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AN INVESTIGATION OF THE SOLUTIONS AND THE DYNAMIC BEHAVIOR OF SOME RATIONAL DIFFERENCE EQUATIONS

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Abstract. The main purpose of this work is to find the form of the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(\pm 1 \pm x_{n-2}x_{n-6})}, \quad n = 1, 2, ...,$$

where the initial conditions are arbitrary positive real numbers. Moreover, we gave the solutions of some special cases of this equation, and studied some dynamic behavior of these equations. At the end we illustrated our results by presenting some numerical examples to the equations are given.

1. Introduction

In the last few decades, there has been a major interest in studying a qualitative behavior of the solutions of rational difference equations. The reasons of this interest comes from the fact that these equations are powerful tool for applications since difference equations plays an important role in mathematics to describe and model a real life situations such as population dynamics, statistical problem, stochastic time series, number theory, biology, economic, probability theory, genetics, psychology, etc. [1]-[5]. It is well known that the field of difference equations is old and it has been developed incrementally, and the rational difference equations is important category of difference equations where they occupies a good place in applicable analysis, which has encouraged the mathematical researchers to continue investigating the qualitative properties of the solution of rational difference equations and the systems of difference equations.

Recently, Abo-Zeid [6] solved and studied the global behavior of the well defined solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{Ax_{n-2} + Bx_{n-3}},$$

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Elsayed [7] have obtained the solution and also he studied the behavior of the following rational difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_n + dx_{n-1}}.$$

Cinar [8]-[10] have investigated the positive solutions of the following difference equations

$$x_{n+1} = \frac{\alpha x_{n-1}}{1 + b_n x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{1 + \alpha x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + \alpha x_n x_{n-1}}.$$

Ibrahim [11] got the solutions of the rational difference equation:

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}.$$

Bozkurt [12] was investigated the local and global behavior of the positive solutions of the following difference equation

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}}.$$

Simsek et. al. [13] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Xian and L. Wei [14] investigated the global asymptotic stability of the following difference equation

$$x_{n+1} = \frac{p + qx_n}{1 + rx_{n-k}}.$$

Karatas et. al. [15] studied study the positive solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2} x_{n-5}}.$$

For other papers related to study the dynamic behavior of difference, we refer to [16]-[28].

Our goal is to study the dynamic behaviors of the solutions of the following difference equations.

$$x_{n+1} = \frac{x_{n-2} x_{n-6}}{x_{n-3}(\pm 1 \pm x_{n-2} x_{n-6})}, \tag{1.1}$$

where the initial conditions $x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary nonzero real numbers.

2. Preliminaries

Here, we review some results which will be useful in our investigation of the difference equation (1.1).

Definition 2.1. Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of the initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2} \dots, x_{n-k}), \qquad n = 0, 1, ..., \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 2.2. A point $x^* \in I$ is called an **equilibrium point** of Eq. (2.1) if

$$x^* = F(x^*, x^*, x^*, ...).$$

That is,$x_n = x^*$, for $n \geq 0$, is a solution of Eq. (2.1), or equivalently, x^* is a fixed point of F.

Definition 2.3. Let x^* be an equilibrium point of (2.1).

(**i**) The equilibrium point x^* of Eq. (2.1) is called **locally stable** if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_0 \in I$ with

$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + ... + |x_0 - x^*| < \delta,$$

we have,

$$|x_n - x^*| < \epsilon \quad for\ all \qquad n \geq -k.$$

(**ii**) The equilibrium point x^* of Eq. (2.1) is called **locally asymptotically stable** if it is locally stable, and if there exists $\gamma > 0$ such that if $x_{-k}, x_{-k+1}, ..., x_0 \in I$ with

$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + \cdots + |x_0 - x^*| < \gamma,$$

we have,

$$\lim_{n\to\infty} x_n = x^*.$$

(**iii**) The equilibrium point x^* of Eq. (2.1) is called **a global attractor** if for every solution $x_{-k}, x_{-k+1}, ..., x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = x^*.$$

(**iv**) The equilibrium point x^* of Eq. (2.1) is called **a global asymptotically stable** if it is locally stable and global attractor of Eq. (2.1).

(**v**) The equilibrium point x^* of Eq. (2.1) is called **unstable** if x^* is not locally stable.

The linearized equation of Eq. (2.1) about the equilibrium point x^* is the linear difference equation

$$z_{n+1} = \sum_{i=1}^{k} \frac{\partial F(\hat{x}, \dots, \hat{x})}{\partial x_{n-i}} z_{n-i}.$$

Definition 2.4. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with periodic p if $x_{n+p} = x_n$ for all $n \geq -k$.

Theorem 2.1. [30]. Assume that $p_0, p_1, \ldots, p_k \in R$, and $k \in \{0, 1, 2, \ldots\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1, \tag{2.2}$$

is a sufficient condition for the asymptotic stability of the difference equation:

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \quad \text{n=0,1,...}.$$

3. On the difference equation $x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(1+x_{n-2}x_{n-6})}$

In this section, we obtain a specific form of the solution of the first case of the equation (1.1):

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(1 + x_{n-2}x_{n-6})}. \tag{3.1}$$

Theorem 3.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of equation (1.1). Then for $n = 0, 1, \ldots,$

$$x_{24n-6} = g \prod_{i=0}^{n-1} \frac{(1 + (8i+2)ae)(1 + (8i+5)bf)(1 + (8i)cg)}{(1 + (8i+6)ae)(1 + (8i+1)bf)(1 + (8i+4)cg)},$$

$$x_{24n-5} = f \prod_{i=0}^{n-1} \frac{(1 + (8i+5)ae)(1 + (8i)bf)(1 + (8i+3)cg)}{(1 + (8i+1)ae)(1 + (8i+4)bf)(1 + (8i+7)cg)},$$

$$x_{24n-4} = e \prod_{i=0}^{n-1} \frac{(1 + (8i)ae)(1 + (8i+3)bf)(1 + (8i+6)cg)}{(1 + (8i+4)ae)(1 + (8i+7)bf)(1 + (8i+2)cg)},$$

$$x_{24n-3} = d \prod_{i=0}^{n-1} \frac{(1 + (8i+3)ae)(1 + (8i+6)bf)(1 + (8i+1)cg)}{(1 + (8i+7)ae)(1 + (8i+2)bf)(1 + (8i+5)cg)},$$

$$x_{24n-2} = c \prod_{i=0}^{n-1} \frac{(1 + (8i+6)ae)(1 + (8i+1)bf)(1 + (8i+4)cg)}{(1 + (8i+2)ae)(1 + (8i+5)bf)(1 + (8i+8)cg)},$$

$$x_{24n-1} = b \prod_{i=0}^{n-1} \frac{(1 + (8i+1)ae)(1 + (8i+4)bf)(1 + (8i+7)cg)}{(1 + (8i+5)ae)(1 + (8i+8)bf)(1 + (8i+3)cg)},$$

$$x_{24n} = a \prod_{i=0}^{n-1} \frac{(1 + (8i+4)ae)(1 + (8i+7)bf)(1 + (8i+2)cg)}{(1 + (8i+8)ae)(1 + (8i+3)bf)(1 + (8i+6)cg)},$$

$$x_{24n+1} = \frac{cg}{d(1+cg)} \prod_{i=0}^{n-1} \frac{(1 + (8i+7)ae)(1 + (8i+2)bf)(1 + (8i+5)cg)}{(1 + (8i+3)ae)(1 + (8i+6)bf)(1 + (8i+9)cg)},$$

$$x_{24n+2} = \frac{bf}{c(1+bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+4)cg)},$$

$$x_{24n+3} = \frac{ae}{b(1+ae)} \prod_{i=0}^{n-1} \frac{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+3)cg)}{(1+(8i+9)ae)(1+(8i+4)bf)(1+(8i+7)cg)},$$

$$x_{24n+4} = \frac{cg}{a(1+2cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+8)ae)(1+(8i+3)bf)(1+(8i+6)cg)}{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+10)cg)},$$

$$x_{24n+5} = \frac{bdf(1+cg)}{cg(1+2bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)},$$

$$x_{24n+6} = \frac{ace(1+bf)}{bf(1+2ae)} \prod_{i=0}^{n-1} \frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+4)cg)}{(1+(8i+10)ae)(1+(8i+5)bf)(1+(8i+8)cg)},$$

$$x_{24n+7} = \frac{bcg(1+ae)}{ae(1+3cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+9)ae)(1+(8i+4)bf)(1+(8i+7)cg)}{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+11)cg)},$$

