Eastern Anatolian Journal of Science
Volume VIII, Issue II, 2022, 1-5

Eastern Anatolian Journal of Science

Coefficient Bound Estimates and Fekete-Szegö Problem for a Certain Class of Analytic Functions

Nizami MUSTAFA[1*] and Semra KORKMAZ[2]

[1] *Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey,* nizammstafa@gmail.com (ORCID:0000-0002-2758-0274)

[2] *Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Turkey,* korkmazsemra36@outlook.cm

Abstract

In this study, we introduce and examine a certain subclass of analytic functions on the open unit disk in the complex plane. Here, we give coefficient bound estimates and investigate the Fekete-Szegö problem for this class. Some interesting special cases of the results obtained here are also discussed.

Keywords: Coefficient estimates, Fekete-Szegö problem, Univalent function.

1. Introduction

Let $U=\{z\in\mathbb{C}:|z|<1\}$ and $H(U)$ denote the class of analytic functions in U. By A we denote the class of all functions $f\in H(U)$ given by

$$f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$$
$$=z+\sum_{n=2}^{\infty}a_nz^n,\ z\in\mathbb{C} \tag{1}$$

Let S denote the class of all univalent functions in A. For $\alpha\in[0,1)$, some of the important and well-investigated subclasses of S include the classes $S^*(\alpha)$ and $C(\alpha)$, respectively, starlike and convex function classes of order α in U. By definition, we have

$$S^*(\alpha)=\left\{f\in S:\operatorname{Re}\frac{zf'(z)}{f(z)}>\alpha,\ z\in U\right\}\text{ and}$$

$$C(\alpha)=\left\{f\in S:\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\ z\in U\right\}.$$

For the functions f and g which are analytic in U, f is said to be subordinate to g and denoted as $f(z)\prec g(z)$ if there exists an analytic function ω such that

$$\omega(0)=0,\ |\omega(z)|<1\text{ and } f(z)=g(\omega(z)).$$

As it is known that the coefficient problem is one of the important subjects of the theory of geometric functions. Many researchers have introduced and investigated several interesting subclasses of analytic functions and they have found some estimates on the first two coefficients of the functions belonging to these subclasses (see Brannan, D.A. and Clunie, J. 1980, Brannan, D.A. and Taha, T.S. 1986, Lewin, M. 1967, Netanyahu, E. 1969, Srivastava, H.M., Mishra, A.K. and Gochhayat, P. 2010, Xu, Q.H., Xiao, G. and Srivastava, H.M. 2012).

The functional $H_2(1)=a_3-a_2^2$ is known as the Fekete-Szegö functional and one usually considers the further generalized functional $H_2(1)=a_3-\mu a_2^2$, where μ is a complex or real number (see Fekete, M.

Received:28.06.2022
Accepted:24.10.2022
Published:15.12.2022
Corresponding author: Nizami MUSTAFA, *Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars Turkey.*
E-mail: nizamimustafa@gmail.com
Cite this article as: N. Mustafa, Coefficient Bound Estimates and Fekete-Szegö Problem for a Certain Class of Analytic Functions, Eastern Anatolian Journal of Science, Vol. 8, Issue 2, 1-5, 2022.

and Szegö, G. 1993), and is an important tool in the theory of analytic functions. Estimating the upper bound of $\left|a_3-\mu a_2^2\right|$ is known as the Fekete-Szegö problem in the theory of analytic functions. The Fekete-Szegö problem has been investigated by many mathematicians for several subclasses of analytic functions (see Mustafa, N. 2017, Mustafa, N. and Gündüz, M.C. 2019, Zaprawa, P. 2014). Very recently, Mustafa and Mrugusundaramoorthy (Mustafa, N. and Murugusundaramoorthy, G. 2021) introduced a subclass of bi-univalent functions related to shell shaped region and they examined the Fekete-Szegö problem for this subclass.

Now, let we define the following subclass of analytic and univalent functions.

