

Loxodromes On Twisted Surfaces in Euclidean 3-Space

Mustafa ALTIN^{1*}

¹ Technical Sciences Vocational School, Bingöl University, Bingöl, Türkiye

*1 maltin@bingol.edu.tr

(Geliş/Received: 28/06/2022;

Kabul/Accepted: 31/08/2022)

Abstract: In the present paper, loxodromes, which cut all meridians and parallels of twisted surfaces (that can be considered as a generalization of rotational surfaces) at a constant angle, have been studied in Euclidean 3-space and also some examples have been constructed to visualize and support our theory.

Key words: Loxodromes, meridian, twisted surface.

Öklidyen 3-Uzayında Twisted Yüzeyler Üzerinde Loksodromlar

Öz: Bu makalede, Öklidyen 3-uzayında twisted yüzeylerin (ki bunlar dönel yüzeylerin bir genelleştirilmesi olarak düşünülebilir) tüm meridyen ve paralellerini sabit bir açı ile kesen loksodromlar çalışıldı ve ayrıca görselleştirmek ve teorimizi desteklemek için bazı örnekler inşa edildi.

Anahtar kelimeler: Loksodromlar, meridyen, twisted yüzeyler.

1. General Information and Basic Concepts

Loxodromes (also known as rhumb lines) correspond to the curves which intersect all of the meridians at a constant angle on the Earth (see Figure 1). An aircraft flying and a ship sailing on a fixed magnetic compass course move along a curve. Here the course is a rhumb and the curve is a loxodrome. Generally, a loxodrome is not a great circle, thus it does not measure the shortest distance between two points on the Earth. However, loxodromes are important in navigation and they should be known by aircraft pilots and sailors [1].

If the shape of the Earth is approximated by a sphere, then the loxodrome is a logarithmic spiral that cuts all meridians at the same angle and asymptotically approaches the Earth's poles but never meets them. Since maritime surface navigation defines the course as the angle between the current meridian and the longitudinal direction of the ship, it may be concluded that the loxodrome is the curve of the constant course, which means that whenever navigating on an unchanging course we are navigating according to a loxodrome [11].

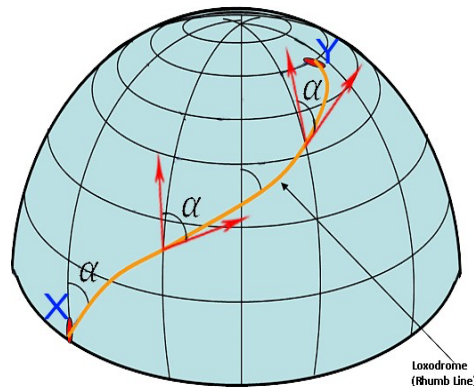


Figure 1. Loxodrome on Earth

In this context, there are lots of studies about loxodromes in Euclidean and Minkowskian spaces. For instance, the differential equations of loxodromes on a sphere, spheroid, rotational surface, helicoidal surface and canal surface in Euclidean 3-space have been given in [11], [13], [12], [2] and [3], respectively. Also, in [4] and [5], spacelike and timelike loxodromes on rotational surface and in [6], differential equations of the spacelike

Loxodromes On Twisted Surfaces in Euclidean 3-Space

loxodromes on the helicoidal surfaces in Lorentz-Minkowski 3-space have been given.

Now, we recall some basic notions about curves and twisted surfaces in Euclidean 3-space E^3 .

For two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in E^3 the inner product of these vectors and the norm of the vector \vec{u} are defined by

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 \quad (1)$$

and

$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}, \quad (2)$$

respectively. We say that \vec{u} is a unit vector, if $\|\vec{u}\| = 1$.

The arc-length of a regular curve $\alpha: I \subset \mathbb{R} \rightarrow E^3, s \rightarrow \alpha(s)$, between s_0 and s is

$$t(s) = \int_{s_0}^s \|\alpha'(s)\| ds. \quad (3)$$

Then the parameter $t \in J \subset \mathbb{R}$ is determined as $\|\alpha'(s)\| = 1$.

Also, the angle ϕ ($0 < \phi < \pi$) between the vectors \vec{u} and \vec{v} is

$$\cos \phi = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}. \quad (4)$$

Here, we recall the definition and parametrization of twisted surfaces in E^3 (for detail, see [8]).

