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Inverse Scattering Problem for the Sturm-Liouville Equation with Infinite Range of Discontinuous Conditions

Rauf Kh Amirov[1,*], Selma Gülyaz Özyurt[1]

[1]*Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, 58140 Sivas, Turkey.*

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Abstract. In this paper, we construct the new integral representation of the Jost solution of Sturm-Liouville equation with impuls in the semi axis $[0, +\infty)$ and we give this type of relation, examine the properties of the Kernel function and their partial derivatives with x and t, constructed integral representation and obtain the partial differential equation provided by this Kernel function. Finally, in the paper we prove uniqueness of the determination of the potential by the scattering data.

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1. Introduction

The boundary value problems for Sturm-Liouville equations with discontinuities inside the interval often appear in mathematical physics, geophysics, electromagnetic, elasticity and other branches of engineering and natural sciences. For example, we take a rod consisting of two homogeneous parts that have different modules of elasticity with cross section and connected at the point $x = a$. The vibration problem of this rod can be expressed as follows

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(\rho(x)\frac{\partial u(x,t)}{\partial x}\right), \qquad x \in (0,a) \cup (a,+\infty),\ t > 0,$$
$$u(0,t) = 0,$$
$$u(a-0,t) = u(a+0,t),$$
$$\rho(a-0)u_x(a-0,t) - \rho(a+0)u_x(a+0,t) = Mu_{tt}(a,t)\ ,$$

where M is a condensed mass at point $x = a$, $\rho(x)$ is the coefficient of elasticity of the rod and the piecewise continuous function

$$\rho(x) = \begin{cases} 1, & x < a, \\ \alpha^2, & x > a. \end{cases}$$

Moreover, the boundary value problems with discontinuities in an interior point arise in geophysical models for oscillations of the Earth [19, 20], in heat and mass transfer problems [7] and in vibrating stirring problems which an interior point is under the action of a damping force [12], in addition, for the various applications of discontinuous boundary value problems, the works [5, 9, 10, 13, 24–27] can be given. From the point of view applications in quantum

*Corresponding Author
Email addresses: emirov@cumhuriyet.edu.tr (R. Amirov), sgulyaz@cumhuriyet.edu.tr (S. Gülyaz Özyurt)
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mechanics [23], of interest is the study of direct and inverse problems for the Sturm-Liouville differential equation on a semi axis $[0, +\infty)$.

In recent years, Sturm-Liouville problems with discontinuity conditions (or transmission conditions) at interior points of the interval and semi axis $[0, +\infty)$ began to attract a great deal of attention because of the application of these problems in physics, mechanics and engineering. Hereby, this type of problems were widely investigated by many authors, also the investigations were continued and developed in many directions. For example, some aspect of the direct spectral problems were studied in [1–4, 6–10, 12–23] and the references therein, moreover, the inverse spectral problems (according to Weyl function and/or spectral data and etc.) are examined in the works [4, 7–9, 22] where further references can be found. In particular, in the paper [11] on noncompact A-graph consider scattering problem for Sturm-Liouville operator with standard matching conditions in the internal vertices with the following continuity condition $\lim_{x\to v, x\in intr} y(x) = \lim_{x\to v, x\in intr'} y(x)$ for any two edges $r, r' \in I(v)$ and establish some properties of the spectral characteristics and investigate the inverse problem of recovering the operator from the scattering data and uniqueness theorem has been proven according to the Weyl function for such inverse problem.

In this present paper, as different from other studies, we construct the new integral representation of the Jost solution of Sturm-Liouville equation with discontinuity conditions in the semi axis $[0, +\infty)$. In the special cases with discontinuity conditions in the point $x = a \in [0, \pi]$, the integral representation for the Sturm-Liouville equation is obtained in [4] and solved inverse problems by the spectral data and by the Weyl function. By the way, we note that the integral representations and transformation operators or used for the solution of inverse problems of spectral analysis, especially, the relation between the potential function of the problem and the kernel function of the integral representations and transformation operator plays the central role in the solution of the inverse problems. Therefore, in this paper, we give this type of relation , examining the properties of the kernel function and their partial derivatives with x and t, constructed integral representation and obtain the partial differential equation provided by this kernel function. Additionally, unlike the other works [8, 11, 16], it is shown that the kernel function is real valued and has a discontinuity along the line $t = 2a - x$ for $x > a$. Finally, in the paper we prove uniqueness of the determination of the potential by the scattering data.

2. Preliminaries

Consider the boundary value problem generated on the semi axis $0 \le x < \infty$ by the differential equation:

$$l(y) := -y'' + q(x)y = \lambda^2 y, \tag{2.1}$$

with discontinuity conditions at a point $a \in (0, \infty)$

$$y(a-0) = \alpha y(a+0), \quad y'(a-0) = \alpha^{-1} y'(a+0), \tag{2.2}$$

and boundary condition

$$y(0) = 0, \tag{2.3}$$

where $\alpha\,(\alpha > 0, \alpha \neq 1)$is a real constant, λ is a complex parameter, $q(x)$ is a real valued function with

$$\int_0^\infty x\,|q(x)|\,dx < \infty.$$

We call the Jost solution of the equation (2.1) with conditions (2.2) the solution $e(\lambda, x)$ satisfying the condition at infinity

$$\lim_{x\to\infty} e(\lambda, x)e^{-i\lambda x} = 1.$$

Let us denote

$$\sigma(x) = \int_x^\infty |q(s)|\,ds, \qquad \sigma_1(x) = \int_x^\infty \sigma(s)ds.$$

It can be easy shown that if $q(x) \equiv 0$, then the Jost solution is

$$e_0(\lambda, x) = \begin{cases} e^{i\lambda x}, & a < x, \\ A^+ e^{i\lambda x} + A^- e^{i\lambda(2d-x)}, & x < a, \end{cases}$$

where $A^{\pm} = \dfrac{1}{2}\left(\alpha \pm \dfrac{1}{\alpha}\right)$.