$$x_{24n+8} = \frac{abf(1+2cg)}{cg(1+3bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+10)cg)}{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+6)cg)},$$

$$x_{24n+9} = \frac{aceg(1+2bf)}{bdf(1+cg)(1+3ae)} \prod_{i=0}^{n-1} \frac{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)}{(1+(8i+11)ae)(1+(8i+6)bf)(1+(8i+9)cg)},$$

$$x_{24n+10} = \frac{bfg(1+2ae)}{ae(1+bf)(1+4cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+10)ae)(1+(8i+5)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)},$$

$$x_{24n+11} = \frac{aef(1+3cg)}{cg(1+ae)(1+4bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+11)cg)}{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)},$$

$$x_{24n+12} = \frac{ceg(1+3bf)}{bf(1+2cg)(1+4ae)} \prod_{i=0}^{n-1} \frac{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+6)cg)}{(1+(8i+12)ae)(1+(8i+7)bf)(1+(8i+10)cg)},$$

$$x_{24n+13} = \frac{bdf(1+3ae)(1+cg)}{ae(1+2bf)(1+5cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+11)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+13)cg)},$$

$$x_{24n+14} = \frac{ae(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)} \prod_{i=0}^{n-1} \frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)},$$

$$x_{24n+15} = \frac{cg(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)},$$

$$x_{24n+16} = \frac{bf(1+4ae)(1+2cg)}{e(1+3bf)(1+6cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+12)ae)(1+(8i+7)bf)(1+(8i+10)cg)}{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+14)cg)},$$

$$x_{24n+17} = \frac{ae(1+2bf)(1+5cg)}{d(1+3ae)(1+6bf)(1+cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+13)cg)}{(1+(8i+11)ae)(1+(8i+14)bf)(1+(8i+9)cg)},$$

where $x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a$ are arbitrary nonzero real numbers.

Proof. The result holds for $n = 0$. Now, assume that $n > 0$ and our assumption holds for $n-1$. Then,

$$x_{24n-30} = g \prod_{i=0}^{n-2} \frac{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i)cg)}{(1+(8i+6)ae)(1+(8i+1)bf)(1+(8i+4)cg)},$$

$$x_{24n-29} = f \prod_{i=0}^{n-2} \frac{(1+(8i+5)ae)(1+(8i)bf)(1+(8i+3)cg)}{(1+(8i+1)ae)(1+(8i+4)bf)(1+(8i+7)cg)},$$

$$x_{24n-28} = e \prod_{i=0}^{n-2} \frac{(1+(8i)ae)(1+(8i+3)bf)(1+(8i+6)cg)}{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+2)cg)},$$

$$x_{24n-27} = d \prod_{i=0}^{n-2} \frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)},$$

$$x_{24n-26} = c \prod_{i=0}^{n-2} \frac{(1+(8i+6)ae)(1+(8i+1)bf)(1+(8i+4)cg)}{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i+8)cg)},$$

$$x_{24n-25} = b \prod_{i=0}^{n-2} \frac{(1+(8i+1)ae)(1+(8i+4)bf)(1+(8i+7)cg)}{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+3)cg)},$$

$$x_{24n-24} = a \prod_{i=0}^{n-2} \frac{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+2)cg)}{(1+(8i+8)ae)(1+(8i+3)bf)(1+(8i+6)cg)},$$

$$x_{24n-23} = \frac{cg}{d(1+cg)} \prod_{i=0}^{n-2} \frac{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)},$$

$$x_{24n-22} = \frac{bf}{c(1+bf)} \prod_{i=0}^{n-2} \frac{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+4)cg)},$$

$$x_{24n-21} = \frac{ae}{b(1+ae)} \prod_{i=0}^{n-2} \frac{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+3)cg)}{(1+(8i+9)ae)(1+(8i+4)bf)(1+(8i+7)cg)},$$

$$x_{24n-20} = \frac{cg}{a(1+2cg)} \prod_{i=0}^{n-2} \frac{(1+(8i+8)ae)(1+(8i+3)bf)(1+(8i+6)cg)}{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+10)cg)},$$

$$x_{24n-19} = \frac{bdf(1+cg)}{cg(1+2bf)} \prod_{i=0}^{n-2} \frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)},$$

$$x_{24n-18}=\frac{ace(1+bf)}{bf(1+2ae)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+4)cg)}{(1+(8i+10)ae)(1+(8i+5)bf)(1+(8i+8)cg)},$$

$$x_{24n-17}=\frac{bcg(1+ae)}{ae(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+4)bf)(1+(8i+7)cg)}{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+11)cg)},$$

$$x_{24n-16}=\frac{abf(1+2cg)}{cg(1+3bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+4)ae)(1+(8i+7)bf)(1+(8i+10)cg)}{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+6)cg)},$$

$$x_{24n-15}=\frac{aceg(1+2bf)}{bdf(1+cg)(1+3ae)}\prod_{i=0}^{n-2}\frac{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)}{(1+(8i+11)ae)(1+(8i+6)bf)(1+(8i+9)cg)},$$

$$x_{24n-14}=\frac{bfg(1+2ae)}{ae(1+bf)(1+4cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+10)ae)(1+(8i+5)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)},$$

$$x_{24n-13}=\frac{aef(1+3cg)}{cg(1+ae)(1+4bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+5)ae)(1+(8i+8)bf)(1+(8i+11)cg)}{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)},$$

$$x_{24n-12}=\frac{ceg(1+3bf)}{bf(1+2cg)(1+4ae)}\prod_{i=0}^{n-2}\frac{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+6)cg)}{(1+(8i+12)ae)(1+(8i+7)bf)(1+(8i+10)cg)},$$

$$x_{24n-11}=\frac{bdf(1+3ae)(1+cg)}{ae(1+2bf)(1+5cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+11)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+13)cg)},$$

$$x_{24n-10}=\frac{ae(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)},$$

$$x_{24n-9}=\frac{cg(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)},$$

$$x_{24n-8}=\frac{bf(1+4ae)(1+2cg)}{e(1+3bf)(1+6cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+12)ae)(1+(8i+7)bf)(1+(8i+10)cg)}{(1+(8i+8)ae)(1+(8i+11)bf)(1+(8i+14)cg)},$$

$$x_{24n-7}=\frac{ae(1+2bf)(1+5cg)}{d(1+3ae)(1+6bf)(1+cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+13)cg)}{(1+(8i+11)ae)(1+(8i+14)bf)(1+(8i+9)cg)}.$$

Now, it follows from equation (3.1) that

$$x_{24n-6}=\frac{x_{24n-9}x_{24n-13}}{x_{24n-10}(1+x_{24n-9}x_{24n-13})}$$

$$=\frac{\frac{ae}{(1+5ae)}\prod_{i=0}^{n-2}\frac{1+(8i+5)ae}{1+(8i+13)ae}}{\frac{ae(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{1+\frac{ae}{1+5ae}\prod_{i=0}^{n-2}\frac{1+(8i+5)ae}{1+(8i+13)ae}\}}$$

$$=\frac{\frac{1}{(1+5ae)}\prod_{i=0}^{n-2}\frac{1+(8i+5)ae}{1+(8i+13)ae}}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{1+\frac{ae}{1+5ae}\prod_{i=0}^{n-2}\frac{1+(8i+5)ae}{1+(8i+13)ae}\}}$$

$$= \frac{\frac{1}{(1+5ae)}\left\{\frac{(1+5ae)(1+13ae)...(1+(8n-11)ae)}{(1+13ae)(1+21ae)...(1+(8n-3)ae)}\right\}}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\left\{1+\frac{ae}{(1+5ae)}\left\{\frac{(1+5ae)(1+13ae)...(1+(8n-11)ae)}{(1+13ae)(1+21ae)...(1+(8n-3)ae)}\right\}\right\}}$$

$$= \frac{\frac{1}{(1+(8n-3)ae)}}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\left\{1+\frac{ae}{(1+(8n-3)ae)}\right\}}$$

$$= \frac{1}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}(1+(8n-3)ae)\left\{1+\frac{ae}{(1+(8n-3)ae)}\right\}}$$

$$= \frac{1}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{(1+(8n-3)ae)+ae)\}}$$

$$= \frac{1}{\frac{(1+bf)(1+4cg)}{g(1+2ae)(1+5bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)}{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}\{(1+(8n-2)ae\}}$$

$$= \frac{g(1+2ae)(1+5bf)}{(1+bf)(1+4cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+10)ae)(1+(8i+13)bf)(1+(8i+8)cg)}{(1+(8i+6)ae)(1+(8i+9)bf)(1+(8i+12)cg)\{(1+(8n-2)ae\}}.$$

