ \frac{1}{s^\beta} \mathcal{L} \left[ a \sum_{n=0}^{\infty} \psi_n + \gamma (\nabla^2 \sum_{n=0}^{\infty} \phi_n) - \sum_{n=0}^{\infty} \mathcal{A}_n (\psi, \phi) \right] \right\}.
\end{align*}
\] (26)
Therefore from (26), the following procedure can be defined
\[
\psi_0(x, \eta, \zeta) = h(x, \eta) + \frac{b_x \zeta}{\Gamma(\alpha + 1)},
\]
\[
\psi_{n+1}(x, \eta, \zeta) = \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[ - (a + 1) \psi_n + \gamma (\nabla^2 \psi_n) + \mathcal{A}_n \right] \right\}, \quad n \geq 0.
\]
(27)
\[
\phi_0(x, \eta, \zeta) = j(x, \eta),
\]
\[
\phi_{n+1}(x, \eta, \zeta) = \mathcal{L}^{-1} \left\{ \frac{1}{s^\beta} \mathcal{L} \left[ a \psi_n + \gamma (\nabla^2 \phi_n) - \mathcal{A}_n \right] \right\}, \quad n \geq 0.
\]
These are the first values of \( \mathcal{A}_n \)
\[
\mathcal{A}_0 = \psi_0^\beta \phi_0,
\]
\[
\mathcal{A}_1 = 2 \psi_0 \psi_1 \phi_0 + \psi_0^2 \phi_1,
\]
\[
\mathcal{A}_2 = (2 \psi_0 \psi_2 + \psi_1^2) \phi_0 + 2 \psi_0 \psi_1 \phi_1 + \psi_0^2 \phi_2,
\]
\[
\mathcal{A}_3 = (2 \psi_1 \psi_2 + 2 \psi_0 \psi_1) \phi_0 + (2 \psi_0 \psi_2 + \psi_1^2) \phi_1 + 2 \psi_0 \psi_1 \phi_2 + \psi_0^2 \phi_3,
\]
\[
\mathcal{A}_4 = (2 \psi_0 \psi_4 + 2 \psi_1 \psi_3 + \psi_2^2) \phi_0 + (2 \psi_1 \psi_2 + 2 \psi_0 \psi_3) \phi_1 + (2 \psi_0 \psi_2 + \psi_1^2) \phi_2 + 2 \psi_0 \psi_1 \phi_3 + \psi_0^2 \phi_4,
\]
\[
\vdots
\]
6. Numerical results and discussion

Example 1: [15, 16] Consider the (RDBS) (1), for \( a = 1, \quad b = 0, \quad \text{and} \quad \gamma = \frac{1}{4} \),
\[
\psi_\zeta^\beta = -2 \psi + \psi^2 \phi + \frac{1}{4} \left( \psi_{xx} + \psi_{\eta \eta} \right),
\]
(28)
\[
\phi_\zeta^\beta = \psi - \psi^2 \phi + \frac{1}{4} \left( \phi_{xx} + \phi_{\eta \eta} \right),
\]
with the following initial conditions
\[
\psi(x, \eta, 0) = e^{-(x+\eta)}, \quad \phi(x, \eta, 0) = e^{(x+\eta)}.
\]
(29)
Therefore from (26), the following procedure can be defined
\[
\sum_{n=0}^{\infty} \psi_n = e^{-(x+\eta)} + \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[ -2 \sum_{n=0}^{\infty} \psi_n + \frac{1}{4} \left( \nabla^2 \sum_{n=0}^{\infty} \psi_n \right) + \sum_{n=0}^{\infty} \mathcal{A}_n (\psi, \phi) \right] \right\},
\]
(30)
\[
\sum_{n=0}^{\infty} \phi_n = e^{(x+\eta)} + \mathcal{L}^{-1} \left\{ \frac{1}{s^\beta} \mathcal{L} \left[ \sum_{n=0}^{\infty} \psi_n + \frac{1}{4} \left( \nabla^2 \sum_{n=0}^{\infty} \phi_n \right) - \sum_{n=0}^{\infty} \mathcal{A}_n (\psi, \phi) \right] \right\}.
\]
Thus, we obtain \( \psi_0(x, \eta, \zeta) = e^{-(x+\eta)}, \phi_0(x, \eta, \zeta) = e^{(x+\eta)} \), and a recurrence relation as shown below
\[
\psi_{n+1}(x, \eta, \zeta) = \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[ -2 \psi_n + \frac{1}{4} (\nabla^2 \psi_n) + \mathcal{A}_n \right] \right\}, \quad n \geq 0.
\]
(31)
\[
\phi_{n+1}(x, \eta, \zeta) = \mathcal{L}^{-1} \left\{ \frac{1}{s^\beta} \mathcal{L} \left[ \psi_n + \frac{1}{4} (\nabla^2 \phi_n) - \mathcal{A}_n \right] \right\}, \quad n \geq 0.
By applying (31), we obtain