A twisted surface in E^3 is obtained by rotating a planar curve α in its supporting plane while this plane itself is rotated about some containing straight line. Without loss of generality, the coordinate system can be chosen in such a way that the xz -plane corresponds with the plane supporting the planar curve with the z -axis as its containing rotation axis and that the straight line through the point $(a, 0, 0)$ parallel with the y -axis acts as rotation axis for the planar curve.

Firstly, let us apply the rotation about the straight line through the point $(a, 0, 0)$ parallel with the y -axis to the profile curve $\alpha(y) = (f(y), 0, g(y))$, (f and g are real-valued functions) and next apply the rotation about the z -axis to the obtained surface. Then, up to a transformation, we get the parametrization of the twisted surface in E^3

$$T(x, y) = \begin{pmatrix} (a + f(y) \cos(bx) - g(y) \sin(bx)) \cos x, \\ (a + f(y) \cos(bx) - g(y) \sin(bx)) \sin x, \\ f(y) \sin(bx) + g(y) \cos(bx) \end{pmatrix}. \quad (5)$$

Here, the presence of the factor $b \in \mathbb{R}$ allows for differences in the rotation speed of both rotations and it is obvious from the construction that, if we take $b = 0$, then the twisted surface reduces to a surface of revolution. Thus, the twisted surfaces can be considered as generalizations of surfaces of revolution.

After giving the definition of the twisted surfaces, twisted surfaces with null rotation axis in Minkowski 3-space have been studied in [9] and twisted surfaces with vanishing curvature in Galilean 3-space have been classified in [7]. Also, in [10], the twisted surfaces in pseudo-Galilean space have been studied.

In this paper, we have studied loxodromes on twisted surfaces in Euclidean 3-space and also we have constructed some examples to visualize and support our results.

2. Loxodromes on Twisted Surfaces in E^3

In this section, we obtain the equations of loxodromes on the twisted surfaces in E^3 .

Let T be the twisted surface which parametrized as (5). Then the coefficients of first fundamental form of the twisted surface T are obtained by

$$g_{11} = \frac{1}{2} \begin{pmatrix} 2a^2 + (1 + 2b^2 + \cos(2bx))f^2 + (1 + 2b^2 - \cos(2bx))g^2, \\ -4agsin(bx) + 4f \cos(bx)(a - g \sin(bx)) \end{pmatrix},$$

$$\begin{aligned} g_{12} &= g_{21} = b(fg' - f'g), \\ g_{22} &= f'^2 + g'^2, \end{aligned} \tag{6}$$

where $g_{11} = \langle T_x, T_x \rangle$, $g_{12} = g_{21} = \langle T_x, T_y \rangle$, $g_{22} = \langle T_y, T_y \rangle$, $f = f(y)$, $g = g(y)$, $f' = \frac{df(y)}{dy}$ and $g' = \frac{dg(y)}{dy}$.

Also, we know that the first fundamental form in the base $\{M_x, M_y\}$ for a surface $M(x, y)$ is given by

$$ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2, \tag{7}$$

where g_{ij} are the coefficients of the first fundamental form of M . So, from (6) and (7), we can write the first fundamental form of the twisted surface (5) as

$$ds^2 = \left(\begin{aligned} &a^2 - 2agsin(bx) + 2fcos(bx)(a - gsin(bx)) \\ &+ \frac{1}{2}((1 + 2b^2 + cos(2bx))f^2 + (1 + 2b^2 - cos(2bx))g^2) \end{aligned} \right) dx^2 \\ + 2b(fg' - f'g)dxdy + (f'^2 + g'^2)dy^2 \tag{8}$$

and from (8), the arc-length of any curve on the twisted surface between x_1 and x_2 is given by

$$s = \left| \int_{x_1}^{x_2} \sqrt{\begin{aligned} &a^2 - 2agsin(bx) + 2fcos(bx)(a - gsin(bx)) \\ &+ \frac{1}{2}((1 + 2b^2 + cos(2bx))f^2 + (1 + 2b^2 - cos(2bx))g^2) \\ &+ 2b(fg' - f'g)\frac{dy}{dx} + (f'^2 + g'^2)\left(\frac{dy}{dx}\right)^2 \end{aligned}} dx \right|. \tag{9}$$

Furthermore, a curve γ is called a loxodrome on the twisted surface T in E^3 if it cuts all meridians (y constant) (or parallels (x constant)) of T at a constant angle.

