Comparison of Three-Parameter Weibull Distribution Parameter Estimators with the Maximum Likelihood Method

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Abstract

Important distributions used to model and analyse data in various real-life sciences such as natural sciences, engineering, and medicine are the Weibull, Weibull exponential, and Weibull Rayleigh distribution. The main objective of this paper is to determine the best evaluators and compare them for the distribution with three-parameters of Weibull, Weibull Rayleigh and Exponential Weibull. The methods under consideration for comparing the parameter estimators for these distributions is that of maximum likelihood using the statistical program R for the application of real data. Based on the results obtained from this study, the maximum likelihood approach used in estimating the parameters is the comparison between these distributions.

Keywords: Exponential weibull distribution, maximum likelihood, parameters, weibull distribution, weibull-rayleigh distribution

1. INTRODUCTION

In probability theory and statistics, the Weibull distribution is a continuous probability distribution. The widespread interest in study of reliability is due to the fast development of the world, especially in the field of technology. The estimating parameters is the key to the life model, it can predict the life of product accurately in the reliability. The process of estimating three-parameter Weibull distribution is important because of the difficulty of obtaining the estimated parameters. The estimation process by using the maximum likelihood function requires iterative methods and therefore requires considerable time and effort [1].

In this paper, first describbes the Weibull distribution with three-parameters, where further the way of generating certain distributions is given, which is taken as the basic distribution of the Weibull distribution, then for a certain distribution such as Weibull-Rayleigh, the method of generation of the new

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distribution, then proceeds with the main properties of this distribution such as probability density function, distribution function, confidence interval for estimated parameters, asymptotic behavior of parameter estimators with maximum likelihood method using simulations for generating it data. Estimates of the parameters of these distributions can be found graphically via the probability bar graph, or analytically, using either the smallest squares (linear regression) or the method for calculating the maximum likelihood (MLE). In this paper, we will only use MLE for estimates of Weibull, Weibull-Rayleigh and Exponentiated Weibull Distribution parameters [2].

2. MATERIALS AND METHODS

2.1. Maximum Likelihood Estimator: Exponentiated Weibull Distribution (EW)

Let$X_1, X_2, \ldots, X_n$ random selection of size n with observed values $x_1, x_2, \ldots, x_n$ that follows the Exponentiated-Weibull distribution and let $\Psi = (\alpha, \beta, \theta)^T$ the parameter vector of the model under consideration. The EW distribution has the function of cumulative distribution (CDF) with three- parameters:

$$F(x; \alpha, \beta, \gamma) = \left(1 - e^{-\beta x^{\gamma}}\right)^{\alpha}, x > 0 \qquad (1)$$

and probability density function (PDF) with:

$$f(x; \alpha, \beta, \gamma) = \alpha\beta\gamma x^{\beta-1} e^{-\beta x^{\gamma}} \left(1 - e^{-\beta x^{\gamma}}\right)^{\alpha-1}, x > 0 \qquad (2)$$

Where, $\alpha > 0$ and $\gamma > 0$ are form parameters and $\lambda > 0$ is the scale parameter.

Gupta and Kundu (1999) considered a special case of EW distribution when $\gamma = 1$ and referred to it as Generalized Exponential Distribution (GE). GE distribution has received considerable attention in recent years. Readers refer to a review article by Gupta and Kundu (2001) for a current account on Generalized Exponential Distribution and a book length treatment of the various distributions expressed by Al-Hussaini and Ahsanullah (2015) in their works [3, 4].

Mudholkar et al. (1996) presented a family with three Generalized Weibull (GW) parameters containing unimodal and bath-shaped Hazard Distribution. They showed that distributions in this household are analytical and manageable computationally. Modelling and data analysis using this family of distributions are discussed and illustrated in the analysis section [5].