The following theorem and lemma are obtained from [3] by some formal changes.

Theorem 2.1. [*[3], Theorem 1.*] *Equation (2.1) with the discontinuity conditions (2.2) has the Jost solution $e(\lambda, x)$ which is represented as*

$$e(\lambda, x) = e_0(\lambda, x) + \int_x^{+\infty} K(x,t) e^{i\lambda t} dt \tag{2.4}$$

for all λ from the closed upper half plane $Im\lambda \geq 0$. Here the kernel $K(x,t)$ satisfies the inequalities

$$|K(x,t)| \leq \frac{c}{2}\sigma\left(\frac{x+t}{2}\right) e^{c\sigma_1(x)}, \quad 0 < \ |x-a| < t-a, \tag{2.5}$$

$$|K(x,t)| \leq \left\{\frac{c}{2}\sigma\left(\frac{x+t}{2}\right) + \frac{|A^-|}{2}\sigma\left(\frac{2a+x-t}{2}\right)\right\} e^{c\sigma_1(x)}, \quad |t-a| < x-a, \tag{2.6}$$

where $C = A^+ + |A^-|$. Moreover, the function $K(x,t)$ is continuous at $t \neq 2a - x$, $x \neq a$ and the following relations are satisfied:

$$K(x,x) = \frac{A^+}{2}\int_x^{\infty} q(t)dt, \quad x < a, \tag{2.7}$$

$$K(x,x) = \frac{1}{2}\int_x^{\infty} q(t)dt, \quad x > a, \tag{2.8}$$

$$K(x, 2a-x+0) - K(x, 2a-x-0) = \frac{A^-}{2}\left(\int_a^{\infty} q(t)dt - \int_x^{a} q(t)dt\right), \qquad x < a. \tag{2.9}$$

There exists both first order partial derivatives of the function $K(x,t)$ when $t \neq 2a - x$, $x \neq a$. In addition

$$\left|\frac{\partial K(x_1,x_2)}{\partial x_i} + \frac{1}{4}q\left(\frac{x_1+x_2}{2}\right)\right| \leq \frac{1}{2}\sigma(x_1)\sigma\left(\frac{x_1+x_2}{2}\right), \ x_1 > a,$$

$$\left|\frac{\partial K(x_1,x_2)}{\partial x_i} + \frac{A^+}{4}q\left(\frac{x_1+x_2}{2}\right) + (-1)^i\frac{A^-}{4}q\left(\frac{x_2+2a-x_1}{2}\right)\right|$$
$$\leq \frac{C^2}{2}\sigma(x_1)\sigma\left(\frac{x_1+x_2}{2}\right)e^{C\sigma_1(x)}, \ x_2 > 2a - x_1, \ x_1 < a,$$

$$\left|\frac{\partial K(x_1,x_2)}{\partial x_1} + \frac{A^+}{4}q\left(\frac{x_1+x_2}{2}\right)\right.$$
$$\left. +(-1)^{i+1}\frac{A^-}{4}q\left(\frac{2a+x_1-x_2}{2}\right) + (-1)^i\frac{A^-}{4}q\left(\frac{2a+x_2-x_1}{2}\right)\right|$$
$$\leq \frac{C}{2}\left[C\sigma\left(\frac{x_1+x_2}{2}\right) + |A^-|\,\sigma\left(\frac{2a+x_1-x_2}{2}\right)\right]$$
$$\times\sigma(x_1)e^{C\sigma_1(x)}, \ x_1 \leq x_2 \leq 2a - x_1.$$

Also note that the kernel $K(x,t)$ and the derivative $K_x(x,t)$ satisfy the following discontinuity conditions at $x = a, t > a$;

$$K(a-0,t) = \alpha K(a+0,t), \tag{2.10}$$

$$K_x(a-0,t) = \alpha^{-1} K_x(a+0,t). \tag{2.11}$$

If the function $q(x)$ is differentiable, the function $K(x,t)$ has the second partial derivatives and the equation

$$\frac{\partial^2 K(x,t)}{\partial x^2} - \frac{\partial^2 K(x,t)}{\partial t^2} = q(x)K(x,t) \tag{2.12}$$

is satisfied for $t \neq 2a - x$, $x \neq a$. Moreover

$$\lim_{x+t\to\infty} \frac{\partial K(x,t)}{\partial x} = \lim_{x+t\to\infty} \frac{\partial K(x,t)}{\partial t} = 0. \tag{2.13}$$

Conversely, if the function is the solution of the partial derivative equation (2.12) with conditions (2.7)-(2.9), (2.10), (2.11) and (2.13) then the function $e(\lambda,x)$ constructed by the formula (2.4) is the Jost solution of equation (2.1) with discontinuity conditions (2.2). Now, we investigate some additional properties of solutions of the equation (2.1) with discontinuity conditions (2.2).

Lemma 2.2. *The solution $e(\lambda,x)$ is an analytic function of the variable λ on the upper half plane $Im\lambda \geq 0$ and if is continuous for $\lambda \in (-\infty,+\infty)$. For all λ from the closed upper half plane $Im\lambda \geq 0$ we have*

$$|e(\lambda,x)| \leq 2C\,(1+\sigma_1(x))\, e^{-Im\lambda x + C\sigma_1(x)},$$

$$\left|e'(\lambda,x) - e_0'(\lambda,x)\right| \leq 2C^2\,(1+\sigma_1(x))\,\sigma(x) e^{-Im\lambda x + C\sigma_1(x)},$$

$$|e(\lambda,x) - e_0(\lambda,x)| \leq C_1\,(1+\sigma_1(x))\left(\sigma_1(x) - \sigma_1(x + \frac{1}{|\lambda|})\right) e^{-Im\lambda x + C\sigma_1(x)}, \quad x > a,$$

$$\begin{aligned} |e(\lambda,x) - e_0(\lambda,x)| \leq \quad & C_1\,(1+\sigma_1(x))\Big[A^+\left(\sigma_1(x) - \sigma_1(x+\tfrac{1}{|\lambda|})\right) \\ & + |A^-|\left(\sigma_1(2a-x) - \sigma_1(2a-x+\tfrac{1}{|\lambda|})\right) \qquad x < a. \\ & + |A^-|\,|\lambda|^{-1}\,\sigma(a)\Big], \end{aligned}$$