Definition 1.1. A function $f\in S$ is said to be in the class $S^*(\varphi)$ if the following condition is satisfied

$$\frac{zf'(z)}{f(z)}\prec\varphi(z),\ z\in U.$$

In this definition, $\varphi(z)=z+\sqrt{1+z^2}$ and the branch of the square root is chosen with the initial value $\varphi(0)=1.$ It can be easily seen that the function $\varphi(z)=z+\sqrt{1+z^2}$ maps the unit disc U onto a shell shaped region on the right half plane and it is analytic and univalent in U. The range $\varphi(U)$ is symmetric respect to real axis and φ is a function with positive real part in U with $\varphi(0)=\varphi'(0)=1.$ Moreover, it is a starlike domain with respect to point $\varphi(0)=1.$

Let, P be the set of the functions $p(z)$ analytic in U and satisfying $\operatorname{Re}\left(p(z)\right)>0,\ z\in U$ and $p(0)=1$, with power series

$$\begin{aligned}p(z)&=1+p_1z+p_2z^2+p_3z^3+\cdots+p_nz^n+\cdots\\&=1+\sum_{n=1}^{\infty}p_nz^n,\ z\in U.\end{aligned}$$

In order to prove our main results, we shall need the following lemmas concerning functions with positive real part (see Duren, P.L. 1983, Grenander, U. and Szegö, G. 1958).

Lemma 1.2. Let $p\in\mathrm{P}$, then $\left|p_n\right|\le 2,\ n=1,2,3,\ldots$ and

$$\begin{aligned}\left|p_2-\frac{c}{2}p_1^2\right|&\le 2\cdot\max\left\{1,\left|c-1\right|\right\}\\&=2\cdot\begin{cases}1 & if\quad c\in[0,2],\\ \left|c-1\right| & elsewhere.\end{cases}\end{aligned}$$

Lemma 1.3. Let $p\in\mathrm{P}$, then $\left|p_n\right|\le 2$ for every $n=1,2,3,\ldots$ and

$$2p_2=p_1^2+\left(4-p_1^2\right)x,$$

$$\begin{aligned}4p_3=&\ p_1^3+2\left(4-p_1^2\right)p_1x-2\left(4-p_1^2\right)p_1x^2\\&+2\left(4-p_1^2\right)\left(1-|x|^2\right)z\end{aligned}$$

for some x and z with $|x|<1$ and $|z|<1$.

Lemma 1.4. Let $p\in\mathrm{P}$, $B\in[0,1]$ and $B(2B-1)\le D\le B$, then

$$\left|p_3-2Bp_1p_2+Dp_1^3\right|\le 2.$$

Remark 1.5. As it can be seen from the serial expansion of the function φ given in Definition 1.1, this function belongs to the class P.

In this paper, we give an upper bound estimate for the coefficients of the functions belonging to the class $S^*(\varphi)$ and examine the Fekete-Szegö problem for this class.

2. *Main results*

In this section, firstly we give the following theorem on the coefficient bound estimates for the class $S^*(\varphi)$.

Theorem 2.1. Let the function $f \in H(U)$ given by (1) be in the class $S^*(\varphi)$. Then,

$$|a_2| \le 1,\ |a_3| \le \frac{3}{4} \text{ and } |a_4| \le \frac{5}{6}.$$

Proof. Let the function $f \in H(U)$ given by (1) be in the class $S^*(\varphi)$. Then, according to Definition 1.1 there is an analytic function $\omega : U \to U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying the following condition

$$\frac{zf'(z)}{f(z)} = \varphi(\omega(z)) = \omega(z) + \sqrt{1+\omega^2(z)},\ z \in U. \tag{2}$$

Now, let us define the function $p \in \mathrm{P}$ as follows

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$
$$= 1 + \sum_{n=1}^{\infty} p_n z^n,\ z \in U.$$

From here, we get the following expression for the function ω

$$\omega(z) = \frac{p(z)-1}{p(z)+1}$$
$$= \frac{1}{2}\begin{bmatrix} p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 \\ + \left(p_3 - p_1 p_2 + \frac{p_1^2}{4}\right) z^3 + \cdots \end{bmatrix}, z \in U. \tag{3}$$

If we use series expansion of the function $\sqrt{1+\omega^2(z)}$ and change the expression of the function ω in (2) with the expression in (3), from (2), we obtain

$$\frac{zf'(z)}{f(z)} = 1 + \frac{p_1}{2} z + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right) z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{4}\right) z^3 + \cdots,\ z \in U. \tag{4}$$