2.2. Maximum Likelihood Estimator: Weibull-Rayleigh Distribution

In this section we have studied the three-parameter Weibull Rayleigh (WR) Distribution [6, 7]. Use $G(x)$ and $g(x)$ in in the formula:

$$f(x) = \sum_{i=o}^{\infty} \sum_{j=o}^{\infty} \omega_{i,j}\, g(x;\zeta) \left(G(x;\zeta)\right)^{\beta(i+1)+j-1} \qquad (3)$$

to be cdf and pdf of

$$g(x, \theta) = \theta x e^{-\frac{\theta}{2}x^2}, x > 0, \theta > 0 \qquad (4)$$

$$G(x, \theta) = 1 - e^{-\frac{\theta}{2}x^2}, x > 0, \theta > 0 \qquad (5)$$

Cdf of Weibull-Rayleigh Distribution provided by:

$$F(x; \alpha, \beta, \gamma) = 1 - e^{-\alpha\left(e^{\frac{\gamma}{2}x^2 - 1}\right)^{\beta}}, x > 0 \qquad (6)$$

Relevant PDF of the WR Distribution is provided by:

$$f(x; \alpha, \beta, \gamma) = \alpha\beta\gamma x e^{\frac{\gamma}{2}x^2} \left(e^{\frac{\gamma}{2}x^2} - 1\right)^{\beta-1} e^{-\alpha\left(e^{\frac{\gamma}{2}x^2} - 1\right)^{\beta}} \qquad (7)$$

2.3. Maximum Likelihood Estimator: Weibull Distribution

Weibull Distribution is a generalization of Exponential Distribution and Weibull-Rayleigh Distribution. For random selection $X_1, X_2, \ldots, X_n$ of size n with the observed values $x_1, x_2, \ldots, x_n$ that follows the Weibull Distribution and let it be $\Psi = (\alpha, \beta, \theta)^T$ the parameter vector of the model under consideration. The cumulative function (CDF) with three Weibull Distribution parameters and the probability density function (pdf) are [8-10]:

$$F(x,\alpha,\beta,\theta) = 1 - e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}}, x > 0. \quad (8)$$

and

$$f(x,\alpha,\beta,\theta) = \alpha\beta\theta x e^{\frac{\theta}{2}x^2}\left(e^{\frac{\theta}{2}x^2} - 1\right)^{\beta-1} e^{-\alpha\left(e^{\frac{\theta}{2}x^2}-1\right)^{\beta}}$$

$$x > 0, \quad (9)$$

The function of likeness is:

$$L(x_i,\alpha,\beta,\theta) = f(x_1,x_2,\ldots,x_n:\alpha,\beta,\theta) = \prod_{i=1}^{n} \alpha\beta\theta x_i e^{\frac{\alpha}{2}{x_i}^2}\left(e^{\frac{\alpha}{2}{x_i}^2} - 1\right)^{\beta-1} e^{-\alpha\left(e^{\frac{\alpha}{2}{x_i}^2}-1\right)^{\beta}}$$

Taking the natural logarithm, we have:

$$l = \ln L = n\log\alpha + n\log\beta + n\log\theta + \sum_{i=1}^{n}\log x_i + \frac{\theta}{2}\sum_{i=1}^{n} x_i^2 + (\beta - 1)\sum_{i=1}^{n}\log\left(e^{\frac{\theta}{2}x_i^2} - 1\right) - \alpha\sum_{i=1}^{n}\left[e^{\frac{\theta}{2}x_i^2} - 1\right]^{\beta} \quad (10)$$