\[
\psi_1(x, \eta, \zeta) = \frac{1}{2} e^{-x+\eta} \xi^\alpha \\
\phi_1(x, \eta, \zeta) = \frac{1}{2} \xi^\beta \\
\psi_2(x, \eta, \zeta) = \frac{1}{4} e^{-x+\eta} \left( \frac{\xi^{2\alpha}}{\Gamma(1+\alpha+\beta)} + \frac{2\xi^{\alpha+\beta}}{\Gamma(1+2\alpha)} \right) \\
\phi_2(x, \eta, \zeta) = \frac{1}{4} \left( -2e^{-x+\eta} + e^{x+\eta} \right) \xi^{2\beta} \\
\psi_3(x, \eta, \zeta) = -\frac{1}{2} \frac{e^{-(x+\eta)}}{\Gamma(1+\alpha+\beta)} \xi^{2\alpha+\beta} + \frac{1}{4} \frac{e^{-(x+\eta)}}{\Gamma(1+\alpha+\beta)^2} \xi^{3\alpha} - \frac{1}{8} \frac{e^{-(x+\eta)}}{\Gamma(1+3\alpha)} \xi^{3\alpha} \\
\phi_3(x, \eta, \zeta) = \frac{1}{4} \frac{e^{-(x+\eta)}}{\Gamma(1+\alpha+\beta)} \xi^{2\alpha+\beta} + \frac{1}{4} \frac{e^{-(x+\eta)}}{\Gamma(1+\alpha+\beta)^2} \xi^{2\alpha+\beta} + \frac{1}{4} \frac{e^{-(x+\eta)}}{\Gamma(1+\alpha+\beta+2\alpha)} \xi^{2\alpha+\beta} \\
+ \frac{1}{8} \frac{e^{-(x+\eta)}}{\Gamma(1+\alpha+2\beta)} \xi^{3\alpha} - \frac{1}{8} \frac{e^{-(x+\eta)}}{\Gamma(1+\alpha+2\beta)} \xi^{3\alpha} - \frac{1}{8} \frac{e^{-(x+\eta)}}{\Gamma(1+\alpha+2\beta)} \xi^{3\alpha}.
\]

In this way, the remaining components of the solution can be obtained.

The solution in series form when \( \alpha = \beta = 1 \), is given by

\[
\psi(x, \eta, \zeta) = e^{-x+\eta} \left( 1 - \frac{1}{2} \xi^\alpha + \frac{1}{8} \xi^{2\alpha} - \frac{1}{48} \xi^{3\alpha} + \cdots \right) = e^{-x+\eta} \xi^\alpha,
\]

\[
\phi(x, \eta, \zeta) = e^{x+\eta} \left( 1 + \frac{1}{2} \xi^\beta + \frac{1}{8} \xi^{2\beta} + \frac{1}{48} \xi^{3\beta} + \cdots \right) = e^{x+\eta} \xi^\beta.
\]

**Example 2:** \([10, 12]\) Consider the (RDBS) (1), for \( a = \frac{1}{2}, \ b = 1, \ and \ \gamma = \frac{1}{500} \):

\[
\psi_\xi = 1 - \frac{3}{2} \psi + \psi^2 \phi + \frac{1}{500} (\psi_{xx} + \psi_{\eta \eta}), \\
\phi_\xi = \frac{1}{2} \psi - \psi^2 \phi + \frac{1}{500} (\phi_{xx} + \phi_{\eta \eta}),
\]

with the following initial conditions

\[
\psi(x, \eta, 0) = x^2, \ \phi(x, \eta, 0) = \eta^2.
\]