Hence,

$$x_{24n-6} = g\prod_{i=0}^{n-1}\frac{(1+(8i+2)ae)(1+(8i+5)bf)(1+(8i)cg)}{(1+(8i+6)ae)(1+(8i+1)bf)(1+(8i+4)cg)}.$$

Similarly, we have

$$x_{24n-5} = \frac{x_{24n-8}x_{24n-12}}{x_{24n-9}(1+x_{24n-8}x_{24n-12})}$$

$$= \frac{\frac{cg}{(1+6cg)}\prod_{i=0}^{n-2}\frac{1+(8i+6)cg}{1+(8i+4)cg}}{\frac{cg(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\left\{1+\frac{cg}{(1+6cg)}\prod_{i=0}^{n-2}\frac{1+(8i+6)cg}{1+(8i+4)cg}\right\}}$$

$$= \frac{\frac{1}{(1+6cg)}\prod_{i=0}^{n-2}\frac{1+(8i+6)cg}{1+(8i+4)cg}}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\left\{1+\frac{cg}{(1+6cg)}\prod_{i=0}^{n-2}\frac{1+(8i+6)cg}{1+(8i+4)cg}\right\}}$$

$$= \frac{\frac{1}{(1+6cg)}\left\{\frac{(1+6cg)(1+14cg)...(1+(8n-10)cg)}{(1+14cg)(1+22cg)...(1+(8n-2)cg)}\right\}}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\left\{1+\frac{cg}{(1+6cg)}\left\{\frac{(1+6cg)(1+14cg)...(1+(8n-10)cg)}{(1+14cg)(1+22cg)...(1+(8n-2)cg)}\right\}\right\}}$$

$$= \frac{\frac{1}{(1+(8n-3)cg)}}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\left\{1+\frac{cg}{(1+(8n-3)cg)}\right\}}$$

$$= \frac{1}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}(1+(8n-2)ae)\{1+\frac{cg}{(1+(8n-2)cg)}\}}$$

$$= \frac{1}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\{(1+(8n-2)cg)+cg\}}$$

$$= \frac{1}{\frac{(1+ae)(1+4bf)}{f(1+5ae)(1+3cg)}\prod_{i=0}^{n-2}\frac{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}\{(1+(8n-1)cg)\}}$$

$$= \frac{f(1+5ae)(1+3cg)}{(1+ae)(1+4bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+13)ae)(1+(8i+8)bf)(1+(8i+11)cg)}{(1+(8i+9)ae)(1+(8i+12)bf)(1+(8i+7)cg)}\frac{1}{\{(1+(8n-1)cg)\}}.$$

Then, we have

$$x_{24n-5} = f\prod_{i=0}^{n-1}\frac{(1+(8i+5)ae)(1+(8i)bf)(1+(8i+3)cg)}{(1+(8i+1)ae)(1+(8i+4)bf)(1+(8i+7)cg)}.$$

Again, applying the same steps,

$$x_{24n+1} = \frac{x_{24n-2}x_{24n-6}}{x_{24n-3}(1+x_{24n-2}x_{24n-6})}$$

$$= \frac{cg\prod_{i=0}^{n-1}\frac{1+(8i)cg}{1+(8i+8)cg}}{d\prod_{i=0}^{n-1}d\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+cg\prod_{i=0}^{n-1}\frac{1+(8i)cg}{1+(8i+8)cg}\}}$$

$$= \frac{cg\{\frac{(1+8cg)(1+16cg)...(1+(8n-16)cg)(1+(8n-8)cg)}{(1+8cg)(1+16cg)...(1+(8n-8)cg)(1+(8n)cg)}\}}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+cg\{\frac{(1+8cg)(1+16cg)...(1+(8n-16)cg)(1+(8n-8)cg)}{(1+8cg)(1+16cg)...(1+(8n-8)cg)(1+(8n)cg)}\}\}}$$

$$= \frac{\frac{cg}{(1+(8n)cg)}}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+\frac{cg}{(1+(8n)cg)}\}}$$

$$= \frac{cg}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}(1+(8n)cg)\{1+\frac{cg}{(1+(8n)cg)}\}}$$

$$= \frac{cg}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+(8n)cg+cg\}}$$

$$= \frac{cg}{d\prod_{i=0}^{n-1}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+1)cg)}{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}\{1+(8n)cg+cg\}}.$$

Hence,

$$x_{24n+1} = \frac{cg}{d(1+cg)} \prod_{i=0}^{n-1} \frac{(1+(8i+7)ae)(1+(8i+2)bf)(1+(8i+5)cg)}{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)}.$$

Consequently, we can easily obtain the solutions of the other relations. Thus, the proof is completed. □

Theorem 3.2. Equation (3.1) has a unique equilibrium point $x^* = 0$ which is not locally asymptotically stable.

Proof. For the equilibrium points of equation (3.1), we can write

$$x^* = \frac{x^{*2}}{x^*(1+x^{*2})},$$

$$\Rightarrow \quad x^{*2}(1+x^{*2}) = x^{*2} \qquad \Rightarrow \quad 1+x^{*2} = 1.$$

Thus, the equilibrium point of equation (3.1) is $x^* = 0$.
Now, let F be a function define by

$$F(u,v,w) = \frac{uw}{v(1+uw)}.$$

Therefore,

$$F_u(u,v,w) = \frac{w}{v(1+uw)^2}, \quad F_v(u,v,w) = \frac{-uw}{v^2(1+uw)}, \quad F_w(u,v,w) = \frac{u}{v(1+uw)^2}.$$

Then,

$$F_u(x^*,x^*,x^*) = 1, \quad F_v(x^*,x^*,x^*) = -1, \quad F_w(x^*,x^*,x^*) = 1.$$

It follows from Theorem (2.1) that equation (3.1) is not asymptotically stable. □

Numerical Examples

To confirm the result of the first subsection, we assume the following numerical examples which illustrate difference types of solutions to equation (3.1).

Example 3.1. We put $x_{-6} = 0.43,\ x_{-5} = 0.22,\ x_{-4} = 0.1,\ x_{-3} = 0.4,\ x_{-2} = 0.33,\ x_{-1} = 0.7,\ x_0 = 0.5$ in equation (3.1). So from Figure 1, we can see the behavior of the solution of equation equation (3.1), where the solution dose not converge to zero which prove the fact that the equilibrium point 0 is not locally asymptotically stable .

Example 3.2. In Figure 2, since $x_{-6} = 7,\ x_{-5} = 6,\ x_{-4} = 5,\ x_{-3} = 4,\ x_{-2} = 3,\ x_{-1} = 2,\ x_0 = 1$, we assure the same result of the previous example.

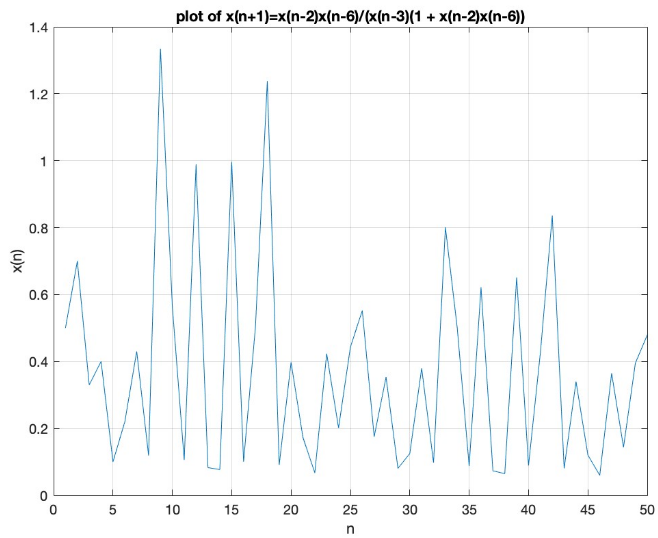

FIGURE 1

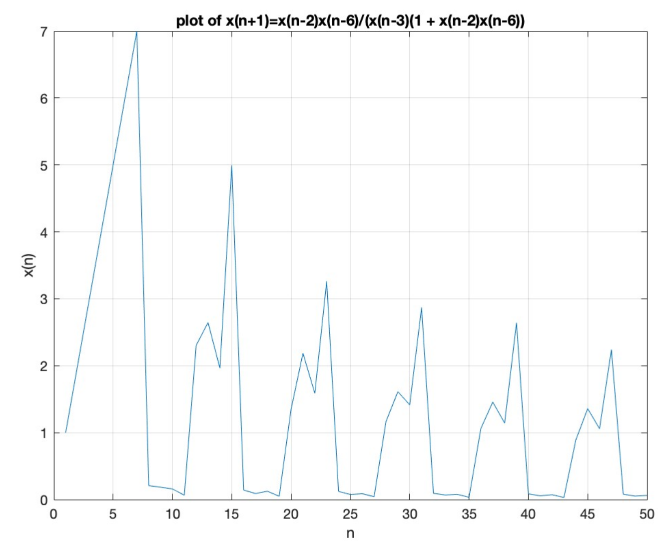

FIGURE 2

4. On the difference equation $x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(-1+x_{n-2}x_{n-6})}$

In this section, we study the second following case of the equation (1.1) in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(-1 + x_{n-2}x_{n-6})}. \tag{4.1}$$