Deriving (10) based on parameters α, β, and θ, as well as equalizing the results to zero, we get the maximum of this function. Partial derivatives of $L(\phi)$ with respect to each parameter are [11]:

$$U_n(\phi) = \frac{\partial(L)}{\partial\alpha}, \frac{\partial(L)}{\partial\beta}, \frac{\partial(L)}{\partial\theta}$$

where

$$\frac{\partial(L)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left[e^{\frac{\theta}{2}x_i^2} - 1\right]^{\beta} = 0, \quad (11)$$

$$\frac{\partial(L)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left(e^{\frac{\theta}{2}x_i^2} - 1\right) - \alpha\sum_{i=1}^{n}\left[e^{\frac{\theta}{2}x_i^2} - 1\right]^{\beta}\log\left[e^{\frac{\theta}{2}x_i^2} - 1\right] = 0 \quad (12)$$

and

$$\frac{\partial(L)}{\partial\theta} = \frac{n}{\theta} + \frac{1}{2}\sum_{i=1}^{n} x_i^2 + (\beta - 1)\sum_{i=1}^{n}\frac{x_i^2 e^{\frac{\theta}{2}x_i^2}}{2\left(e^{\frac{\theta}{2}x_i^2} - 1\right)} - \frac{\alpha\beta}{2}\sum_{i=1}^{n} x_i^2 e^{\frac{\theta}{2}x_i^2}\left[e^{\frac{\theta}{2}x_i^2} - 1\right]^{\beta-1} = 0 \quad (13)$$

Evaluation of parameters with the method of maximum similarity for parameters α, β, and θ, të themi $\hat{\alpha}, \hat{\beta}$, and $\hat{\theta}$ are obtained by solving the equations [12-14]:

$$\frac{\partial(L)}{\partial\alpha} = \frac{\partial(L)}{\partial\beta} = \frac{\partial(L)}{\partial\theta} = 0.$$

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\theta}\end{pmatrix} \sim N\left[\begin{pmatrix}\alpha\\ \beta\\ \theta\end{pmatrix}\right], \begin{pmatrix}\widehat{V\alpha\alpha} & \widehat{V\alpha\beta} & \widehat{V\alpha\theta}\\ \widehat{V\beta\alpha} & \widehat{V\beta\beta} & \widehat{V\beta\theta}\\ \widehat{V\theta\alpha} & \widehat{V\theta\beta} & \widehat{V\vartheta\vartheta}\end{pmatrix} \quad (14)$$

$$V^{-1} = -E\begin{bmatrix}\widehat{V\alpha\alpha} & \widehat{V\alpha\beta} & \widehat{V\alpha\theta}\\ \widehat{V\beta\alpha} & \widehat{V\beta\beta} & \widehat{V\beta\theta}\\ \widehat{V\theta\alpha} & \widehat{V\theta\beta} & \widehat{V\vartheta\vartheta}\end{bmatrix} \quad (15)$$

where,

(

$$V\alpha\alpha = \frac{\partial^2 L}{\partial \alpha^2} = -\frac{n}{\alpha^2} \quad (16)$$

$$V_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2} = -\frac{n}{\beta^2} - \alpha \sum_{i=1}^{n} \left(e^{1/2\theta x_i^2} - 1\right)^{\beta} \left(ln\left(e^{1/2\theta x_i^2} - 1\right)\right)^2$$

$$V_{\theta\theta} = \frac{\partial^2 L}{\partial \theta^2} = -\frac{n}{\theta^2} - (\beta - 1) \sum_{i=1}^{n} 1/4 \frac{x_i^4 e^{1/2\theta x_i^2}}{\left(e^{1/2\theta x_i^2} - 1\right)^2} - \frac{\alpha}{4} \sum_{i=1}^{n} \frac{\left(e^{1/2\theta x_i^2} - 1\right)^{\beta} \beta x_i^4 e^{1/2\theta x_i^2} \left(\beta e^{1/2\theta x_i^2} - 1\right)}{\left(e^{1/2\theta x_i^2} - 1\right)^2}$$

$$V_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} \left(e^{1/2\theta x_i^2} - 1\right)^{\beta} ln\left(e^{1/2\theta x_i^2} - 1\right)$$

$$V_{\alpha\theta} = \frac{\partial^2 L}{\partial \alpha \partial \theta} = -\frac{1}{2} \sum_{i=1}^{n} \frac{\left(e^{1/2\theta x_i^2} - 1\right)^{\beta} \beta x_i^2 e^{1/2\theta x_i^2}}{e^{1/2\theta x_i^2} - 1}$$