By using (36), we get

\[
\sum_{n=0}^{\infty} \psi_n = x^2 + \frac{\xi^\alpha}{\Gamma(\alpha+1)}, \quad \phi_n \psi_n = \eta^2 \text{ and a recurrence relation as shown below}
\]

\[
\psi_{n+1}(x, \eta, \zeta) = \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[ \frac{\psi_n + 1}{500} (\nabla^2 \sum_{n=0}^{\infty} \psi_n) + \sum_{n=0}^{\infty} \mathcal{A}_n(\psi, \phi) \right] \right\}, \quad n \geq 0.
\]

\[
\phi_{n+1}(x, \eta, \zeta) = \mathcal{L}^{-1} \left\{ \frac{1}{s^\beta} \mathcal{L} \left[ \frac{\phi_n + 1}{500} (\nabla^2 \phi_n) - \sum_{n=0}^{\infty} \mathcal{A}_n(\psi, \phi) \right] \right\}, \quad n \geq 0.
\]
Substituting $\psi_0$ and $\phi_0$ into (36), we get

$$\psi_1(x, \eta, \zeta) = \frac{\Gamma(1+2\alpha)\eta^2\zeta^{2\alpha}}{(\Gamma(1+\alpha))^2(1+3\alpha)} + \frac{1}{250} \left(1 - 375x^2 + 250x^4\eta^2\right)\zeta^{2\alpha} + \frac{1}{2} \left(4x^2\eta^2 - 3\right)\zeta^{2\alpha},$$

$$\phi_1(x, \eta, \zeta) = \frac{\Gamma(1+2\alpha)\eta^2\zeta^{2\alpha+\beta}}{(\Gamma(1+\alpha))^2(1+2\alpha+\beta)} + \frac{1}{250} \left(1 - 250x^2 - 250x^4\eta^2\right)\zeta^{2\alpha+\beta} + \frac{1}{2} \left(-2x^2\eta^2 + 1\right)\zeta^{2\alpha+\beta},$$

and so on. In this manner the rest of the iterative components can be found.

7. Tables and Figures

Below we know the approximate solution of the four order ($\Psi_4, \Phi_4$), which we use in the four molar tables to compare with the exact solution and calculate the absolute error for various values of $\alpha$ and $\beta$.

$$\Psi_4 = \psi_0 + \psi_1 + \psi_2 + \psi_3,$$

$$\Phi_4 = \phi_0 + \phi_1 + \phi_2 + \phi_3.$$

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<th>$\zeta$</th>
<th>$\psi$ Exact</th>
<th>$\alpha = \beta = 1$</th>
<th>$\alpha = \beta = 0.9$</th>
<th>$\alpha = \beta = 0.8$</th>
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<td>0.6</td>
<td>0.2725317930340126</td>
<td>0.2724147261874530</td>
<td>0.2672528759832714</td>
<td>0.2637083996441059</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$\phi$ Exact</th>
<th>$\alpha = \beta = 1$</th>
<th>$\alpha = \beta = 0.9$</th>
<th>$\alpha = \beta = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.857651118063164</td>
<td>2.85765040308997</td>
<td>2.902768403574928</td>
<td>2.961911368085330</td>
</tr>
<tr>
<td>0.2</td>
<td>3.004166023946433</td>
<td>3.004154467418655</td>
<td>3.073643397863855</td>
<td>3.159324321821366</td>
</tr>
<tr>
<td>0.3</td>
<td>3.15819290689768</td>
<td>3.158133806826573</td>
<td>3.24586670787438</td>
<td>3.350234564546649</td>
</tr>
<tr>
<td>0.4</td>
<td>3.320116922736547</td>
<td>3.319928206491313</td>
<td>3.422208983952619</td>
<td>3.540549820548468</td>
</tr>
<tr>
<td>0.5</td>
<td>3.49034295746198141</td>
<td>3.489877451641429</td>
<td>3.603890449326122</td>
<td>3.732756435818485</td>
</tr>
<tr>
<td>0.6</td>
<td>3.669296666719244</td>
<td>3.668321327505480</td>
<td>3.791718269201408</td>
<td>3.928207358812717</td>
</tr>
</tbody>
</table>