If we make the necessary operations and simplifications in Equation (4) and the coefficients of the same order terms on the left and right sides of the equality are equalized, then the following expressions are obtained for the coefficients a_2, a_3 and a_4

$$2a_2 = p_1,\ 2a_3 - a_2^2 = \frac{p_2}{2} - \frac{p_1^2}{8},$$

$$3a_4 - 3a_2 a_3 + a_2^3 = \frac{p_3}{2} - \frac{p_1 p_2}{4}.$$

From these equalities, we can write

$$a_2 = \frac{p_1}{2}, \tag{5}$$

$$a_3 = \frac{1}{2} a_2^2 + \frac{1}{4}\left(p_2 - \frac{p_1^2}{4}\right), \tag{6}$$

$$a_4 = a_2 a_3 - \frac{1}{3} a_2^3 + \frac{1}{6}\left(p_3 - \frac{p_1 p_2}{2}\right). \tag{7}$$

By applying Lemma 1.2 to the equality (5), we immediately obtain the first result of the theorem.

Now firstly, considering (5), then using the Lemma 1.3 and applying the triangle inequality to the equality (6), we get the following inequality for $|a_3|$

$$|a_3| \le \frac{3}{16} t^2 + \frac{4-t^2}{8} \xi,\ \xi \in (0,1)$$

with $\xi = |x| < 1$. By maximizing the right-hand side of this inequality with respect to parameter ξ, we obtain

$$|a_3| \le \frac{3}{16} t^2 + \frac{4-t^2}{8},\ t \in [0,2].$$

Therefore,

$$|a_3| \le \frac{t^2}{16} + \frac{1}{2},\ t \in [0,2].$$

From the last inequality, we obtain the second result of the theorem.

Now let's find an upper bound estimate for the coefficient a_4. From equalities (5) - (7), we get

$$a_4 = \frac{p_1}{8}\left(p_2 - \frac{p_1^2}{4}\right) + \frac{1}{6}\left(p_3 - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right);$$

that is,

$$a_4 = \frac{p_1}{8}\left(p_2 - \frac{c}{2}p_1^2\right) + \frac{1}{6}\left(p_3 - 2Bp_1p_2 + Dp_1^3\right),$$

with $c = \frac{1}{2},\ B = \frac{1}{4}$ and $D = \frac{1}{8}$.

Applying the triangle inequality to the last equality, we obtain

$$|a_4| \le \frac{|p_1|}{8}\left|p_2 - \frac{c}{2}p_1^2\right| + \frac{1}{6}\left|p_3 - 2Bp_1p_2 + Dp_1^3\right|.$$

From this, applying Lemma 1.2 and Lemma 1.4, we obtain the desired upper bound estimate for $|a_4|$. With this, the proof of Theorem 2.1 is completed.

Now, we give the following theorem on the Fekete-Szegö problem for the class $S^*(\varphi)$.

Theorem 2.2. Let the function $f \in H(U)$ given by (1) be in the class $S^*(\varphi)$ and $\mu \in \mathbb{C}$. Then,

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{4}\cdot\begin{cases} 2 & if\ \ 2|1-2\mu| \le 1, \\ 2|1-2\mu|+1 & if\ \ 2|1-2\mu| > 1. \end{cases}$$

Proof. Let the function $f \in H(U)$ given by (1) be in the class $S^*(\varphi)$ and $\mu \in \mathbb{C}$. Then, from the expressions of the coefficients a_2 and a_3, obtained in the equalities (5) and (6), we write the following expression for $a_3 - \mu a_2^2$

$$a_3 - \mu a_2^2 = \frac{1}{2}(1-2\mu)a_2^2 + \frac{1}{4}\left(p_2 - \frac{p_1^2}{4}\right).$$

Considering (5) and using Lemma 1.3, the above expression can be written as follows

$$a_3 - \mu a_2^2 = \frac{1}{8}\left[(1-2\mu)\,p_1^2 + \frac{p_1^2}{2} + \left(4 - p_1^2\right)x\right]$$

for some x with $|x| < 1$. From this, using the triangle inequality we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{8}\left\{\left[|1-2\mu| + \frac{1}{2}\right]t^2 + \left(4 - t^2\right)\xi\right\},$$

$\xi \in (0,1)$,

with $\xi = |x|$. If we maximize the right-hand side of this inequality with respect to the parameter ξ, we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{8}\left\{\left[|1-2\mu| - \frac{1}{2}\right]t^2 + 4\right\},\ t \in [0,2]$$

.