Theorem 4.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of equation (4.1). Then the solutions of equation (4.1) are periodic of period 24 and given by:

$$x_{24n-6} = g, \qquad x_{24n-5} = f,$$

$$x_{24n-4} = e, \qquad x_{24n-3} = d,$$

$$x_{24n-2} = c, \qquad x_{24n-1} = b,$$

$$x_{24n} = a, \qquad x_{24n+1} = \frac{cg}{d(-1+cg)},$$

$$x_{24n+2} = \frac{bf}{c(-1+bf)}, \qquad x_{24n+3} = \frac{ae}{b(-1+ae)},$$

$$x_{24n+4} = \frac{cg}{a}, \qquad x_{24n+5} = \frac{bdf(-1+cg)}{cg},$$

$$x_{24n+6} = \frac{ace(-1+bf)}{bf}, \qquad x_{24n+7} = \frac{bcg(-1+ae)}{ae(-1+cg)},$$

$$x_{24n+8} = \frac{abf}{cg(-1+bf)}, \qquad x_{24n+9} = \frac{(ae)(cg)}{bdf(-1+ae)(-1+cg)},$$

$$x_{24n+10} = \frac{(bf)g}{(ae)(-1+bf)}, \qquad x_{24n+11} = \frac{(aef)(-1+cg)}{(cg)(-1+ae)},$$

$$x_{24n+12} = \frac{ceg(-1+bf)}{bf}, \qquad x_{24n+13} = \frac{bdf(-1+ae)}{ae},$$

$$x_{24n+14} = \frac{ae}{g}, \qquad \mathrm{x}_{24n+15} = \frac{cg}{f(-1+cg)},$$

$$x_{24n+16} = \frac{bf}{e(-1+bf)}, \qquad x_{24n+17} = \frac{ae}{d(-1+ae)},$$

where $x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b, x_0 = a$ are arbitrary nonzero real numbers with initial conditions $x_{-2}x_{-6} \neq 1,\ \ x_{-1}x_{-5} \neq 1,$ $x_0 x_{-4} \neq 1$.

Proof. For $n = 0$ the conclusion holds. Now, suppose that $n > 0$ and our assumption holds for $n - 1$. Then,

$$x_{24n-30} = g, \qquad x_{24n-29} = f,$$

$$x_{24n-28} = e, \qquad x_{24n-27} = d,$$

$$x_{24n-26} = c, \qquad x_{24n-25} = b,$$

$$x_{24n-24} = a, \qquad x_{24n-23} = \frac{cg}{d(-1+cg)},$$

$$x_{24n-22} = \frac{bf}{c(-1+bf)}, \qquad x_{24n-21} = \frac{ae}{b(-1+ae)},$$

$$x_{24n-20} = \frac{cg}{a}, \qquad x_{24n-19} = \frac{bdf(-1+cg)}{cg},$$

$$x_{24n-18} = \frac{ace(-1+bf)}{bf}, \qquad x_{24n-17} = \frac{bcg(-1+ae)}{ae(-1+cg)},$$

$$x_{24n-16} = \frac{abf}{cg(-1+bf)}, \qquad x_{24n-15} = \frac{(ae)(cg)}{bdf(-1+ae)(-1+cg)},$$

$x_{24n-14} = \frac{bfg}{ae(-1+bf)}, \qquad x_{24n-13} = \frac{aef(-1+cg)}{cg(-1+ae)},$

$x_{24n-12} = \frac{ceg(-1+bf)}{bf}, \qquad x_{24n-11} = \frac{bdf(-1+ae)}{ae},$

$x_{24n-10} = \frac{ae}{g}, \qquad x_{24n-9} = \frac{cg}{f(-1+cg)},$

$x_{24n-8} = \frac{bf}{e(-1+bf)}, \qquad x_{24n-7} = \frac{ae}{d(-1+ae)}.$

Now, we proof some of the relations of equation (4.1).

$$x_{24n-6} = \frac{x_{24n-9}x_{24n-13}}{x_{24n-10}(-1+x_{24n-9}x_{24n-13})}$$

$$= \frac{\frac{cg}{f(-1+cg)}\frac{aef(-1+cg)}{cg(-1+ae)}}{\frac{ae}{g}\{-1+\{\frac{cg}{f(-1+cg)}\frac{aef(-1+cg)}{cg(-1+ae)}\}\}} = \frac{\frac{ae}{-1+ae}}{\frac{ae}{g}\{-1+\{\frac{ae}{-1+ae}\}\}}$$

$$= \frac{1}{\frac{1}{g}(-1+ae)\{-1+\{\frac{ae}{-1+ae}\}\}}v = \frac{g}{1-ae+ae} = g.$$

Similarly,

$$x_{24n+7} = \frac{x_{24n+4}x_{24n}}{x_{24n+3}(-1+x_{24n+4}x_{24n})} = \frac{\frac{cg}{a}(a)}{\frac{ae}{b(-1+ae)}\{-1+\frac{cg}{a}(a)\}}$$

$$= \frac{cg}{\frac{ae}{b(-1+ae)}\{-1+cg\}} = \frac{bcg(-1+ae)}{ae\{-1+cg\}}.$$

Also,

$$x_{24n+12} = \frac{x_{24n+9}x_{24n+5}}{x_{24n+8}(-1+x_{24n+9}x_{24n+5})}$$

$$= \frac{\frac{(ae)(cg)}{bdf(-1+ae)(-1+cg)}\frac{bdf(-1+cg)}{cg}}{\frac{abf}{cg(-1+bf)}\{-1+\{\frac{(ae)(cg)}{bdf(-1+ae)(-1+cg)}\frac{bdf(-1+cg)}{cg}\}}$$

$$= \frac{\frac{ae}{-1+ae}}{\frac{abf}{cg(-1+bf)}\{-1+\frac{ae}{-1+ae}\}} = \frac{ae}{\frac{abf}{cg(-1+bf)}} = \frac{ecg(-1+bf)}{bf}.$$

Hence, we can easily proof the other relations. Thus, the proof has been done. $\square$

Theorem 4.2. Equation (4.1) has three equilibrium points which are 0 and $\pm\sqrt{2}$, where they are not locally asymptotically stable.

Proof. By using equation (4.1), and for the equilibrium points of (4.1) we can write

$$x^* = \frac{x^{*2}}{x^*(-1+x^{*2})}.$$

Then we have,

$$x^{*2}(-1+x^{*2}) = x^{*2},$$

or

$$x^{*2}(x^{*2}-2) = 0.$$

Thus, 0, $\pm\sqrt{2}$ are the equilibrium points.

Now, let F be a function define by

$$F(u,v,w) = \frac{uw}{v(-1+uw)}.$$

Therefore,

$$F_u(u,v,w) = \frac{-w}{v(-1+uw)^2},\ \ F_v(u,v,w) = \frac{-uw}{v^2(-1+uw)},\ \ F_w(u,v,w) = \frac{-u}{v(-1+uw)^2}.$$

Then,

$$F_u(x^*,x^*,x^*) = -1,\quad F_v(x^*,x^*,x^*) = \pm 1,\quad F_w(x^*,x^*,x^*) = -1.$$

Furthermore, we see from Theorem (2.1) that equation (4.1) is not asymptotically stable. □

Numerical Examples.

Conforming the result of the second subsection, we consider the following numerical examples which illustrate difference types of solutions to equation (4.1).

Example 4.1. In Figure 3 if we take the initial conditions as $x_{-6} = 5,\ x_{-5} = 3,\ x_4 = 4,\ x_{-3} = 1,\ x_{-2} = 1,\ x_{-1} = 3,\ x_0 = 4$, then we see that the behavior of the solution of equation (4.1) doesn't converge to the equilibrium points zero or $\pm\sqrt{2}$, which confirm the result of Theorem (4.2.).