Proof. If $x > a$ then equation (2.5) is satisfied, so from equation (2.4) we have

$$|e(\lambda,x)| \leq Ce^{-Im\lambda x + C\sigma_1(x)}\,(1+\sigma_1(x))\,. \tag{2.14}$$

Since the solution $e(\lambda,x)$ also satisfies the integral equation

$$e(\lambda,x) = e_0(\lambda,x) + \int_x^{+\infty} S_0(x,t,\lambda) q(t) e(\lambda,t)dt, \tag{2.15}$$

where

$$S_0(x,t,\lambda) = \begin{cases} \dfrac{\sin\lambda(t-x)}{\lambda}, & a < x < t \text{ or } x < t < a, \\ A^+\dfrac{\sin\lambda(t-x)}{\lambda} + A^-\dfrac{\sin\lambda(t-2a+x)}{\lambda}, & x < a < t, \end{cases}$$

then $x > a$ the equation (2.14) takes a form of

$$e(\lambda,x) = e_0(\lambda,x) + \int_x^{+\infty} \frac{\sin\lambda(t-x)}{\lambda} q(t) e(\lambda,t)dt. \tag{2.16}$$

From (2.16),(2.14) and (2.15) we obtain

$$\begin{aligned}\left|e(\lambda,x)-e^{i\lambda x}\right| &\le \int_x^{+\infty}\left|\frac{\sin\lambda(t-x)}{\lambda}\right||q(t)|\,C\,(1+\sigma_1(t))\,e^{-Im\lambda t+C\sigma_1(x)}dt\\ &= \int_x^{+\infty}\left|\frac{\sin\lambda(t-x)}{\lambda}e^{i\lambda(t-x)}\right||q(t)|\,dtC\,(1+\sigma_1(x))\,e^{-Im\lambda x+C\sigma_1(x)}\\ &\le \left[\int_x^{x+\frac{1}{|\lambda|}}(t-x)\,|q(t)|\,dt+\frac{1}{|\lambda|}\int_x^{+\infty}|q(t)|\,dt\right]C\,(1+\sigma_1(x))\,e^{-Im\lambda x+C\sigma_1(x)}\\ &= C\left(\sigma_1(x)-\sigma_1\left(x+\tfrac{1}{|\lambda|}\right)\right)(1+\sigma_1(x))\,e^{-Im\lambda x+C\sigma_1(x)},\end{aligned}$$

i.e.

$$\left|e(\lambda,x)-e^{i\lambda x}\right|\le C\left(\sigma_1(x)-\sigma_1\left(x+\frac{1}{|\lambda|}\right)\right)(1+\sigma_1(x))\,e^{-Im\lambda x+C\sigma_1(x)}.$$

Similarly from (2.16),(2.14) and (2.15) we find

$$\left|e'(\lambda,x)-i\lambda e^{i\lambda t}\right|\le C\,(1+\sigma_1(x))\,\sigma(x)e^{-Im\lambda x+C\sigma_1(x)}.$$

Let now $x<a$. In this case equation (2.5) satisfied for $t>2a-x$ and equation (2.6) is satisfied for $t<2a-x$. So, we have from equation (2.4)

$$|e(\lambda,x)|\le 2C\,(1+\sigma_1\,(x))^{-Im\lambda x+C\sigma_1(x)}. \tag{2.17}$$

Further, from the equation (2.15) we have for $x<a$

$$\begin{aligned}e(\lambda,x) &= A^+e^{i\lambda x}+A^-e^{i\lambda(2a-x)}+\int_x^a\frac{\sin\lambda(t-x)}{\lambda}q(t)e\,(t,\lambda)\,dt\\ &+\int_a^{+\infty}\left[A^+\frac{\sin\lambda(t-x)}{\lambda}+A^-\frac{\sin\lambda(t-2a+x)}{\lambda}\right]q(t)e\,(t,\lambda)\,dt.\end{aligned} \tag{2.18}$$

Therefore by (2.14) and (2.17)

$$\begin{aligned}|e(\lambda,x)-e_0(\lambda,x)| &\le \int_x^a\left|\frac{\sin\lambda(t-x)}{\lambda}\right||q(t)|\,2C\,(1+\sigma_1\,(t))\\ &\times e^{-Im\lambda t+C\sigma_1(t)}dt+\int_a^{+\infty}\left[A^+\left|\frac{\sin\lambda(t-x)}{\lambda}\right|\right.\\ &\left.+A^-\left|\frac{\sin\lambda(t-2a+x)}{\lambda}\right|\right]|q(t)|\\ &\times C\,(1+\sigma_1\,(t))\,e^{-Im\lambda t+C\sigma_1(t)}dt\\ &\le 2C_1\,(1+\sigma_1\,(x))\,e^{-Im\lambda tx+C\sigma_1(x)}\\ &\times\left[A^+\left(\sigma_1(x)-\sigma_1\left(x+\tfrac{1}{|\lambda|}\right)\right)\right.\\ &\left.+|A^-|\left(\sigma_1(2a-x)-\sigma_1\left(2a-x+\tfrac{1}{|\lambda|}\right)\right)\right]\\ &+\frac{|A^-|}{\lambda}\sigma(a).\end{aligned}$$

From the equation (2.18), we also have

$$\begin{aligned}e'(\lambda,x)-e_0'(\lambda,x) &= \int_x^a\cos\lambda(t-x)q(t)e\,(t,\lambda)\,dt\\ &+\int_x^a\left[A^-\cos\lambda(t-2a+x)-A^+\cos\lambda(t-x)\right]q(t)e\,(t,\lambda)\,dt.\end{aligned}$$