$$V_{\beta\theta} = \frac{\partial^2 L}{\partial b \partial \theta} = \sum_{i=1}^{n} 1/2 \frac{x_i^2 e^{1/2\theta x_i^2}}{e^{1/2\theta x_i^2} - 1} - \frac{\alpha}{2} \sum_{i=1}^{n} \frac{\left(e^{\frac{1}{2\theta x_i^2}} - 1\right)^{\beta} x_i^2 e^{\frac{1}{2\theta x_i^2}} \left(\beta\, ln\left(e^{\frac{1}{2\theta x_i^2}} - 1\right) + 1\right)}{e^{\frac{1}{2\theta x_i^2}} - 1} \quad (17)$$

Finding the inverse matrix of the above matrix we will obtain the asymptotic values of variance and covariance for the estimators $\hat{\alpha}, \hat{\beta},$ and $\hat{\theta}$. Using (15), we approximate $100(1-\gamma)\%$ confidence interval for α, β and θ like below:

$$\hat{\alpha} \pm z_{\frac{\gamma}{2}} \sqrt{\widehat{V\alpha\alpha}}, \hat{\beta} \pm z_{\frac{\gamma}{2}} \sqrt{\widehat{V\beta\beta}}, \hat{\theta} \pm z_{\frac{\gamma}{2}} \sqrt{\widehat{V\theta\theta}} \quad (18)$$

where, z_{γ} is 100γ standardized normal distribution percentage.

3. RESULTS

3.1. Weibull Distribution, Exponential-Weibull and Weibull-Rayleigh Applications with Three- Parameters

In this section we will compare the data [15] adjustment results with Weibull-Rayleigh, Exponential-Weibull and Weibull Distributions. Data are taken from the study conducted by Kamberi, Iljazi and Orhani (2021) [16]. The main purpose of the study is to compare the results of the students who are using information technology in learning with those students who are not using it. Therefore, we here will take the results of student failure before and after the tests developed with the integration of technology and without its use. These data were analysed by means of the R program for our study conducted for the real data. Based on the results obtained from this study, the maximum likelihood approach used in estimating the parameters is the comparison between these distributions. Student data according to failure points are presented as follows:

Sample 1: Failure results according to the test that students have not used information technology:

45, 64, 26, 38, 11, 5, 11, 34, 28, 51, 5, 23, 11, 26, 30, 17, 59, 26, 26, 21

Sample 2: Failure results according to the test that students have used information technology:

75, 41, 18, 23, 8, 5, 13, 29, 34, 45, 0, 18, 11, 18, 23, 34, 56, 18, 26, 11

In this section, we obtained two sets of data, representing the results of student failure from the reference study. To test the fit of the two data samples for Weibull Distribution with three-parameters, the Kolmogorov-Smirnov test was used with the resulting values 0.2 and the respective P values is 0.7537. It therefore refers that the three-parameter Weibull Distribution can fit into both data samples.

For the purpose of estimating the reliability function of the three-parameter Weibull Distribution, the methods described in this paper were used. In order to obtain the best estimate, we are presenting the results in the following table:

Table 1 Evaluating maximum likelihood estimation

Model	Parameters	ML – Sample 1	ML – Sample 2
Weibull Distribution	$\hat{\alpha}$	1.775309	1.728758
	$\hat{\beta}$	31.4732	32.86591
	$\hat{\theta}$	-0.120412	-3.89424
Exponentiated Weibull Distribution	$\hat{\alpha}$	1.7251257	1.719023
	$\hat{\beta}$	31.057	32.8991
	$\hat{\theta}$	-0.140	-3.99023
Weibull-Rayleigh Distribution	$\hat{\alpha}$	1.7109	1.7201
	$\hat{\beta}$	31.657	32.8098
	$\hat{\theta}$	-0.121223	-3.70982