Example 4.2. Consider $x_{-6} = 0.1,\ x_{-5} = 0.2,\ x_{-4} = 0.3,\ x_{-3} = 0.4,\ x_{-2} = 0.5,\ x_{-1} = 0.6,\ x_0 = 0.7$. In Figure 4, we get the same result of Example 4.1.

5. On the difference equation $x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(1-x_{n-2}x_{n-6})}$

In this section, we get the expressions of the solution of the third case of the equation (1.1):

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(1-x_{n-2}x_{n-6})}. \tag{5.1}$$

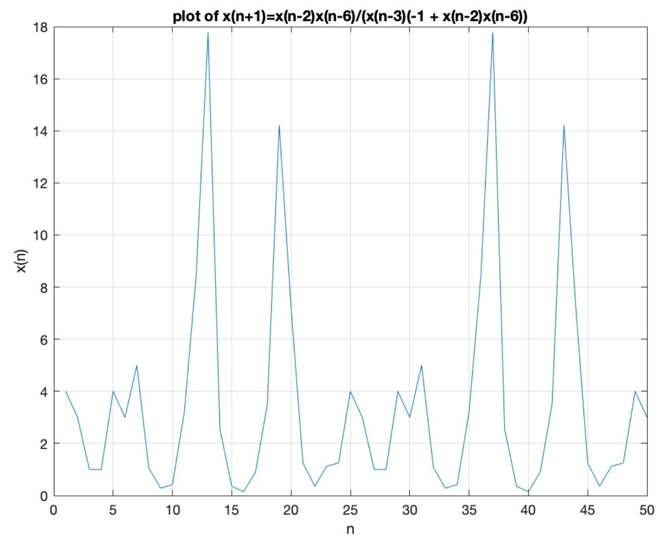

FIGURE 3

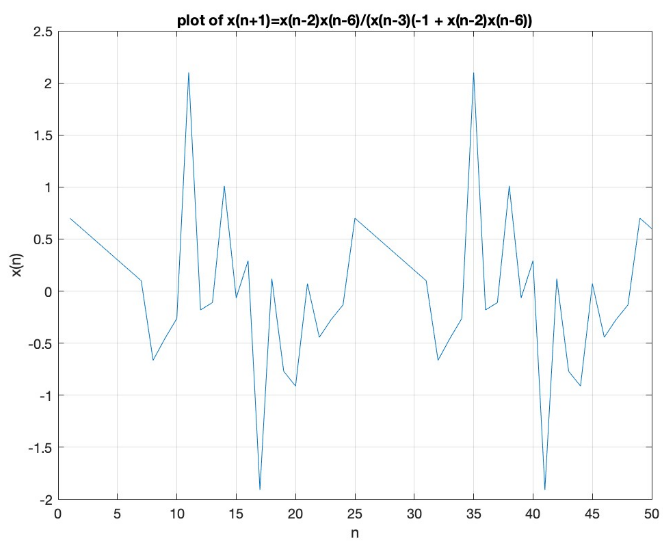

FIGURE 4

Theorem 5.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of equation (5.1). Then

$$x_{24n-6} = g\prod_{i=0}^{n-1}\frac{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i)cg)}{(1-(8i+6)ae)(1-(8i+1)bf)(1-(8i+4)cg)},$$

$$x_{24n-5} = f\prod_{i=0}^{n-1}\frac{(1-(8i+5)ae)(1-(8i)bf)(1-(8i+3)cg)}{(1-(8i+1)ae)(1-(8i+4)bf)(1-(8i+7)cg)},$$

$$x_{24n-4} = e\prod_{i=0}^{n-1}\frac{(1-(8i)ae)(1-(8i+3)bf)(1-(8i+6)cg)}{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+2)cg)},$$

$$x_{24n-3} = d\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)},$$

$$x_{24n-2} = c\prod_{i=0}^{n-1}\frac{(1-(8i+6)ae)(1-(8i+1)bf)(1-(8i+4)cg)}{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i+8)cg)},$$

$$x_{24n-1} = b\prod_{i=0}^{n-1}\frac{(1-(8i+1)ae)(1-(8i+4)bf)(1-(8i+7)cg)}{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+3)cg)},$$

$$x_{24n} = a\prod_{i=0}^{n-1}\frac{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+2)cg)}{(1-(8i+8)ae)(1-(8i+3)bf)(1-(8i+6)cg)},$$

$$x_{24n+1} = \frac{cg}{d(1-cg)}\prod_{i=0}^{n-1}\frac{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+9)cg)},$$

$$x_{24n+2} = \frac{bf}{c(1-bf)}\prod_{i=0}^{n-1}\frac{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i+8)cg)}{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+4)cg)},$$

$$x_{24n+3} = \frac{ae}{b(1-ae)}\prod_{i=0}^{n-1}\frac{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+3)cg)}{(1-(8i+9)ae)(1-(8i+4)bf)(1-(8i+7)cg)},$$

$$x_{24n+4} = \frac{cg}{a(1-2cg)}\prod_{i=0}^{n-1}\frac{(1-(8i+8)ae)(1-(8i+3)bf)(1-(8i+6)cg)}{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+10)cg)},$$

$$x_{24n+5} = \frac{bdf(1-cg)}{cg(1-2bf)}\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+9)cg)}{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+5)cg)},$$

$$x_{24n+6} = \frac{ace(1-bf)}{bf(1-2ae)}\prod_{i=0}^{n-1}\frac{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+4)cg)}{(1-(8i+10)ae)(1-(8i+5)bf)(1-(8i+8)cg)},$$

$$x_{24n+7} = \frac{bcg(1-ae)}{ae(1-3cg)}\prod_{i=0}^{n-1}\frac{(1-(8i+9)ae)(1-(8i+4)bf)(1-(8i+7)cg)}{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+11)cg)},$$

$$x_{24n+8} = \frac{abf(1-2cg)}{cg(1-3bf)}\prod_{i=0}^{n-1}\frac{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+10)cg)}{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+6)cg)},$$

$$x_{24n+9} = \frac{aceg(1-2bf)}{bdf(1-cg)(1-3ae)}\prod_{i=0}^{n-1}\frac{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+5)cg)}{(1-(8i+11)ae)(1-(8i+6)bf)(1-(8i+9)cg)},$$

$$x_{24n+10} = \frac{bfg(1-2ae)}{ae(1-bf)(1-4cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+10)ae)(1-(8i+5)bf)(1-(8i+8)cg)}{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+12)cg)},$$

$$x_{24n+11} = \frac{aef(1-3cg)}{cg(1-ae)(1-4bf)} \prod_{i=0}^{n-1} \frac{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+11)cg)}{(1-(8i+9)ae)(1-(8i+12)bf)(1-(8i+7)cg)},$$

$$x_{24n+12} = \frac{ceg(1-3bf)}{bf(1-2cg)(1-4ae)} \prod_{i=0}^{n-1} \frac{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+6)cg)}{(1-(8i+12)ae)(1-(8i+7)bf)(1-(8i+10)cg)},$$

$$x_{24n+13} = \frac{bdf(1-3ae)(1-cg)}{ae(1-2bf)(1-5cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+11)ae)(1-(8i+6)bf)(1-(8i+9)cg)}{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+13)cg)},$$

$$x_{24n+14} = \frac{ae(1-bf)(1-4cg)}{g(1-2ae)(1-5bf)} \prod_{i=0}^{n-1} \frac{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+12)cg)}{(1-(8i+10)ae)(1-(8i+13)bf)(1-(8i+8)cg)},$$

$$x_{24n+15} = \frac{cg(1-ae)(1-4bf)}{f(1-5ae)(1-3cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+9)ae)(1-(8i+12)bf)(1-(8i+7)cg)}{(1-(8i+13)ae)(1-(8i+8)bf)(1-(8i+11)cg)},$$

$$x_{24n+16} = \frac{bf(1-4ae)(1-2cg)}{e(1-3bf)(1-6cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+12)ae)(1-(8i+7)bf)(1-(8i+10)cg)}{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+14)cg)},$$

$$x_{24n+17} = \frac{ae(1-2bf)(1-5cg)}{d(1-3ae)(1-6bf)(1-cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+13)cg)}{(1-(8i+11)ae)(1-(8i+14)bf)(1-(8i+9)cg)}.$$

where $x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a$ are arbitrary nonzero real numbers.