So, using again the equation (2.14) and (2.17) we find that

$$\begin{aligned}
&\left|e'(\lambda,x)-e_0'(\lambda,x)\right| \le \int_x^a |\cos\lambda(t-x)|\,|q(t)|\,2C\,(1+\sigma_1(t))\,e^{-Im\lambda t+C\sigma_1(t)}dt\\
&+\int_a^{+\infty} A^+\,|\cos\lambda(t-x)|\,|q(t)|\,C\,(1+\sigma_1(t))\,e^{-Im\lambda t+C\sigma_1(t)}dt\\
&+|A^-|\int_a^{2a-x} |\cos\lambda(2a-x-t)|\,|q(t)|\,C\,(1+\sigma_1(t))\,e^{-Im\lambda t+C\sigma_1(t)}dt\\
&+|A^-|\int_{2a-x}^{+\infty} |\cos\lambda(t-2a+x)|\,|q(t)|\,C\,(1+\sigma_1(t))\,e^{-Im\lambda t+C\sigma_1(t)}dt\\
&\le 2C(1+\sigma_1(x))e^{-Im\lambda x+C\sigma_1(x)}\left[A^+\int_a^{+\infty}|q(t)|\,dt\right.\\
&\left.+|A^-|\int_a^{2a-x}|q(t)|\,dt+|A^-|\int_{2a-x}^{+\infty}|q(t)|\,dt\right]\\
&\le 2C^2(1+\sigma_1(x))e^{-Im\lambda x+C\sigma_1(x)}\sigma(x)\,.
\end{aligned}$$

Note that for all real $\lambda \neq 0$ the functions $e(\lambda,x)$ and $e(-\lambda,x)$ are linearly independent solutions ($x \neq a$) of equation (2.1) with the wronskian

$$\omega\left[e(\lambda,x),e(-\lambda,x)\right] = e(\lambda,x), e'(-\lambda,x) - e'(\lambda,x), e(-\lambda,x) = -2i\lambda.$$

Let the function $\omega(\lambda,x,\infty) = O(x)$, $x \to 0$ satisfies the integral equation

$$\omega(\lambda,x,\infty) = S_0(x,\lambda) + \int_0^x S\,(x,t,\lambda)\,q(t)\omega(\lambda,t,\infty)dt, \tag{2.19}$$

where

$$S_0(x,\lambda) = \begin{cases} \dfrac{\sin\lambda x}{\lambda}, & 0 \le x < a,\\ A^+\dfrac{\sin\lambda x}{\lambda} - A^-\dfrac{\sin\lambda(2a-x)}{\lambda}, & x > a,\end{cases}$$

$$S\,(x,t,\lambda) = \begin{cases} \dfrac{\sin\lambda\,(x-t)}{\lambda}, & a < t \le x, \text{ or } 0 \le t \le x < a,\\ A^+\dfrac{\sin\lambda\,(x-t)}{\lambda} + A^-\dfrac{\sin\lambda(x+t-2a)}{\lambda}, & 0 \le t \le a < x.\end{cases}$$

It is clear that $\omega(\lambda,x,\infty)$ is the solution of the equation (2.1) with the discontinuity conditions (2.2) and the conditions

$$\omega(\lambda,x,\infty) = x\,(1+o(1))\,,\ \omega'(\lambda,x,\infty) = 1+o(1), \quad x \to 0$$

(see [17]) .

Let us show that the solution $\omega(\lambda,x,\infty)$ is a entire function of the parameter λ and satisfies the inequalities the following inequality for all $Im\lambda \ge 0$, $x \neq a$;

$$\left|\omega(\lambda,x,\infty)e^{i\lambda x}\right| \le Cxe^{C\int_0^x t|q(t)|dt}, \tag{2.20}$$

$$\left|\left[\lambda\omega(\lambda,x,\infty) - \lambda S_0(x,\lambda)\right]e^{i\lambda x}\right| \le C\left[\sigma_1(0) - \sigma_1\left(\frac{1}{|\lambda|}\right)\right]e^{C\int_0^x t|q(t)|dt}, \tag{2.21}$$

where $C = A^+ + |A^-|$.

Let

$$\omega(\lambda,x,\infty) = xe^{-i\lambda x}Z(\lambda,x), \quad Im\lambda \ge 0. \tag{2.22}$$

Then, we have the following equation for the function

$$Z(\lambda, x) = \frac{S_0(x,\lambda)e^{i\lambda x}}{x} + \int_0^x \frac{S(x,,\lambda)e^{i\lambda(x-t)}}{x} tq(t)Z(\lambda,t)dt.$$

We apply the method of successive approximation to the integral equation to find the solution $Z(\lambda, x)$. Put

$$Z(\lambda, x) = \sum_{k=0}^{\infty} Z_k(\lambda, x), \tag{2.23}$$

$$Z_0(\lambda, x) = \frac{S_0(x,\lambda)e^{i\lambda x}}{x},$$

$$Z_k(\lambda, x) = \int_0^x \frac{S(x,t,\lambda)e^{i\lambda(x-t)}}{x} tq(t)Z_{k-1}(\lambda,t)dt, \quad k = 1, 2, \ldots. \tag{2.24}$$

$0 \le x \le a$, we have

$$|Z_0(\lambda, x)| = \left|\frac{\sin \lambda x}{\lambda x} e^{i\lambda x}\right| \le 1, \quad (Im\lambda \ge 0)$$

and if $x > a$ we have for $Im\lambda \ge 0$

$$\begin{aligned} |Z_0(\lambda, x)| &= \left|A^+ \frac{\sin \lambda x}{\lambda x} e^{i\lambda x} - A^- \frac{\sin \lambda(2a-x)}{\lambda x} e^{i\lambda x}\right| \\ &\le A^+ + |A^-| = C. \end{aligned}$$

Therefore, we have for all $Im\lambda \ge 0$, $x \ne a$

$$|Z_0(\lambda, x)| \le C, \tag{2.25}$$

where $C = A^+ + |A^-| = \max\left(\alpha, \frac{1}{\alpha}\right)$.