Estimating parameters of three-parameter Weibull Distribution, Exponential-Weibull Distribution and Weibull-Rayleigh is an important factor in reliability. The estimation of Weibull Distribution parameters using traditional techniques such as maximum likelihood function is difficult because it is nonlinear functions. Therefore, from the results of the study we are noticing that the Weibull Distribution model for sample 1 shows the findings for the parameters $\hat{\alpha} = 1.77$, $\hat{\beta} = 31.47$ and $\hat{\theta} = -0.12$, whereas for sample 2 displays these findings for the parameters $\hat{\alpha} = 1.73$, $\hat{\beta} = 31.87$ and $\hat{\theta} = -3.89$. On the other hand, from the results of the study we are noticing that the Exponential-Weibull Distribution model for sample 1 shows the findings for the parameters $\hat{\alpha} = 1.73$, $\hat{\beta} = 31.06$ and $\hat{\theta} = -0.14$, whereas for sample 2 displays these findings for the parameters $\hat{\alpha} = 1.72$, $\hat{\beta} = 32.89$ and $\hat{\theta} = -3.99$. And finally, from the results of the study we are noticing that the Weibull-Rayleigh Distribution model for sample 1 shows the findings for the parameters $\hat{\alpha} = 1.71$, $\hat{\beta} = 31.66$ and $\hat{\theta} = -0.12$, whereas for sample 2 displays these findings for the parameters $\hat{\alpha} = 1.72$, $\hat{\beta} = 32.81$ and $\hat{\theta} = -3.71$. From Table 1 we notice that the best estimators for choices 1 and 2 are for $\hat{\alpha}$ and $\hat{\beta}$ parameters compared to $\hat{\theta}$ parameter, since their values are approximate for the three distributions with the maximum depth method. The eight parameter values for choice 1 and 2 are the best estimators, therefore the Weibull-Rayleigh distribution represents the closest distribution to the Weibull distribution, compared to the Exponential Weibull distribution.

Comparing the results from sample 1 and sample 2 for Weibull Distribution, we are noticing that for sample 1 shape is 1.76, standard deviation with 0.31, as well as scale with 31.33 and standard deviation with 4.18. On the other hand, we are noticing that for the sample 2 shape is 1.18, standard deviation with 0.12, as well as scale with 26.4 and standard deviation with 5.16. These findings are presented below:

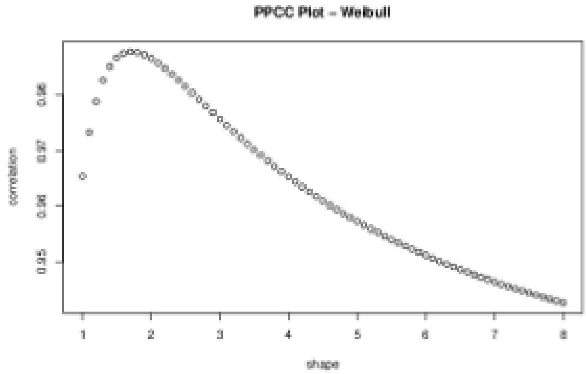

Figure 1 Weibull Distribution for sample 1

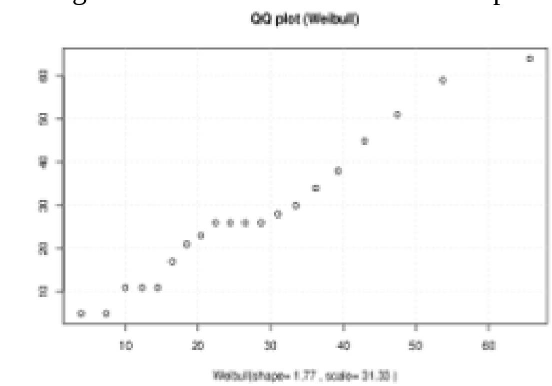

Figure 2 Weibull Distribution for sample 1

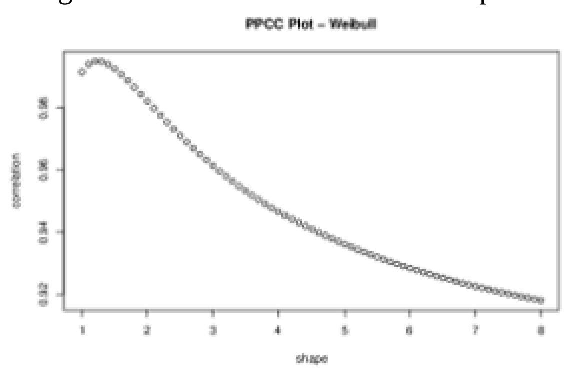

Figure 3 Weibull Distribution for sample 2

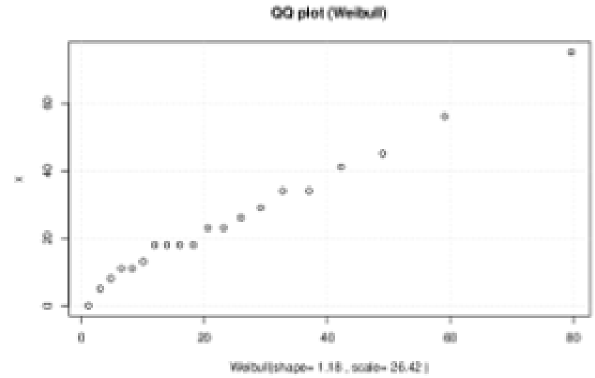

Figure 4 Weibull Distribution for sample 2

Figures 1, 2 and 3, 4 show that the data are distributed approximately with the Weibull distribution. Figure 1, 2 has a distribution that closely approximates the Weibull distribution for the sample 1 data compared to the sample 2 data.

Comparing the results from sample 1 and sample 2 for Exponential-Weibull Distribution, we are noticing that for sample 1 the Estimated Value rate is 0.04 and the standard deviation is 0.008. On the other hand, we are noticing that for sample 2 the rate of Estimated Value is like the result of sample 1 with 0.04 and standard deviation with 0.009. These findings are presented below:

