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Lacunary Strongly Invariant Convergence in Fuzzy Normed Spaces

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Abstract
In this study, firstly, we defined the notions of lacunary invariant convergence and lacunary invariant Cauchy sequence in fuzzy normed spaces. Then, we introduced lacunary strongly invariant convergence in fuzzy normed spaces and we investigated some properties of these new concepts.

Keywords: *Fuzzy normed space; invariant convergence; lacunary convergence.*

AMS Subject Classification (2020): *Primary: 03E72 ; Secondary: 46S40; 40A05; 40A30.*

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1. Introduction

The idea of fuzzy sets initially introduced by Zadeh [1] to deal with imprecise phenomena as an alternative to classical set theory. After that, several classical concepts were reconstructed. Fuzzy topological spaces [2, 3], fuzzy metric [4–6], fuzzy norm [7–10] are just some of the examples. Felbin's fuzzy norm [9], which is associated with Kaleva and Seikkala [5] type metric space by assigning a non-negative fuzzy real number to each element of a linear space, forms the basis of this study. Das and Das [11] studied fuzzy topology generated by fuzzy norm. Diamond and Kloeden [12] investigated the metric spaces of fuzzy sets-theory and applications. Fang and Huang [13] studied on the level convergence of a sequence of fuzzy numbers. Recently Yalvaç and Dündar [14] defined the notions of invariant convergence and invariant Cauchy sequences with some properties and inclusions in fuzzy normed spaces. Also, some other authors [15–18] studied the concepts related to fuzzy numbers and fuzzy normed space.

Banach [19] defined the generalized limit, an application of Hahn-Banach theorem on the set of all bounded real valued sequences. It is also known as Banach limit. Later, Lorentz [20] offered that if all Banach limits of the given bounded sequence are equal, it is called almost convergent. In further studies [21, 22], invariant mean and invariant convergence are given as more general cases of Banach limit and almost convergence. Also, several authors including Schaefer [23], Mursaleen and Edely [24], Mursaleen [25, 26], Savaş [27, 28] had significant studies on invariant convergence.

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Freedman et al.[29] gave the relation between strong Cesaro convergent space and the sequence of integers (2^r) and offered lacunary convergence by taking lacunary sequences instead of (2^r). Further studies on this convergence were done by several authors [30, 31].

Now, we recall the basic notions and some essential definitions used in our paper (See [1, 7–10, 15, 16, 18, 20–26, 28–36]).

A fuzzy number is a fuzzy set provided that

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$ for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$;

(iii) u is upper semi-continuous;

(iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is a compact set.

The set of all fuzzy numbers is denoted by $L(\mathbb{R})$. $\mathbb{R}$ can be embedded in $L(\mathbb{R})$ since each $r \in \mathbb{R}$ considered a fuzzy real number $\tilde{r}$ defined by $\widetilde{r}(t) = 1$ if $t = r$ and $\widetilde{r}(t) = 0$ if $t \neq r$.

For $u \in L(\mathbb{R})$, the α-level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1], \\ cl\,\{x \in \mathbb{R} : u(x) > \alpha\}, & \text{if } \alpha = 0. \end{cases}$$

The α-level set of a fuzzy number, denoted by $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, is a non-empty, bounded and closed interval for each $\alpha \in [0,1]$ where $u_\alpha^- = -\infty$ and $u_\alpha^+ = \infty$ are also admissible.

If $u \in L(\mathbb{R})$ and $u(x) = 0$ for $x < 0$, then u is called a non-negative fuzzy number. The set of all non-negative fuzzy numbers is denoted by $L^*(\mathbb{R})$. It is easy to see $\tilde{0} \in L^*(\mathbb{R})$.

A partial ordering $\preceq$ in $L(\mathbb{R})$ is defined by for $u, v \in L(\mathbb{R})$,

$$u \preceq v \quad \text{iff} \quad u_\alpha^- \leq v_\alpha^- \quad \text{and} \quad u_\alpha^+ \leq v_\alpha^+ \quad \text{for all} \quad \alpha \in [0,1].$$

Arithmetic equations addition, multiplication and multiplication with a scaler on $L(\mathbb{R})$ are defined by

(i) $(u \oplus v)(t) = \sup_{s \in \mathbb{R}} \{u(s) \wedge v(t-s)\}, \quad t \in \mathbb{R}$,

(ii) $(u \odot v)(t) = \sup_{s \in \mathbb{R}, s \neq 0} \{u(s) \wedge v(t/s)\}, \quad t \in \mathbb{R}$,

(iii) For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \tilde{0}$, $t \in \mathbb{R}$.