By the similar way we obtain for all $Im\lambda \ge 0$, $0 \le t \le x$ and $x \ne a$ that

$$\left|\frac{S(x,t,\lambda)}{x} e^{i\lambda(x-t)}\right| \le 1 - \frac{t}{x} \le 1 \text{ if } a < t \le x \text{ or } 0 \le t \le x < a$$

and

$$\left|\frac{S(x,t,\lambda)}{x} e^{i\lambda(x-t)}\right| \le C, \text{ if } 0 \le t \le a \le x.$$

Therefore,

$$\left|\frac{S(x,t,\lambda)}{x} e^{i\lambda(x-t)}\right| \le C, \tag{2.26}$$

for all $Im\lambda \ge 0$, $0 \le t \le x$ and $x \ne a$.

Because of (2.25) and (2.26) we obtain from the (2.24) that

$$|Z_k(\lambda, x)| \le \frac{C^{k+1}}{k!} \left(\int_0^x t\,|q(t)|\,dt\right)^k, \; Im\lambda \ge 0, \; x > 0, \; x \ne a.$$

Then, we observe that the series (2.23) uniformly converges in the region $Im\lambda \ge 0$, $x \in [0,a) \cup (a,b]$ for any $b > 0$. For the $Z(\lambda, x)$ of the series (2.23) we have

$$|Z(\lambda, x)| \le Ce^{C\int_0^x t|q(t)|dt}. \tag{2.27}$$

Now, from (2.22) and (2.27) we obtain (2.20). Therefore, the solution $\omega(\lambda, x, \infty)$ is an analytic function of λ ($Im\lambda > 0$) and it is continuous in the closed upper plane $Im\lambda \ge 0$.

For the solution similar way, we prove that the solution $\omega(\lambda, x, \infty)$ is analytic function of the parameter λ in the half plane $Im\lambda < 0$ and it is continuous in the closed half plane $Im\lambda \le 0$. Hence the solution $\omega(\lambda, x, \infty)$ is an entire function of the parameter λ.

Now, let us prove the estimation (2.21). From the equation (2.19) and the estimation (2.20) we have

$$|\omega(\lambda,x,\infty)-S_0(x,\lambda)| \le Cx\int_0^x t\,|q(t)|\,dt e^{|Im\lambda x|+C\int_0^x t|q(t)|dt},$$

$$\left|\omega'(\lambda,x,\infty)-C_0(x,\lambda)\right| \le C\int_0^x t\,|q(t)|\,dt e^{|Im\lambda x|+C\int_0^x t|q(t)|dt},$$

where

$$C_0(x,\lambda)=\begin{cases}\cos\lambda x, & 0\le x<a\\ A^+\cos\lambda x+A^-\cos\lambda(2a-x), & x>a.\end{cases}$$

Using again the (2.19) and (2.27), we obtain in $Im\lambda\ge 0$

$$\begin{aligned}|\omega(\lambda,x,\infty)-S_0(x,\lambda)| &\le \int_0^x \left|S(x,t,\lambda)e^{i\lambda(x-t)}\right| t\,|q(t)|\,Z(\lambda,t)dt\\ &\le C\int_0^x t\,|q(t)|\,e^{C\int_0^t s|q(s)|ds}dt\\ &= e^{C\int_0^x t|q(t)|dt}-1.\end{aligned}$$

In particular, we have

$$\left|[\lambda\omega(\lambda,x,\infty)-S_0(x,\lambda)]\,e^{i\lambda x}\right| \le \left(C\int_0^x t\,|q(t)|\,dt\right)e^{C\int_0^x t|q(t)|dt}, \tag{2.28}$$

$$\left|\lambda\omega(\lambda,x,\infty)e^{i\lambda x}\right| \le e^{C\int_0^x t|q(t)|dt}. \tag{2.29}$$

Now, let $Im\lambda\ge 0$ and $|\lambda|^{-1}<x$.

Then, from (2.19), (2.27) and (2.29) we have

$$\begin{aligned}\left|[\lambda\omega(\lambda,x,\infty)-S_0(x,\lambda)]\,e^{i\lambda x}\right| &\le \int_0^x \left|\lambda S(x,t,\lambda)e^{i\lambda(x-t)}\right| |q(t)|\,e^{i\lambda t}\omega(\lambda,t,\infty)dt\\ &\le \left[C\int_0^{|\lambda|^{-1}} t\,|q(t)|\,dt+C\,|\lambda|^{-1}\int_{|\lambda|^{-1}}^x |q(t)|\,dt\right]e^{C\int_0^x t|q(t)|dt}\\ &= C\left(\sigma_1(0)-\sigma_1\left(\frac{1}{|x|}\right)\right)e^{C\int_0^x t|q(t)|dt}.\end{aligned}$$

Hence, the estimation (2.21) holds for $x>|\lambda|^{-1}$. Let $x\le|\lambda|^{-1}$. Then, the estimation (2.21) is obtained from the (2.28) because of

$$\begin{aligned}\int_0^x t\,|q(t)|\,dt &= -x\sigma(x)+\int_0^x \sigma(t)dt\\ &\le \sigma_1(0)-\sigma_1(x)\le\sigma_1(0)-\sigma_1\left(\frac{1}{|\lambda|}\right).\end{aligned}$$

Consequently, we have proved the following theorem. □

Theorem 2.3. *For all values of the parameter λ the equation (2.1) has solution $\omega(\lambda,x,\infty)$ satisfying the discontinuity conditions (2.2) and the conditions*

$$\omega(\lambda,x,\infty)=x\,(1+o(1))\,,\ \omega_x(\lambda,x,\infty)=1+o(1),\ x>0,\ \ x\to 0.$$

The solution $\omega(\lambda,x,\infty)$ is an entire function with respect to λ and for $Im\lambda\ge 0$ the inequalities (2.20) and (2.21) are hold.