Proof. The result holds for $n = 0$. Now, assume that $n > 0$ and our assumption holds for $n-1$. Then,

$$x_{24n-30} = g \prod_{i=0}^{n-2} \frac{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i)cg)}{(1-(8i+6)ae)(1-(8i+1)bf)(1-(8i+4)cg)},$$

$$x_{24n-29} = f \prod_{i=0}^{n-2} \frac{(1-(8i+5)ae)(1-(8i)bf)(1-(8i+3)cg)}{(1-(8i+1)ae)(1-(8i+4)bf)(1-(8i+7)cg)},$$

$$x_{24n-28} = e \prod_{i=0}^{n-2} \frac{(1-(8i)ae)(1-(8i+3)bf)(1-(8i+6)cg)}{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+2)cg)},$$

$$x_{24n-27} = d \prod_{i=0}^{n-2} \frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)},$$

$$x_{24n-26} = c\prod_{i=0}^{n-2}\frac{(1-(8i+6)ae)(1-(8i+1)bf)(1-(8i+4)cg)}{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i+8)cg)},$$

$$x_{24n-25} = b\prod_{i=0}^{n-2}\frac{(1-(8i+1)ae)(1-(8i+4)bf)(1-(8i+7)cg)}{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+3)cg)}$$

$$x_{24n-24} = a\prod_{i=0}^{n-2}\frac{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+2)cg)}{(1-(8i+8)ae)(1-(8i+3)bf)(1-(8i+6)cg)},$$

$$x_{24n-23} = \frac{cg}{d(1-cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+9)cg)},$$

$$x_{24n-22} = \frac{bf}{c(1-bf)}\prod_{i=0}^{n-2}\frac{(1-(8i+2)ae)(1-(8i+5)bf)(1-(8i+8)cg)}{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+4)cg)},$$

$$x_{24n-21} = \frac{ae}{b(1-ae)}\prod_{i=0}^{n-2}\frac{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+3)cg)}{(1-(8i+9)ae)(1-(8i+4)bf)(1-(8i+7)cg)},$$

$$x_{24n-20} = \frac{cg}{a(1-2cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+8)ae)(1-(8i+3)bf)(1-(8i+6)cg)}{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+10)cg)},$$

$$x_{24n-19} = \frac{bdf(1-cg)}{cg(1-2bf)}\prod_{i=0}^{n-2}\frac{(1+(8i+3)ae)(1+(8i+6)bf)(1+(8i+9)cg)}{(1+(8i+7)ae)(1+(8i+10)bf)(1+(8i+5)cg)},$$

$$x_{24n-18} = \frac{ace(1-bf)}{bf(1-2ae)}\prod_{i=0}^{n-2}\frac{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+4)cg)}{(1-(8i+10)ae)(1-(8i+5)bf)(1-(8i+8)cg)},$$

$$x_{24n-17} = \frac{bcg(1-ae)}{ae(1-3cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+9)ae)(1-(8i+4)bf)(1-(8i+7)cg)}{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+11)cg)},$$

$$x_{24n-16} = \frac{abf(1-2cg)}{cg(1-3bf)}\prod_{i=0}^{n-2}\frac{(1-(8i+4)ae)(1-(8i+7)bf)(1-(8i+10)cg)}{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+6)cg)},$$

$$x_{24n-15} = \frac{aceg(1-2bf)}{bdf(1-cg)(1-3ae)}\prod_{i=0}^{n-2}\frac{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+5)cg)}{(1-(8i+11)ae)(1-(8i+6)bf)(1-(8i+9)cg)},$$

$$x_{24n-14} = \frac{bfg(1-2ae)}{ae(1-bf)(1-4cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+10)ae)(1-(8i+5)bf)(1-(8i+8)cg)}{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+12)cg)},$$

$$x_{24n-13} = \frac{aef(1-3cg)}{cg(1-ae)(1-4bf)}\prod_{i=0}^{n-2}\frac{(1-(8i+5)ae)(1-(8i+8)bf)(1-(8i+11)cg)}{(1-(8i+9)ae)(1-(8i+12)bf)(1-(8i+7)cg)},$$

$$x_{24n-12} = \frac{ceg(1-3bf)}{bf(1-2cg)(1-4ae)}\prod_{i=0}^{n-2}\frac{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+6)cg)}{(1-(8i+12)ae)(1-(8i+7)bf)(1-(8i+10)cg)},$$

$$x_{24n-11} = \frac{bdf(1-3ae)(1-cg)}{ae(1-2bf)(1-5cg)}\prod_{i=0}^{n-2}\frac{(1-(8i+11)ae)(1-(8i+6)bf)(1-(8i+9)cg)}{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+13)cg)},$$

$$x_{24n-10} = \frac{ae(1-bf)(1-4cg)}{g(1-2ae)(1-5bf)}\prod_{i=0}^{n-2}\frac{(1-(8i+6)ae)(1-(8i+9)bf)(1-(8i+12)cg)}{(1-(8i+10)ae)(1-(8i+13)bf)(1-(8i+8)cg)},$$

$$x_{24n-9} = \frac{cg(1-ae)(1-4bf)}{f(1-5ae)(1-3cg)} \prod_{i=0}^{n-2} \frac{(1-(8i+9)ae)(1-(8i+12)bf)(1-(8i+7)cg)}{(1-(8i+13)ae)(1-(8i+8)bf)(1-(8i+11)cg)},$$

$$x_{24n-8} = \frac{bf(1-4ae)(1-2cg)}{e(1-3bf)(1-6cg)} \prod_{i=0}^{n-2} \frac{(1-(8i+12)ae)(1-(8i+7)bf)(1-(8i+10)cg)}{(1-(8i+8)ae)(1-(8i+11)bf)(1-(8i+14)cg)},$$

$$x_{24n-7} = \frac{ae(1-2bf)(1-5cg)}{d(1-3ae)(1-6bf)(1-cg)} \prod_{i=0}^{n-2} \frac{(1-(8i+7)ae)(1-(8i+10)bf)(1-(8i+13)cg)}{(1-(8i+11)ae)(1-(8i+14)bf)(1-(8i+9)cg)}.$$

Now, it follows from equation (5.1) that,

$$x_{24n+1} = \frac{x_{24n-2}x_{24n-6}}{x_{24n-3}(1-x_{24n-2}x_{24n-6})}$$

$$= \frac{cg\prod_{i=0}^{n-1}\frac{(1-(8i)cg)}{1-(8i+8)cg}}{d\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}\{1-cg\prod_{i=0}^{n-1}\frac{1-(8i)cg}{1-(8i+8)cg}\}}$$

$$= \frac{cg\{\frac{(1-8cg)(1-16cg)...(1-(8n-16)cg)(1-(8n-8)cg)}{(1-8cg)(1-16cg)...(1-(8n-8)cg)(1-(8n)cg)}\}}{d\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}\{1-cg\{\frac{(1-8cg)(1-16cg)...(1-(8n-16)cg)(1-(8n-8)cg)}{(1-8cg)(1-16cg)...(1-(8n-8)cg)(1-(8n)cg)}\}\}}$$

$$= \frac{\frac{cg}{(1-(8n)cg)}}{d\prod_{i=0}^{n-1}\frac{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+1)cg)}{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}\{1-\frac{cg}{(1+(8n)cg)}\}}$$

$$= \frac{cg}{d(1-cg)} \prod_{i=0}^{n-1} \frac{(1-(8i+7)ae)(1-(8i+2)bf)(1-(8i+5)cg)}{(1-(8i+3)ae)(1-(8i+6)bf)(1-(8i+9)cg)}.$$

We can easily proof the solutions of the other relations. Thus, the proof is completed. □

Theorem 5.2. Equation (5.1) has a unique equilibrium point that is number zero and this equilibrium point is not locally asymptotically stable.

Proof. As the proof of Theorem 3.2, and will be omitted. □

Numerical Examples.

In the next examples we can verify the result of Theorem (5.2.), that the solution does not converge to the equilibrium point 0.

Example 5.1. Assume the initial values of equation (5.1) are $x_{-6}=2,\ x_{-5}=1,\ x_{-4}=2,\ x_{-3}=3,\ x_{-2}=4,\ x_{-1}=2,\ x_0=5$. The behavior in Figure 5 shows that the solution of equation equation (5.1) dose not converge to zero which prove the result of Theorem (5.2.)

Example 5.2. See Figure 6 since (5.1) are $x_{-6}=-1,\ x_{-5}=0.2,\ x_{-4}=-3,\ x_{-3}=0.4,\ x_{-2}=3,\ x_{-1}=-4,\ x_0=-5.$, we got the same result of the previous example.