| $\psi_{Exact} - \Psi_4$ | $|\psi_{Exact} - \Psi_4|$ |
|--------------------------|--------------------------|
| (x, \eta) | \zeta | \alpha = \beta = 1 | \alpha = \beta = 0.9 | \alpha = \beta = 0.8 |
| (0.1, 0.9) | 0.1 | 0.000000948518453 | 0.00052798162303156 | 0.0117607359874232 |
| (0.2, 0.8) | 0.2 | 0.000015026781194 | 0.0069830348936654 | 0.0146857845899624 |
| (0.3, 0.7) | 0.3 | 0.000075328558074 | 0.0074593310360485 | 0.0150174317387870 |
| (0.4, 0.6) | 0.4 | 0.00002357670731814 | 0.007124253821832 | 0.0138558720061726 |
| (0.5, 0.5) | 0.5 | 0.0000570028230515 | 0.0064211075930520 | 0.0116977769047142 |
| (0.6, 0.4) | 0.6 | 0.000117668465596 | 0.0052789170507412 | 0.008823393899067 |
Table 4: Absolute error $|\phi_{Exact} - \Phi_4|$.

<table>
<thead>
<tr>
<th>$(x, \eta)$</th>
<th>$\zeta$</th>
<th>$\alpha = \beta = 1$</th>
<th>$\alpha = \beta = 0.9$</th>
<th>$\alpha = \beta = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1, 0.9)</td>
<td>0.1</td>
<td>0.000000715024167</td>
<td>0.045117286511764</td>
<td>0.104260250022166</td>
</tr>
<tr>
<td>(0.2, 0.8)</td>
<td>0.2</td>
<td>0.00001155652778</td>
<td>0.069477373921922</td>
<td>0.155158297874933</td>
</tr>
<tr>
<td>(0.3, 0.7)</td>
<td>0.3</td>
<td>0.000059102863195</td>
<td>0.087693761097670</td>
<td>0.192041654856881</td>
</tr>
<tr>
<td>(0.4, 0.6)</td>
<td>0.4</td>
<td>0.000188716245234</td>
<td>0.102092061216072</td>
<td>0.220432897811921</td>
</tr>
<tr>
<td>(0.5, 0.5)</td>
<td>0.5</td>
<td>0.000465505820412</td>
<td>0.113547491864281</td>
<td>0.242413478356644</td>
</tr>
<tr>
<td>(0.6, 0.4)</td>
<td>0.6</td>
<td>0.000975340113764</td>
<td>0.122421601582164</td>
<td>0.258910691193473</td>
</tr>
</tbody>
</table>

Figure 1: The behavior of $\psi$ and $\Psi_4$ for Example 1, when $\zeta = 0.3$, $\alpha = \beta = 1$.

Figure 2: The behavior of $\phi$ and $\Phi_4$ for Example 1, when $\zeta = 0.3$, $\alpha = \beta = 1$. 
Conclusion

The approximate solutions to time fractional reaction-diffusion Brusselator system with initial conditions were successfully and flexibly found using the effective method (LADM), although a case study, the order of the fractional time derivatives is different between $\psi$ and $\phi$, i.e. $\psi^\alpha_\zeta, 0 < \alpha \leq 1, \phi^\beta_\zeta, 0 < \beta \leq 1$, however, good approximations were obtained gradually, it has been shown by comparing the numerical results obtained with the exact solution when $\alpha = \beta = 1$, in the first example, it is observed that they are in perfect agreement. In the second example, although there is no known analytic solution, but we were able to find approximate solutions using the present method without any transformation, discretization, perturbation, which proves that the (LADM) is very efficient and reliable, and can be used to solve a wide class of nonlinear fractional order differential equations, especially since most of them do not contain exact analytic solutions.

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Declaration of Competing Interest

The author(s), declares that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Mohamed Zellal: Conceptualization, Methodology, Validation, Formal Analysis, Writing Original Draft. Kacem Belghaba: Investigation, Resources, Visualization, Supervision.

References