3. Derivation of the Fundamental Equation

To derive the fundamental equation, we use the equality

$$-\frac{2i\lambda\omega(\lambda,x)}{e(\lambda,0)} = e(-\lambda,x) - S(\lambda)e(\lambda,x),$$

which was obtained Lemma 3.1.5. in [2]. Here $e(\lambda,x)$ has a Jost solution, regular with respect to λ in the upper half-plane $Im\lambda > 0$, continuous for $Im\lambda \geq 0$ and representable in the form (2.4), so that

$$\begin{aligned} &-2i\lambda\omega(\lambda,x)\left\{\frac{1}{e(\lambda,0)} - \frac{1}{e_0(\lambda,0)}\right\} - \frac{2i\lambda}{e_0(\lambda,0)}\{\omega(\lambda,x) - \omega_0(\lambda,x)\} \\ &= \int_x^\infty K(x,t)e^{-i\lambda t}dt + \{S_0(\lambda) - S(\lambda)\} \\ &\times\left\{e_0(\lambda,x) + \int_x^\infty K(x,t)e^{i\lambda t}dt\right\} - S_0(\lambda)\int_x^\infty K(x,t)e^{i\lambda t}dt, \end{aligned} \tag{3.1}$$

where

$$S_0(\lambda) = \frac{e_0(-\lambda,0)}{e_0(\lambda,0)}, \quad S(\lambda) = \frac{e_0(\lambda,0)}{e_0(-\lambda,0)},$$

$$\omega_0(\lambda,x) = \begin{cases} \dfrac{\sin\lambda x}{\lambda}, & 0 < x \leq a \\ \alpha^+\dfrac{\sin\lambda x}{\lambda} + \alpha^-\dfrac{\sin\lambda(x-2a)}{\lambda} & x > a \end{cases}$$

is the solution of the equation $-y'' = \lambda^2 y$ satisfying conditions (2.2) and $\omega_0(\lambda,0) = 0$, $\omega_0'(\lambda,0) = 1$. As was shown in Lemma 1.3.7 in [2], $S_0(\lambda) - S(\lambda)$ is a Fourier transform of the function

$$F_s(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[S_0(\lambda) - S(\lambda)]\,e^{-i\lambda y}d\lambda. \tag{3.2}$$

Next, since

$$S_0(\lambda) = \frac{\alpha^+ + \alpha^- e^{-2i\lambda a}}{\alpha^+ + \alpha^- e^{2i\lambda a}} = \left(1 + \frac{\alpha^-}{\alpha^+}e^{-2i\lambda a}\right)\left[1 - \frac{\alpha^-}{\alpha^+}e^{2i\lambda a} + \left(\frac{\alpha^-}{\alpha^+}\right)^2 e^{4i\lambda a}\ldots\right]$$

we have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\lambda)e^{i\lambda t}d\lambda = \delta(t) + \frac{\alpha^-}{\alpha^+}\delta(t-2a) - \left(\frac{\alpha^-}{\alpha^+}\right)^2\delta(t) - \\ -\left[\frac{\alpha^-}{\alpha^+} - \left(\frac{\alpha^-}{\alpha^+}\right)^3\right]\delta(t+2a). \end{aligned}$$

Let $i\lambda_k$ $(k = \overline{1,n})$ be the zeros of the function $e(\lambda,0)$, numbered in the order of increase of their modula $(0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n)$, and let m_k^{-1} be the norm of the function $e(i\lambda_k,x)$ in $L_2[0,\infty)$. Moreover as $x \to \infty$ these solutions satisfy the asymptotic formulae;

$$u(\lambda,x) = e^{-i\lambda x} - S(\lambda)e^{i\lambda x} + o(1), \ \ (\lambda \in \mathbb{R}\setminus\{0\}) \tag{3.3}$$

$$u(i\lambda_k,x) = m_k e^{-\lambda_k x}(1 + o(1)), \ \left(k = \overline{1,n}\right). \tag{3.4}$$

A totality of quantities $\{S(\lambda), \lambda_k, m_k\}$ is said to be scattering data of problem (2.1)-(2.3).

Thus, the scattering data completely determines the behavior of the normed eigenfunctions of problem (2.1)-(2.3).

The inverse scattering problem for boundary value problem (2.1)-(2.3) consists of reconstruction of the function $q(x)$ by the scattering data.

As was shown in [2], multiplying the both hand sides of equality (2.1) by $\frac{1}{2\pi}e^{i\lambda y}$ and integrating with respect to $-\infty < \lambda < \infty$ and taking into account (2.2) and (2.3) in the right hand side, we get

$$\begin{aligned} &K(x,y) + F_{1s}(x,y) + \int_x^{+\infty} K(x,t)F_s(t+y)dt- \\ &-\left[1-\left(\frac{\alpha^-}{\alpha^+}\right)^2\right]K(x,-y) - \frac{\alpha^-}{\alpha^+}K(x,2a-y)+ \\ &+\left[\frac{\alpha^-}{\alpha^+}+\left(\frac{\alpha^-}{\alpha^+}\right)^3\right]K(x,-y-2a), \end{aligned} \tag{3.5}$$

where

$$F_{1s}(x,y) = \begin{cases} F_s(x+y), & x > a \\ \alpha^+F_s(x+y) + \alpha^-F_s(2a-x+y), & 0 < x \le a. \end{cases}$$

Hence, for the $y > x$, the expression (3.5) takes the form

$$K(x,y) + F_{1s}(x,y) + \int_x^{+\infty} K(x,t)F_s(t+y)dt + \frac{\alpha^-}{\alpha^+}K(x,2a-y). \tag{3.6}$$