Now, let us suppose that $\gamma(t) = T(x(t), y(t))$; i.e. $\gamma(t)$ is a curve on the twisted surface T . With respect to the local base $\{T_x, T_y\}$, the vector $\gamma'(t)$ has the coordinates (x', y') and the vector T_x has the coordinates $(1, 0)$. At the point $p = T(x, y)$; where the loxodrome cuts the meridians at a constant angle; we get

$$\cos\phi = \frac{\langle T_x, \gamma'(t) \rangle}{\|T_x\| \|\gamma'(t)\|} = \frac{g_{11}dx + g_{12}dy}{\sqrt{g_{11}^2 dx^2 + 2g_{11}g_{12}dxdy + g_{11}g_{22}dy^2}}. \tag{10}$$

Therefore, keeping in mind (10) we get

$$g_{11}^2 \sin^2\phi dx^2 + 2g_{11}g_{12} \sin^2\phi dxdy + (g_{12}^2 - g_{11}g_{22} \cos^2\phi) dy^2 = 0 \tag{11}$$

and so,

$$\frac{dy}{dx} = \frac{-2g_{11}g_{12} \sin^2\phi \mp g_{11} \sqrt{g_{11}g_{12} - g_{11}^2} \sin(2\phi)}{2(g_{12}^2 - g_{11}g_{22} \cos^2\phi)}. \tag{12}$$

Hence from (6) and (12), the loxodrome on the twisted surface (5) must satisfy

$$\frac{dy}{dx} = - \frac{\left(\begin{aligned} &2b(fg' - f'g) \sin\phi \\ &- b^2(fg' - f'g)^2 + \end{aligned} \right) \times \left(\begin{aligned} &\mp 2 \cos\phi \sqrt{(f'^2 + g'^2) \left(\begin{aligned} &a^2 - 2agsin(bx) + 2fcos(bx)(a - gsin(bx)) \\ &+ \frac{1}{2}((1 + 2b^2 + cos(2bx))f^2 \\ &+ (1 + 2b^2 - cos(2bx))g^2) \end{aligned} \right)} \\ &(2a^2 + (1 + 2b^2 + cos(2bx))f^2 + (1 + 2b^2 - cos(2bx))g^2) \sin\phi \\ &- 4agsin(bx) + 4fcos(bx)(a - gsin(bx)) \end{aligned} \right)}{4 \left(\begin{aligned} &b^2(fg' - f'g)^2 - \\ &(f'^2 + g'^2) \left(\begin{aligned} &a^2 - 2agsin(bx) + 2fcos(bx)(a - gsin(bx)) \\ &+ \frac{1}{2}((1 + 2b^2 + cos(2bx))f^2 \\ &+ (1 + 2b^2 - cos(2bx))g^2) \end{aligned} \right) \cos^2\phi \end{aligned} \right)}. \tag{13}$$

Now, we give some examples to support our results that obtained in the paper.

Loxodromes On Twisted Surfaces in Euclidean 3-Space

Example 1. Taking the profile curve as $\alpha(y) = (y, 0, 0)$, the twisted surface (5) becomes

$$T(x, y) = ((a + y \cos(bx)) \cos x, (a + y \cos(bx)) \sin x, y \sin(bx)). \quad (14)$$

From (13) (we take \mp in this equation as $-$), we get the differential equation of the loxodrome on the twisted surface (14) for $a = 0$ and $b = \frac{1}{2}$ as

$$\frac{dy}{dx} = \frac{y \sqrt{2 \cos x + 3 \tan \phi}}{2}. \quad (15)$$

Thus, we write

$$\frac{dy}{y} = \frac{\sqrt{2 \cos x + 3 \tan \phi}}{2} dx$$

and by integrating both sides of this equation, we get

$$\ln y = \int_{x_0}^x \frac{\sqrt{2 \cos x + 3 \tan \phi}}{2} dx. \quad (16)$$

Putting $x_0 = 0$ in (16), we reach that

$$y = y(x) = e^{\sqrt{\frac{5}{3}} \text{EllipticE}\left[\frac{x}{2}, \frac{4}{5}\right] \tan \phi}. \quad (17)$$

Now, if we take $\phi = \frac{\pi}{6}$ and $x \in (-2\pi, 2\pi)$, we get $y \in (0.0476989, 20.9649)$. Thus, the loxodrome which lies on the twisted surface (14) is obtained as

$$\gamma(x) = e^{\sqrt{\frac{5}{3}} \text{EllipticE}\left[\frac{x}{2}, \frac{4}{5}\right]} \cdot \left(\cos\left(\frac{x}{2}\right) \cos x, \cos\left(\frac{x}{2}\right) \sin x, \sin\left(\frac{x}{2}\right) \right). \quad (18)$$

Also, the arc-length of our loxodrome (18) is approximately equal to 41.8343. The twisted surface (14), meridian for $y = 15$ and the loxodrome (18) can be seen in Figure 2.