Let $u, v \in L(\mathbb{R})$. Arithmetic equations in terms of α-level sets are defined by

(i) $[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$,

(ii) $[u \odot v]_\alpha = [u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+]$, $u, v \in L^*(\mathbb{R})$,

(iii) $[ku]_\alpha = k[u]_\alpha = \begin{cases} [ku_\alpha^-, ku_\alpha^+], & k \geq 0, \\ [ku_\alpha^+, ku_\alpha^-], & k < 0. \end{cases}$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined by

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\}.$$

One can see that

$$D\left(u, \tilde{0}\right) = \sup_{0 \leq \alpha \leq 1} \max\left\{\left|u_\alpha^-\right|, \left|u_\alpha^+\right|\right\} = \max\left\{\left|u_0^-\right|, \left|u_0^+\right|\right\}.$$

Obviously, $D\left(u, \tilde{0}\right) = u_0^+$ when $u \in L^*(\mathbb{R})$.

A sequence (u_n) in $L(\mathbb{R})$ is convergent to $u \in L(\mathbb{R})$, denoted by $D - \lim\limits_{n\to\infty} u_n = u$, if $\lim_{n\to\infty} D(u_n, u) = 0$, i.e., for all given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $D(u_n, u) < \varepsilon$, for all $n > n_0$.

Let X be a vector space over $\mathbb{R}$, $\|.\| : X \to L^*(\mathbb{R})$ and $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$, $R(1,1) = 1$.

The quadruple $(X, \|.\|, L, R)$ is called fuzzy normed linear space (FNS) and $\|.\|$ is a fuzzy norm if the following axioms are satisfied

(i) $\|x\| = \widetilde{0}$ iff $x = \theta$,

(ii) $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,

(iii) For all $x, y \in X$,

(a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x+y\|_1^-$,

(b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x+y\|_1^-$.

When $L = \min$ and $R = \max$ are taken in above (iii), triangle inequalities become

$$\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^- \text{ and } \|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$$

for all $\alpha \in (0,1]$. Since they fulfil the other conditions of norm, $\|x\|_\alpha^-$ and $\|x\|_\alpha^+$ can be seen as ordinary norms on X.

Example 1.1. Let $(X, \|.\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|.\|$ on X can be obtained

$$\|x\|(t) = \begin{cases} 0, & \text{if } 0 \le t \le a\|x\|_C \text{ or } t \ge b\|x\|_C, \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a}, & \text{if } a\|x\|_C \le t \le \|x\|_C, \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1}, & \text{if } \|x\|_C \le t \le b\|x\|_C, \end{cases}$$

where $\|x\|_C$ is the ordinary norm of $x\ (\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \widetilde{0}$. Hence $(X, \|.\|)$ is a fuzzy normed linear space.

Throughout paper let $(X, \|.\|)$ be an fuzzy normed linear space (FNS).

Let us consider the topological structure of the space X. For any $\varepsilon > 0$, $\alpha \in [0,1]$ and $x \in X$, the (ε, α)-neighborhood of x is the set $\mathcal{N}_x(\varepsilon, \alpha) := \{y \in X : \|x - y\|_\alpha^+ < \varepsilon\}$.

A sequence (x_n) in X is convergent to x with respect to the fuzzy norm, denoted by $x_n \overset{FN}{\to} x$, if it is provided that $(D) - \lim_{n\to\infty} \|x_n - x\| = \widetilde{0}$, i.e., for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$D\left(\|x_n - x\|, \widetilde{0}\right) = \sup_{\alpha \in [0,1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon,$$

for all $n > n_0$. In terms of neighborhoods, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in \mathcal{N}_x(\varepsilon, 0)$, for all $n > n_0$.