Consequently, the integral of the product of the left-hand side of identity (3.1) and $\frac{1}{2}e^{i\lambda y}$, taken over the real line $-\infty < \lambda < \infty$, must equal (3.6). Let us show that for $x > y$, this integral converges and can be calculated by contour integration. In fact, the first term in the integral,

$$-2i\lambda\omega(\lambda,x)\left\{\frac{1}{e(\lambda,0)} - \frac{1}{e_0(\lambda,0)}\right\}$$

is regular everywhere in the upper plane, apart from a finite number of points $i\lambda_k$ $\left(k=\overline{1,n}\right)$, which are simple zeros of the function $e(\lambda,0)$: moreover it is continuous for real $\lambda \neq 0$ and bounded in a neighborhood of $\lambda = 0$ $(Im\lambda \ge 0)$. Furthermore, since the function $\lambda\omega(\lambda,x)e^{i\lambda x}$ is bounded in the half plane $Im\lambda \ge 0$, and since the function $\frac{1}{e(\lambda,0)} - \frac{1}{e_0(\lambda,0)}$ tends uniformly to zero in this half plane as $|\lambda| \to \infty$, an application of Jordan's Lemma yields, for $y > x$ we get

$$\begin{aligned} I &= \sum_{k=1}^{n} \frac{2i\lambda_k\omega(i\lambda_k,x)}{e(i\lambda_k,0)} = -\sum_{k=1}^{n} m_k^2 e(i\lambda_k,x)e^{-\lambda_k y} \\ &= -\sum_{k=1}^{n} m_k^2 e_0(i\lambda_k,x)e^{-\lambda_k y} - \int_x^{+\infty} K(x,t)\sum_{k=1}^{n} m_k^2 e^{-\lambda_k(t+y)}dt, \end{aligned}$$

the second term in the integral in question $\frac{2i\lambda}{e_0(\lambda,0)}\{\omega(\lambda,x)-\omega_0(\lambda,x)\}$ is an entire function of λ and Jordan's Lemma,

$$\int_{-\infty}^{+\infty} \frac{2i\lambda}{e_0(0,\lambda)}\{\omega(\lambda,y)-\omega_0(\lambda,y)\}e^{-i\lambda y}dy = 0$$

for $y > x$. Equating I and expressions from (3.6) we get the fundamental equations of the inverse problem for $K(x,t)$

$$K(x,y) - \frac{\alpha^-}{\alpha^+}K(x,2a-y) + F_1(x,y) + \int_x^{+\infty} K(x,t)F(t+y)dt = 0,\ y > x, \tag{3.7}$$

where

$$F(y) = F_s(y) + \sum_{k=1}^{n} m_k^{2-\lambda_k y},$$

$$F_1(x,y) = \begin{cases} F(x+y), & x > a \\ \alpha^+F(x+y) + \alpha^-F(2a-x+y), & 0 \le x \le a. \end{cases} \tag{3.8}$$

We have thus proved the following result:

Theorem 3.1. *The kernel $K(x,t)$ of the transformation operator satisfies the fundamental equation (3.7).*

Equation (3.7) plays an important part in solving the inverse problem of the scattering theory. In terms of the scattering data, the potential $q(x)$ is recovered uniquely. In fact, given the scattering data, one can construct the function $F(y)$ via formula (3.8) and then write the fundamental equation (3.7). If the fundamental equation has a unique solution for every $x \geq 0$, then the solution is the kernel $K(x,y)$ of the transformation operator, and hence the potential $q(x)$ is form formulas (2.7), (2.8).

4. Solvability of the Fundamental Equation

From the property of the function $F_s(x)$ and from the form $F(x)$ it follows that for each fixed $x \geq 0$, the operator

$$(F_x f)(y) = \int\limits_x^{+\infty} F(t+y) f(t) dt$$

acting in the space $L_1(x,+\infty)$ (also in $L_2(x,+\infty)$) is completely continuous.

In the fundamental equation we'll consider the kernel $K(x,t)$ as unknown and consider it as Fredholm type equation in the space $L_2(x,+\infty)$ (or $L_1(x,+\infty)$) for each fixed x. At first, we show that for each fixed $x \geq 0$, the operator

$$(M_x f)(y) = \begin{cases} f(y), & x > a \\ f(y) - \frac{\alpha^-}{\alpha^+} f(2a-y) & 0 \leq x \leq a \end{cases}$$

acting in the space $L_2(x,+\infty)$ is invertible.

It suffices to consider the case $0 \leq x \leq a$. Let's consider the equation

$$f(y) - \frac{\alpha^-}{\alpha^+} f(2a-y) = g(y). \tag{4.1}$$

Making substitution $y \to 2a - y$, hence we have

$$f(2a-y) - \frac{\alpha^-}{\alpha^+} f(y) = g(2a-y). \tag{4.2}$$

From the system of equations (4.1)-(4.2) we get

$$f(y) = \frac{\alpha^+}{4} \left[\alpha^+ g(y) + \alpha^- g(2a-y)\right],$$

i.e., the operator M_x has the inverse. From the last formula, we have

$$\int\limits_x^{+\infty} |f(y)|^2 \, dy \leq C \int\limits_x^{+\infty} |g(y)|^2 \, dy,$$

where C are some constants. Thus, we proved that for each fixed $x \geq 0$, the operator M_x is invertible in the space $L_2(x,+\infty)$.

Now, let's denote that the fundamental equation is equivalent to the equation

$$K(x,y) + M_x^{-1} F_1(x,y) + M_x^{-1} \left(F K(x,\cdot)\right)(y) = 0, \quad y > x$$

i.e., to the equation with completely continuous operator, since $M_x^{-1} F$ is a completely continuous operator.

Thus, in order to prove the solvability of the fundamental equation, it suffices to show that the homogeneous equation

$$f_x(y) + M_x^{-1} \left(F f_x\right)(y) = 0,$$

i.e., the equation

$$f_x(y) - \frac{\alpha^-}{\alpha^+} f_x(2a-y) + \int_x^{+\infty} f_x(t)F(t+y)dt = 0, \qquad y > x \tag{4.3}$$

has only a zero solution $f_x(\cdot) \in L_2(x,+\infty)$.