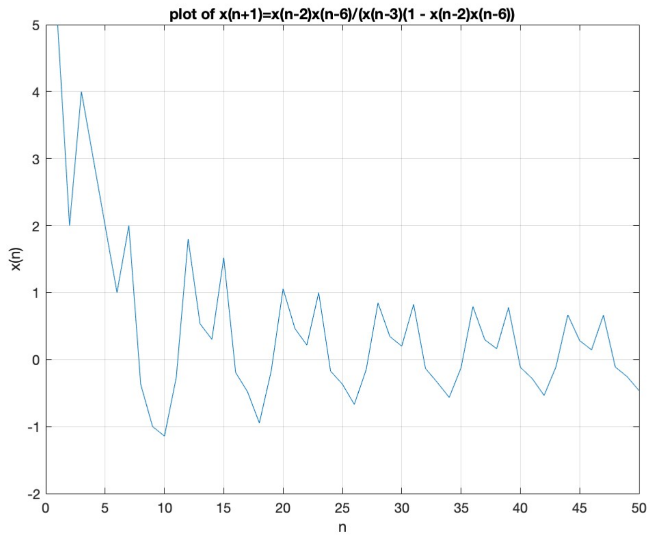

FIGURE 5

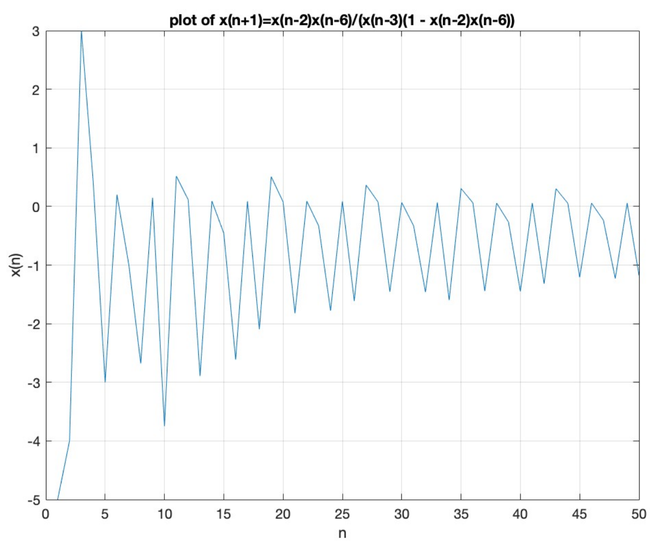

FIGURE 6

6. On the difference equation $x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(-1-x_{n-2}x_{n-6})}$

In this section, we study the last case of the equation (1.1) in the form:

$$x_{n+1} = \frac{x_{n-2}x_{n-6}}{x_{n-3}(-1 - x_{n-2}x_{n-6})}. \tag{6.1}$$

Theorem 6.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of equation (6.1). Then the solutions of equation (6.1) are periodic of period 24 and given by:

$x_{24n-6} = g,$ $\quad x_{24n-5} = f,$

$x_{24n-4} = e,$ $\quad x_{24n-3} = d,$

$x_{24n-2} = c,$ $\quad x_{24n-1} = b,$

$x_{24n} = a,$ $\quad x_{24n+1} = \frac{cg}{d(-1-cg)},$

$x_{24n+2} = \frac{bf}{c(-1-bf)},$ $\quad x_{24n+3} = \frac{ae}{b(-1-ae)},$

$x_{24n+4} = \frac{cg}{a},$ $\quad x_{24n+5} = \frac{bdf(-1-cg)}{cg},$

$x_{24n+6} = \frac{ace(-1-bf)}{bf},$ $\quad x_{24n+7} = \frac{bcg(-1-ae)}{ae(-1-cg)},$

$x_{24n+8} = \frac{abf}{cg(-1-bf)},$ $\quad x_{24n+9} = \frac{(ae)(cg)}{bdf(-1-ae)(-1-cg)},$

$x_{24n+10} = \frac{(bf)g}{(ae)(-1-bf)},$ $\quad x_{24n+11} = \frac{(aef)(-1-cg)}{(cg)(-1-ae)},$

$x_{24n+12} = \frac{ceg(-1-bf)}{bf},$ $\quad x_{24n+13} = \frac{bdf(-1-ae)}{ae},$

$x_{24n+14} = \frac{ae}{g},$ $\quad 24n+15 = \frac{cg}{f(-1-cg)},$

$x_{24n+16} = \frac{bf}{e(-1-bf)},$ $\quad x_{24n+17} = \frac{ae}{d(-1-ae)}.$

where $x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a$ are arbitrary nonzero real numbers with initial conditions $x_{-2}x_{-6} \neq -1,\ x_{-1}x_{-5} \neq -1,$ $x_0x_{-4} \neq -1.$

Proof. For $n = 0$ the conclusion holds. Now, suppose that $n > 0$ and our assumption holds for $n - 1$. Then,

$x_{24n-30} = g,$ $\quad x_{24n-29} = f,$

$x_{24n-28} = e,$ $\quad x_{24n-27} = d,$

$x_{24n-26} = c,$ $\quad x_{24n-25} = b,$

$x_{24n-24} = a,$ $\quad x_{24n-23} = \frac{cg}{d(-1-cg)},$

$x_{24n-22} = \frac{bf}{c(-1-bf)}$, $x_{24n-21} = \frac{ae}{b(-1-ae)}$,

$x_{24n-20} = \frac{cg}{a}$, $x_{24n-19} = \frac{bdf(-1-cg)}{cg}$,

$x_{24n-18} = \frac{ace(-1-bf)}{bf}$, $x_{24n-17} = \frac{bcg(-1-ae)}{ae(-1-cg)}$,

$x_{24n-16} = \frac{abf}{cg(-1-bf)}$, $x_{24n-15} = \frac{(ae)(cg)}{bdf(-1-ae)(-1-cg)}$,

$x_{24n-14} = \frac{bfg}{ae(-1-bf)}$, $x_{24n-13} = \frac{aef(-1-cg)}{cg(-1-ae)}$,

$x_{24n-12} = \frac{ceg(-1-bf)}{bf}$, $x_{24n-11} = \frac{bdf(-1-ae)}{ae}$,

$x_{24n-10} = \frac{ae}{g}$, $x_{24n-9} = \frac{cg}{f(-1-cg)}$,

$x_{24n-8} = \frac{bf}{e(-1-bf)}$, $x_{24n-7} = \frac{ae}{d(-1-ae)}$.

Now, we proof some of the relations of equation (6.1).

$$x_{24n+2} = \frac{x_{24n-1}x_{24n-5}}{x_{24n-2}(-1 - x_{24n-1}x_{24n-5})} = \frac{bf}{c(-1-bf)}.$$

Similarly,

$$x_{24n+9} = \frac{x_{24n+6}x_{24n+2}}{x_{24n+5}(-1 - x_{24n+6}x_{24n+2})} = \frac{\frac{ace(-1-bf)}{bf}\frac{bf}{c(-1-bf)}}{\frac{bdf(-1-cg)}{cg}(-1 - \frac{ace(-1-bf)}{bf}\frac{bf}{c(-1-bf)})}$$

$$= \frac{(ae)(cg)}{bdf(-1-ae)(-1-cg)}.$$

Hence, we can easily proof the other relations. Thus, the proof has been done. □

Theorem 6.2. Equation (6.1) has equilibrium point $x^* = 0$ and it is not locally asymptotically stable.

Proof. The proof is similar to the proof of Theorem 3.2, and will be omitted.

Numerical Examples.

Example 6.1. Figure 7 shows the periodic solution of equation (5.1) where the initial conditions are $x_{-6} = 9,\ x_{-5} = 4,\ x_{-4} = 3,\ x_{-3} = 4,\ x_{-2} = 10,\ x_{-1} = 7,\ x_0 = 9$. Also, it shows that the solution of equation (6.1) doesn't converge to the 0 and this confirms that the equation (6.1) is not asymptotically stable.

Example 6.2. Also in Figure 8 we assure the same results of Example 6.1. where the initial conditions are $x_{-6} = 1,\ x_{-5} = 0.22,\ x_{-4} = 0.3,\ x_{-3} = 7,\ x_{-2} = 1.0,\ x_{-1} = 0.7,\ x_0 = 0.9$.

7. Conclusion

In this article we presents the solution of the difference equation (1.1). First, we obtained the form of the solution of four special cases of the difference equation (1.1) and investigated the existence of the equilibrium point, the global asymptotic behavior and the existence of a periodic solutions of these equations. By the end, we gave some numerical examples of each case with different initial values by using the mathematical program MATLAB to confirm the obtained results.