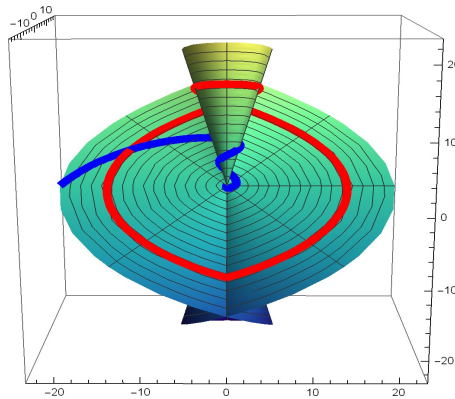


Figure 2. Twisted surface (14), Meridian (red) and Loxodrome (blue)

Example 2. For the profile curve $\alpha(y) = (\cos y, 0, \sin y)$ the twisted surface (5) is

$$T(x, y) = \begin{pmatrix} (a + \cos y \cos(bx) - \sin y \sin(bx)) \cos x, \\ (a + \cos y \cos(bx) - \sin y \sin(bx)) \sin x, \\ \cos y \sin(bx) + \sin y \cos(bx) \end{pmatrix}. \quad (19)$$

If we take $a = 1$ and $b = 0$ then from (13) (we take \mp in this equation as $-$), we have

$$\frac{dy}{dx} = 2 \cos^2\left(\frac{y}{2}\right) \tan \phi \quad (20)$$

and hence

$$\frac{dy}{2 \cos^2\left(\frac{y}{2}\right)} = \tan \phi dx \quad (21)$$

and by integrating both sides of this equation, we get

$$\tan\left(\frac{y}{2}\right) = \int_{x_0}^x \tan \phi dx. \quad (22)$$

Taking $x_0 = 0$ in (22), we arrive at

$$y = y(x) = 2 (\arctan(x \tan \phi) + c\pi), \quad c \in \mathbb{Z}. \quad (23)$$

Here, if we take $c = 0$, $\phi = \frac{\pi}{4}$ and $x \in (-\pi, \pi)$ we have $y \in (-2.52525, 2.52525)$. Thus, the loxodrome which lies on the twisted surface (19) is obtained as

$$\gamma(x) = ((1 + \cos(2\arctan x)) \cos x, (1 + \cos(2\arctan x)) \sin x, \sin(2\arctan x)). \quad (24)$$

Also, the arc-length of our loxodrome (24) is approximately equal to 7.1425. One can see the twisted surface (19), meridian for $y = 1$ and the loxodrome (24) in Figure 3.

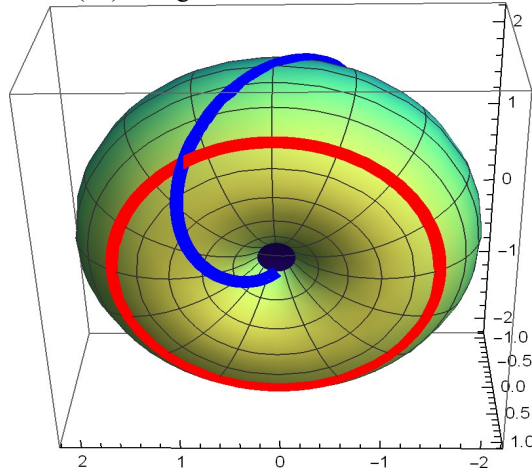


Figure 3. Twisted surface (2.14), Meridian (red) and Loxodrome (blue)

Furthermore, from the definition of the angle θ between the loxodrome and any parallel ($x=\text{constant}$), we have

$$\cos \theta = \frac{\langle \gamma'(t), T_y \rangle}{\|\gamma'(t)\| \|T_y\|} = \frac{g_{12} dx + g_{22} dy}{\sqrt{g_{11}g_{22} dx^2 + 2g_{12}g_{22} dx dy + g_{22}^2 dy^2}}. \quad (25)$$

From (25), we get

$$(g_{11}g_{22} \cos^2 \theta - g_{12}^2) dx^2 - 2g_{12}g_{22} \sin^2 \theta dx dy - g_{22}^2 \sin^2 \theta dy^2 = 0 \quad (26)$$

and so,

Loxodromes On Twisted Surfaces in Euclidean 3-Space

$$\frac{dx}{dy} = \frac{-2g_{12}g_{22} \sin^2 \theta \pm g_{22} \sqrt{g_{11}g_{22} - g_{12}^2} \sin(2\theta)}{2(g_{12}^2 - g_{11}g_{22} \cos^2 \theta)}. \quad (27)$$