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if and only if

(i) $\phi(x) \ge 0$, when the sequence $x = (x_n)$ has $x_n \ge 0$ for all n,

(ii) $\phi(e) = 1$, where $e = (1, 1, 1...)$,

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and satisfied the condition $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m-th iterate of the mapping σ at n. Invariant mean ϕ is a extension of the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. The sequence is called invariant convergent when its invariant means are equal. In case $\sigma(n) = n + 1$, the σ-mean become Banach limit and invariant convergence become almost convergence.

A bounded sequence $x = (x_n)$ is σ-convergent to the number L if $\lim\limits_{m\to\infty} t_{mn} = L$ uniformly in n, where

$$t_{mn} = \frac{x_{\sigma(n)} + x_{\sigma^2(n)} + \cdots + x_{\sigma^m(n)}}{m}.$$

A sequence $x = (x_n)$ in X is invariant convergent to L with respect to fuzzy norm if $(D) - \lim_{m\to\infty} \|t_{mn} - L\| = \widetilde{0}$, uniformly in n. Namely, for given $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$,

$$D(\|t_{mn} - L\|, \widetilde{0}) = \sup_{\alpha \in [0,1]} \|t_{mn} - L\|_\alpha^+ = \|t_{mn} - L\|_0^+ < \varepsilon, \text{ for every } n \in \mathbb{N}.$$

Let $0 < q < \infty$. The sequence $x = (x_n)$ in X is q-strongly invariant convergent to L with respect to fuzzy norm if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \left[D\left(\|x_{\sigma^i(n)} - L\|, \widetilde{0}\right)\right]^q = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \left[\|x_{\sigma^i(n)} - L\|_0^+\right]^q = 0,$$

uniformly in n.

An increasing sequence of non-negative integers $\theta = (k_r)$ with $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ is called lacunary sequence. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is given by q_r.

For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary convergent to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} (x_i - L) = 0.$$

For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary strongly convergent to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - L| = 0.$$

For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary strongly convergent to L with respect to fuzzy norm if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} D\left(\|x_i - L\|, \widetilde{0}\right) = 0.$$

2. Main results

Definition 2.1. For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary invariant convergent to L with respect to fuzzy norm and it is denoted by $x_n \overset{\sigma - FN_\theta}{\longrightarrow} L$ if

$$\lim_{r\to\infty} D\left(\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|, \widetilde{0}\right) = \lim_{r\to\infty} \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ = 0,$$

unifomly in n; i.e., for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$D\left(\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|, \widetilde{0}\right) = \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$.

Definition 2.2. For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary invariant Cauchy sequence with respect to fuzzy norm if for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r, s > r_0$,

$$\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - \frac{1}{h_s}\sum_{j\in I_s} x_{\sigma^j(n)}\right\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$.

Theorem 2.1. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$ be a sequence in X. If x is lacunary invariant convergent to L with respect to fuzzy norm, then x is lacunary invariant Cauchy sequence with respect to fuzzy norm.*

Proof. Assume that the sequence $x = (x_n)$ is lacunary invariant convergent to L with respect to fuzzy norm in X. Then, for every $\varepsilon > 0$ there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$. Therefore for all $r, s > r_0$,

$$\begin{aligned}\left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - \frac{1}{h_s}\sum_{j\in I_s} x_{\sigma^j(n)}\right\|_0^+ &= \left\|\frac{1}{h_r}\sum_{i\in I_r} x_{\sigma^i(n)} - L\right\|_0^+ + \left\|\frac{1}{h_s}\sum_{j\in I_s} x_{\sigma^j(n)} - L\right\|_0^+ \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, x is lacunary invariant Cauchy sequence with respect to fuzzy norm. □

Definition 2.3. For any lacunary sequence $\theta = (k_r)$, the sequence $x = (x_n)$ in X is lacunary strongly invariant convergent to L with respect to fuzzy norm and it is denoted by $x_n \overset{[\sigma - FN]_\theta}{\longrightarrow} L$ if

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} D(\|x_{\sigma^i(n)} - L\|, \widetilde{0}) = \lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L\|_0^+ = 0,$$

unifomly in n; i.e., for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r} D(\|x_{\sigma^i(n)} - L\|, \widetilde{0}) = \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$.