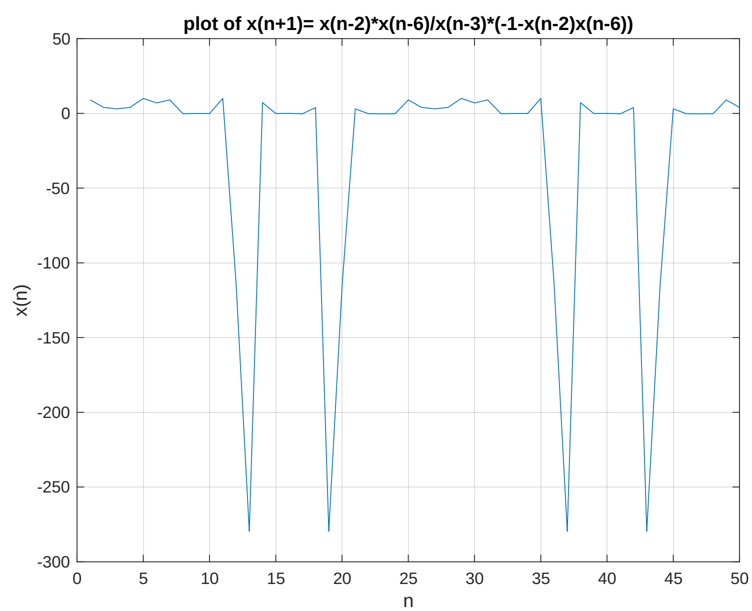

Figure 7

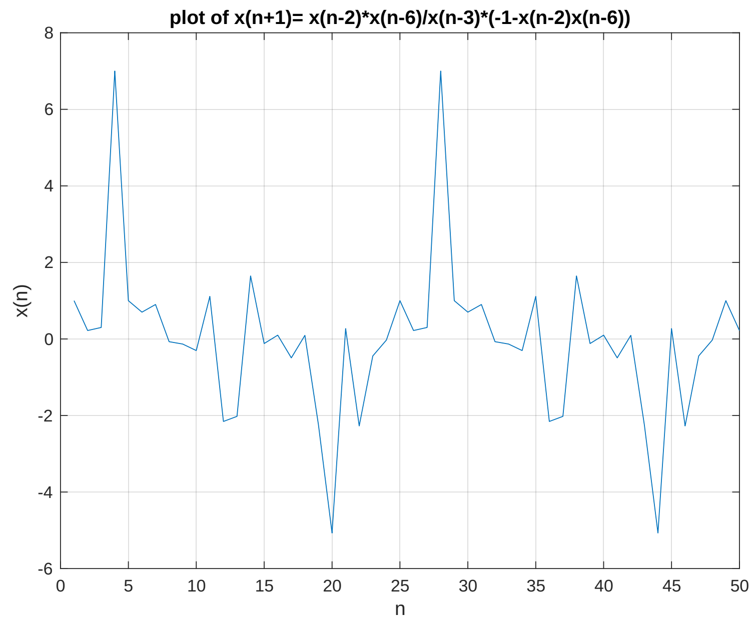

FIGURE 8

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This study does not be necessary ethical committee permission or any special permission.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

References

[1] V.L. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, (1993).

[2] E.A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations Vol. 4, Chapman and Hall/CRC, Boca Raton, (2005).

[3] S. Elaydi, An Introduction to Difference Equations. Springer, New York, NY, USA, (2005).

[4] M.R.S. Kulenovic, G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman Hall-CRC, London, (2002).

[5] R. E. Michens, Difference Equations Theory and Applications, 2nd, Van Nostrand Reinhold, New York, (1990).

[6] R. Abo-Zeid, On the Solutions of a Fourth Order Difference Equation, Universal Journal of Mathematics and Applications, Vol.4, No.2, pp.76-81 (2021).

[7] E. M. Elsayed, Qualitative behavior of a rational recursive sequence, Indagationes Mathematicae, New Series, Vol.19, No.2, pp.189-201 (2008).

[8] C. Cinar, On the positive solutions of the difference equations of the difference equation $x_{n+1} = \frac{\alpha x_{n-1}}{1+b_n x_n x_{n-1}}$, Appl. Math. Comp., Vol.156, pp.587-590 (2004).

[9] C. Cinar, On the positive solutions of the difference equations of the difference equation $x_{n+1} = \frac{x_{n-1}}{1+\alpha x_n x_{n-1}}$, Appl. Math. Comp., Vol.158, No.3, pp.809-812 (2004).

[10] C. Cinar, On the positive solutions of the difference equations of the difference equation $x_{n+1} = \frac{x_{n-1}}{-1+\alpha x_n x_{n-1}}$, Appl. Math. Comp., Vol.158, No.3, pp.793-797 (2004).

[11] T.F. Ibrahim, On the third order rational difference equation $x_{n-+1} = \frac{x_n x_{n-2}}{x_{n-1}(a+b x_n x_{n-2})}$, International Journal of Contemporary Mathematical Sciences, Vol.4, No.25-28, pp.1321-1334 (2009).

[12] F. Bozkurt, Stability analysis of nonlinear difference equation, International Journal of Modern Nonlinear Theory and Application, Vol.2, No.1, pp.1-6 (2013).

[13] D. Simsek, C. Cinar, I. Yalcinkaya, On the recursive sequence $x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$, International Journal of Contemporary Mathematical Sciences, Vol.1, No.9-12, pp.475-480 (2006).

[14] L. Xian, L. Wei, Global asymptotical stability in a rational difference equation, Appl. Math. J. Chinese Univ., Vol.36, No.1, pp.51-59 (2021).

[15] R. Karatas, C. Cinar, D. Simsek, On the positive solutions of the Difference equation $x_{n+1} = \frac{x_{n-5}}{-1+x_{n-2}x_{n-5}}$, Int. J. Contemp. Math. Sci., Vo.1, No.10, pp.495-500 (2006).

[16] H. El-Metwally, E. A. Grove, G. Ladas, R. Levins, M. Radin, On the difference equation $x_{n+1} = \alpha + \beta x_{n-1} e^{-xn}$, Nonlinear Analysis, Vol.47, pp.4623–4634 (2001).

[17] H. El-Metwally, E. A. Grove, G. Ladas, A global Convergence result with applications to periodic solutions, J. Math. Anal. Appl., Vol.245, pp.161-170 (2000).

[18] E. M. Elsayed, On the solution of recursive sequence of order two, Fasciculi Mathematicsa, Vol.40, pp.5-13 (2008).

[19] E.M.Elsayed, On the Difference equation $x_{n+1} = \frac{x_{n-5}}{-1+\alpha x_{n-2}x_{n-5}}$, International Journal of Contemporary Mathematical Sciences, Vol.3, No.3, pp.1657-1664 (2008).

[20] E. M. Elsayed, M. M. El-dessoky, Dynamics and global behavior for a fourth order rational difference equation, Hacettepe Journal of Mathematics and Statistics, Vol.42, No.5, pp.479-494 (2013).

[21] E. M. Elsayed, H. S. Gafel, D Dynamics and Global Stability of Second Order Nonlinear Difference Equation, Pan-American Journal of Mathematics, Vol.1, pp.16 (2022).

[22] T. F. Ibrahim, On the third order rational difference equation $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1(a+bx_n x_{n-2})}}$, , Int. J. Contemp. Math. Sciences, Vol.4, No.27, pp.1321-1334 (2009).

[23] T. F. Ibrahim, Oscillation, non-oscillation and asymptotic behavior for third order nonlinear difference equations, Dynamics of Continuous Discrete and Impulsive System Series, A Mathematical Analysis, Vol.20 No.4, pp.523-532 (2013).

[24] C. Zhang, H. X. Li, Dynamics Of A Rational Difference Equation Of Higher Order, Applied Mathematics E-Notes, Vol.9, pp.80-88 (2009).

[25] M. Saleh, M. Aloqeili, On the difference equation $x_{n+1} = A + \frac{y_n}{y_{n-k}}$ with $A < 0$, App. Math. Comp., Vol.176, No.1, pp.359-363 (2006).

[26] M. Saleh, S. Herzallah, Dynamics and Bifurcation of a Second Order Rational Difference Equation with Quadratic Terms, Journal of Applied Nonlinear Dynamics, Vol.10, No.3, pp.561-576 (2021).

[27] I. Yalcinkaya, On the difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$, Discrete Dynamics in Nature and society, pp.805-460 (2008).

[28] I. Yalcinkaya, On the difference equation, $x_{n+1} = \alpha + \frac{x_{n-2}}{x_n^k}$, Fasciculi Mathematici, Vol.42, pp.133-140 (2009).

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