We multiply the equation (4.3) by $f_x(y)$ and integrate with respect to y in the interval $(x,+\infty)$. As the result, according to (3.2) and (3.8) we get

$$\begin{array}{l} \int_x^{+\infty} |f_x(y)|^2\, dy - \frac{\alpha^-}{\alpha^+} \int_x^{+\infty} f_x(2a-y)\widetilde{f}_x(y)dy \\ +\frac{1}{2\pi} \int_{-\infty}^{+\infty} [S_0(\lambda) - S(\lambda)]\, \widetilde{f}(-\lambda)\widetilde{f}(\lambda)d\lambda + \sum_{k=1}^{n} m_k^2 \left|\widetilde{f}(-i\lambda_k)\right|^2 = 0, \end{array}$$

where

$$\widetilde{f}(\lambda) = \int_{-\infty}^{+\infty} f_x(y)e^{-i\lambda y}dy.$$

Here, taking into account the equality (see (3.2))

$$\frac{\alpha^-}{\alpha^+} \int_x^{+\infty} f_x(2a-y)\widetilde{f}_x(y)dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_0(\lambda)\widetilde{f}(-\lambda)\overline{\widetilde{f}(\lambda)}d\lambda,$$

and also Parseval equality

$$\int_x^{+\infty} |f_x(y)|^2\, dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widetilde{f}(\lambda)\overline{\widetilde{f}(\lambda)}d\lambda.$$

We finally have

$$\sum_{k=1}^{n} m_k^2 \left|\widetilde{f}(-i\lambda_k)\right|^2 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{\widetilde{f}(\lambda) - S(\lambda)\widetilde{f}(-\lambda)\right\} \overline{\widetilde{f}(\lambda)}d\lambda = 0. \tag{4.4}$$

Since $|S(\lambda)| = 1$, then by the Cauchy-Bunyakovskii inequality

$$\left|\int_{-\infty}^{+\infty} S(\lambda)\widetilde{f}(-\lambda)\overline{\widetilde{f}(\lambda)}d\lambda\right| \le \int_{-\infty}^{+\infty} \left|\widetilde{f}(\lambda)^2\right| d\lambda.$$

Consequently, the second term at the left hand side of (4.4) is non-negative. Therefore, from equality (4.4) we have

$$\begin{array}{l} \widetilde{f}(-i\lambda_k) = 0, \qquad k = \overline{1,n}, \\ \int_{-\infty}^{+\infty} \left\{\widetilde{f}(\lambda) - S(\lambda)\widetilde{f}(-\lambda)\right\} \overline{\widetilde{f}(\lambda)}d\lambda = 0. \end{array}$$

Assuming $z(\lambda) = \widetilde{f}(\lambda) - S(\lambda)\widetilde{f}(-\lambda)$, we see that this function is orthogonal to the function $\widetilde{f}(\lambda)$. But then,

$$\left\|\widetilde{f}(\lambda)\right\|^2 = \left\|S(\lambda)\widetilde{f}(-\lambda)\right\|^2 = \left\|\widetilde{f}(\lambda) - z(\lambda)\right\|^2 = \left\|\widetilde{f}(\lambda)\right\|^2 + \|z(\lambda)\|^2,$$

that is possible only for $z(\lambda) = 0$. So, we have

$$\widetilde{f}(-i\lambda_k) = 0, \qquad k = \overline{1,n}, \tag{4.5}$$

$$\widetilde{f}(\lambda) = S(\lambda)\widetilde{f}(-\lambda). \tag{4.6}$$

By definition $S(\lambda)$ substituting at in (4.6), we have

$$\frac{\widetilde{f}(\lambda)}{e(-\lambda,0)} = \frac{\widetilde{f}(-\lambda)}{e(\lambda,0)}, \qquad -\infty < \lambda < +\infty.$$

Hence, it follows that $\frac{\widetilde{f}(\lambda)}{e(-\lambda,0)}$ is a meromorphic function on the whole of complex plane and has poles of first order in the zeros of the function $e(\lambda, 0)$. Consequently,

$$\frac{\widetilde{f}(-\lambda)}{e(\lambda,0)} = \sum_{k=1}^{n} \frac{1}{\lambda - i\lambda_k} \frac{\widetilde{f}(-i\lambda_k)}{e(i\lambda_k,0)} + \psi(\lambda),$$

where $\psi(\lambda)$ is an entire function. It follows from (4.5) that

$$\frac{\widetilde{f}(-\lambda)}{e(\lambda,0)} = \psi(\lambda).$$

Obviously, as $|\lambda| \to \infty$ the left hand side tends to zero. Consequently, $\psi(\lambda) \equiv 0$. Then we have $\widetilde{f}(-\lambda) = 0$, i.e. $f_x(y) = 0$.

We have thus proved the following result:

Theorem 4.1. *Fundamental equation (3.7) has a unique solution $K(x, \cdot) \in L_2(x, +\infty)$.*

Corollary 4.2. *The potential $q(x)$ is uniquely determined by the scattering data.*

Algorithm:Given a collection of scattering data $\{S(\lambda), \lambda_k, m_k\}$satisfying the conditions of (3.3) and (3.4), we construct the function $F_s(y)$ by (3.2) and the functions $F(y)$ and $F_1(x,y)$ by (3.8) and consider the fundamental equation of the inverse problem for K(x,t) by (3.7). Solving integral equation (3.7) we find K(x,t) Next, and find α by (2.10) and (2.11), consequently $q(x)$ by (2.7) and (2.8).

Conflicts of Interest

The authors declare that there no conflicts of interest regarding the publication of this paper.

Authors Contribution Statement

The authors contributed equally to this study. All authors have read and agreed to the published version of the manuscript.

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