Theorem 2.2. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$ be a sequence in X. If x is lacunary strongly invariant convergent to L, then L is unique.*

Proof. Assume that $x_n \overset{[\sigma-FN]_\theta}{\longrightarrow} L_1$, $x_n \overset{[\sigma-FN]_\theta}{\longrightarrow} L_2$ and $L_1 \neq L_2$. Then for every $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that for all $r > r_1$,

$$\frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_1\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$ and for given $\varepsilon > 0$, there exists r_2 such that for all $r > r_2$,

$$\frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_2\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$. Take $r_0 = \max\{r_1, r_2\}$. Then for all $r > r_0$,

$$\begin{aligned}
\|L_1 - L_2\|_0^+ &= \frac{1}{h_r}\sum_{i\in I_r} \|L_1 - L_2\|_0^+ \\
&\leq \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_1\|_0^+ + \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_2\|_0^+ \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

for all $n \in \mathbb{N}$. Since for all $\varepsilon > 0$,

$$\|L_1 - L_2\|_0^+ < \varepsilon,$$

we have $L_1 = L_2$. □

Theorem 2.3. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$, $y = (y_n)$ be sequences in X. If x and y are lacunary strongly invariant convergent to L_1 and L_2, respectively, then the sequence $x + y$ is lacunary strongly invariant convergent to $L_1 + L_2$.*

Proof. Assume that $x_n \overset{[\sigma-FN]_\theta}{\longrightarrow} L_1$ and $y_n \overset{[\sigma-FN]_\theta}{\longrightarrow} L_2$. Then for every $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that for all $r > r_1$,

$$\frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_1\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$ and for given $\varepsilon > 0$, there exists r_2 such that for all $r > r_2$,

$$\frac{1}{h_r}\sum_{i\in I_r} \|y_{\sigma^i(n)} - L_2\|_0^+ < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$. Take $r_0 = \max\{r_1, r_2\}$ then for all $r > r_0$,

$$\begin{aligned}
\frac{1}{h_r}\sum_{i\in I_r} \|(x_{\sigma^i(n)} + y_{\sigma^i(n)}) - (L_1 + L_2)\|_0^+ &\leq \frac{1}{h_r}\sum_{i\in I_r} \|x_{\sigma^i(n)} - L_1\|_0^+ + \frac{1}{h_r}\sum_{i\in I_r} \|y_{\sigma^i(n)} - L_2\|_0^+ \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, we have

$$(x_n + y_n) \overset{[\sigma-FN]_\theta}{\longrightarrow} (L_1 + L_2).$$

□

Theorem 2.4. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$ be sequence in X. If x is strongly lacunary invariant convergent to L and c is a scaler, then the sequence cx is strongly lacunary invariant convergent to cL.*

Proof. Assume that $x_n \overset{[\sigma-FN]_\theta}{\longrightarrow} L$ and c is a scaler. Then for every $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r}\|x_{\sigma^i(n)} - L_1\|_0^+ < \frac{\varepsilon}{|c|},$$

for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r}\|cx_{\sigma^i(n)} - cL_1\|_0^+ &= |c|\frac{1}{h_r}\sum_{i\in I_r}\|x_{\sigma^i(n)} - L_1\|_0^+ \\ &< |c|\frac{\varepsilon}{|c|} = \varepsilon,\end{aligned}$$

for all $n \in \mathbb{N}$. So, we conclude

$$cx_n \overset{[\sigma-FN]_\theta}{\longrightarrow} cL.$$

□

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence and $x = (x_n)$ be sequence in X. If the sequence x is strongly lacunary invariant convergent to L then x is lacunary invariant convergent to L.*

Proof. Assume that $x = (x_n)$ is strongly lacunary invariant convergent to L with respect to fuzzy norm. Then for all $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that for all $r > r_0$,

$$\frac{1}{h_r}\sum_{i\in I_r}\|x_{\sigma^i(n)} - L\|_0^+ < \varepsilon,$$

for all $n \in \mathbb{N}$. Since

$$\begin{aligned}\left\|\frac{1}{h_r}\sum_{i\in I_r}x_{\sigma^i(n)} - L\right\|_0^+ &\leq \frac{1}{h_r}\sum_{i\in I_r}\|x_{\sigma^i(n)} - L\|_0^+ \\ &< \varepsilon,\end{aligned}$$

for all $n \in \mathbb{N}$, then we obtain that x is lacunary invariant convergent to L with respect to fuzzy norm.

□

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