Then, by maximizing the right-hand side of the last inequality with respect to the variable t, we arrive at the result of the theorem.

Thus, the proof of the Theorem 2.2 is completed.

3. *Conclusions*

In this section, we will focus on some special and general cases of the obtained results in the previous section.

In the case of $\mu \in \mathbb{R}$ the following theorem is easily proved.

Theorem 3.1. Let the function $f \in H(U)$ given by (1) be in the class $S^*(\varphi)$ and $\mu \in \mathbb{R}$. Then,

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{4}\cdot\begin{cases} 2 & if\ \ \frac{1}{4} \le \mu \le \frac{3}{4}, \\ 2|1-2\mu|+1 & if\ \ \mu < \frac{1}{4}\ \ or\ \ \mu > \frac{3}{4}. \end{cases}$$

If we take $\mu = 0$ and $\mu = 1$ respectively in Theorem 3.1, we get the following results.

Corollary 3.2. Let $f \in S^*(\varphi)$, then $|a_3| \le \frac{3}{4}$.

Corollary 3.3. Let $f \in S^*(\varphi)$, then

$$\left|a_3 - a_2^2\right| \le \frac{3}{4}.$$

NOTE: We would like to point out that we can also find the results obtained in the study for the class

$$M(\varphi,\beta)=\left\{\begin{array}{l} f\in S:\ (1-\beta)\dfrac{zf'(z)}{f(z)} \\ +\beta\dfrac{zf''(z)}{f'(z)}\prec\varphi(z), z\in U \end{array}\right\},$$

$\beta\in[0,1]$.

It is clear that the $M(\varphi,\beta)$-class is more general version of the class $S^*(\varphi)$. In fact, it is clear that $S^*(\varphi)=M(\varphi,1)$.

References

BRANNAN, D.A. & CLUNIE, J. (1980). Aspects of contemporary complex analysis. Academic Press, London and New York, USA.

BRANNAN, D.A. & TAHA, T.S. (1986). On some classes of bi-univalent functions. Studia Univ. *Babes-Bolyai Mathematics,* 31, 70-77.

DUREN, P.L. (1983). Univalent Functions. In: Grundlehren der Mathematischen Wissenschaften, Band 259, New- York, Berlin, Heidelberg and Tokyo, Springer-Verlag.

GRENANDER, U. & SZEGÖ, G. (1958). Toeplitz Form and Their Applications. California Monographs in Mathematical Sciences, University California Press, Berkeley.

FEKETE, M. & SZEGÖ, G. (1993). Eine Bemerkung Über Ungerade Schlichte Funktionen. *Journal of the London Mathematical Society,* 8, 85-89.

LEWIN, M. (1967). On a coefficient problem for bi-univalent functions. Proceedings of the American Mathematical Society, 18, 63-68.

MUSTAFA, N. (2017). Fekete-Szegö Problem for Certain Subclass of Analytic and Bi-Univalent Functions. *Journal of Scientific and Engineering Reserch*, 4(8), 30-400.

MUSTAFA, N. & GÜNDÜZ, M.C. (2019). The Fekete-Szegö Problem for Certain Class of Analytic and Univalent Functions. *Journal of Scientific and Engineering Reserch,* 6(5), 232-239.

MUSTAFA,N. & MURUGUSUNDARAMOORTHY, G. (2021). Second Hankel Determinant for Mocanu Type Bi-Starlike Functions Related to Shell Shaped Region. *Turkish Journal of Mathematics*, 45, 1270-1286.

NETANYAHU, E. (1969). The minimal distance of the image boundary from the origin and the second coefficient of a univalent function. *Archive for Rational Mechanics and Analysis,* 32, 100-112.

SRIVASTAVA, H.M., MISHRA, A.K. and GOCHHAYAT, P. (2010). Certain subclasses of analytic and bi-univalent functions. *Applied Mathematics Letters,* 23, 1188-1192.

ZAPRAWA, P. (2014). On the Fekete- Szegö Problem for the Classes of Bi-Univalent Functions. *Bulletin of the Belgain Mathematical Society,* 21, 169-178.

XU, Q.H., XIAO, G. and SRIVASTAVA, H.M. (2012). A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. Applied Mathematics and Computation, 218, 11461-11465.