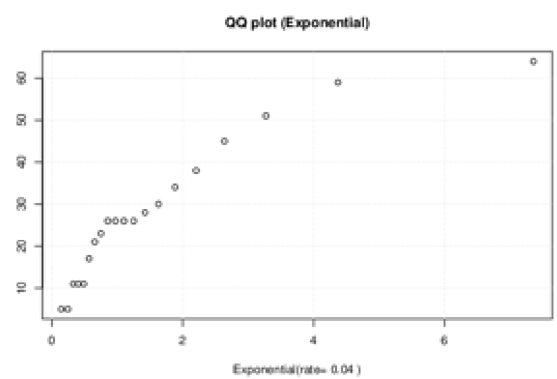

Figure 5 Exponential Weibull Distribution for sample 1

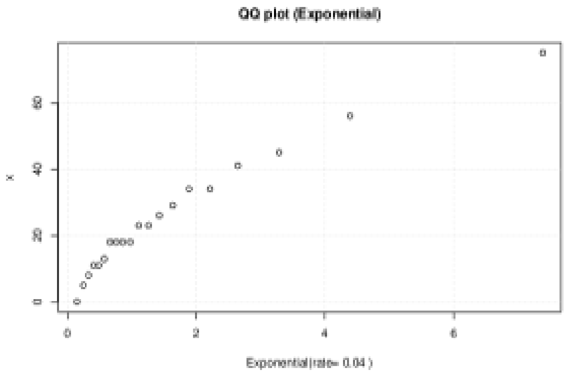

Figure 6 Exponential Weibull Distribution for sample 2

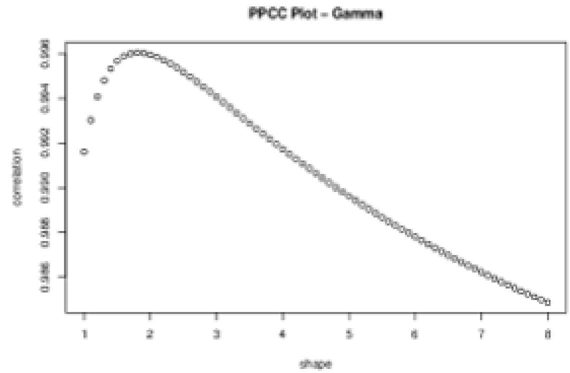

Figure 8 Weibull Rayleigh Distribution for sample 2

On the other hand, Figure 5 and 6 does not give a clear picture of the approximate Weibull Distribution for the data of sample 1 and 2.

The results from sample 1 are showing that for Weibull-Rayleigh Distribution we have for shape a Estimated Value with 2.53 and standard deviation with 0.75, as well as a rate for Estimated Value with 0.09 and a standard deviation with 0.023. On the other hand, the results from sample 2 are showing that for Weibull-Rayleigh Distribution we have for shape an Estimated Value with 1.06 and a standard deviation of 0.29, as well as a rate for Estimated Value with 0.04 and a standard deviation with 0.15. These findings are presented below:

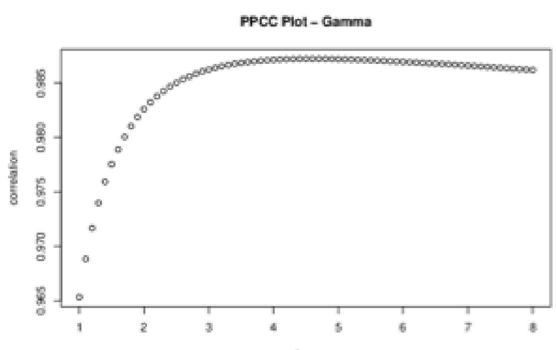

Figure 7 Weibull Rayleigh Distribution for sample 1

However, Figure 7 and 8 gives a better view of the Weibull Distribution for the data in sample 1 and 2, and in particular for sample 2.

4. CONCLUSION AND DISCUSSION

In this paper are given several types of distributions with three-parameters, starting from that of Weibull Distribution, Exponential-Weibull Distribution and Weibull-Rayleigh Distribution. Basic knowledge about these distributions is given first and the theoretical description of the maximum likelihood method estimating (MLE) for assessment the parameters.

Simultaneously in this study the data in the statistical program R were analysed, to see how the statistical parameters are evaluated and depending on it the parameter evaluations were done with the method of maximum likelihood which is shown which distribution is more efficient. Also, an application with real data obtained from the study of the authors Kamberi, Iljazi and Orhani (2021), where it is clearly seen that the distribution of Weibull is one of the distributions that best interprets this data and gives a proper statistical analysis [16, 17].

From the results of Table 1 it is clear that the maximum likelihood (MLE) method is quite clear as a method and can be used efficiently in practice to estimate the unknown parameters of the Weibull (W), Exponential-Weibull (EW), and Weibull-Rayleigh (WR) models.

In this paper we examined the Weibull distribution for the real data set and also compared it to several sub distributions (models) such as the Exponential-Weibull and Weibull-Rayleigh distributions to show its performance. Therefore, the distributions reviewed provide a very good opportunity in practice for data analysis.

The results of Table 1 were analysed using statistical program R. From the parameter values in the table, we conclude that the Weibull-Rayleigh model provides better estimators for the selection data than other models. Also, the model graphs suggest that the Weibull-Rayleigh model provides better parameter estimators compared to other models according to the maximum likelihood (MLE) method.

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Authors' Contribution

The authors contributed equally to the study.

The Declaration of Conflict of Interest/ Common Interest

No conflict of interest or common interest has been declared by the authors.

The Declaration of Ethics Committee Approval

This study does not require ethics committee permission or any special permission.

The Declaration of Research and Publication Ethics

The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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