Therefore, the loxodrome on the twisted surface must

$$\frac{dx}{dy} = \frac{\left(\mp 2 \cos \theta \sqrt{\frac{2b \sin \theta (f'g - fg')}{-b^2(gf' - fg')^2} + \left(\begin{array}{l} a^2 - 2ag \sin(bx) + 2f \cos(bx)(a - g \sin(bx)) \\ + \frac{1}{2} (1 + 2b^2 + \cos(2bx)) f^2 \\ + (1 + 2b^2 - \cos(2bx)) g^2 \end{array} \right) (f'^2 + g'^2)}{2b^2 (f'g - fg')^2} \right) (f'^2 + g'^2) \sin \theta}{\left(\begin{array}{l} 2a^2 - 4ag \sin(bx) + 4f \cos(bx)(a - g \sin(bx)) \\ + (1 + 2b^2 + \cos(2bx)) f^2 + (1 + 2b^2 - \cos(2bx)) g^2 \end{array} \right) (f'^2 + g'^2) \cos^2 \theta} \right)}. \quad (28)$$

Now, let us give an example for the loxodrome which cuts the parallels of the twisted surface at a constant angle.

Example 3. Let us take the profile curve as $\alpha(y) = (\cos^2 y, 0, \sin^2 y)$. Then, the twisted surface (5) is

$$T(x, y) = \begin{pmatrix} (a + \cos^2 y \cos(bx) - \sin^2 y \sin(bx)) \cos x, \\ (a + \cos^2 y \cos(bx) - \sin^2 y \sin(bx)) \sin x, \\ \cos^2 y \sin(bx) + \sin^2 y \cos(bx) \end{pmatrix}. \quad (29)$$

Putting $a = -1$ and $b = 0$, from (28) (we take \mp in this equation as $-$), we get

$$\frac{dx}{dy} = 2\sqrt{2} \cot y \tan \theta \quad (30)$$

which gives

$$dx = 2\sqrt{2} \cot y \tan \theta dy.$$

Thus, by integrating both sides of the equation

$$x = \int_{y_0}^y 2\sqrt{2} \cot y \tan \theta dy. \quad (31)$$

For $y_0 = \frac{\pi}{2}$, we reach that

$$x = x(y) = 2\sqrt{2} \ln(\sin y) \tan \theta. \quad (32)$$

Here, by taking $\theta = \frac{\pi}{3}$ and $y \in (\frac{\pi}{16}, \frac{2\pi}{3})$, we have $x \in (-8.00637, -0.704674)$. Therefore, we obtain the loxodrome which lies on the twisted surface (29) as

$$\delta(y) = (-\sin^2 y \cos(2\sqrt{6} \ln(\sin y)), -\sin^2 y \sin(2\sqrt{6} \ln(\sin y)), \sin^2 y). \quad (33)$$

Also, the arc-length of the loxodrome (33) is approximately equal to 3.42788. One can see the twisted surface (29), meridian for $x = -1$ and the loxodrome (33) in Figure 4.

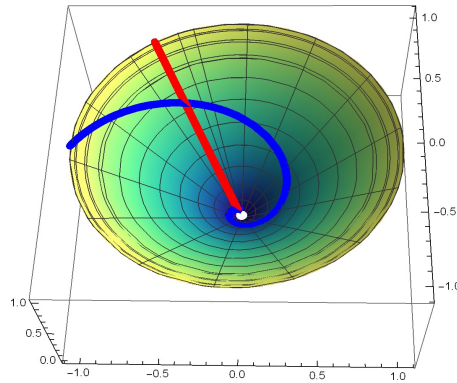


Figure 4. Twisted surface (29), Meridian (red) and Loxodrome (blue)

References

- [1] Alexander J. Loxodromes: a rhumb way to go, *Math. Mag* 2004; 77(5): 349–356.
- [2] Babaarslan M, Yaylı Y. Differential Equation of the Loxodrome on a Helicoidal Surface, *The Journal of Navigation* 2015; 68; 962–970.
- [3] Babaarslan M, Loxodromes on Canal Surfaces in Euclidean 3-Space, *Ann. Sofia Univ. Fac. Math and Inf.* 2016; 103: 97–103.
- [4] Babaarslan M, Yaylı Y. Space-like loxodromes on rotational surfaces in Minkowski 3-space, *J. Math. Anal. Appl* 2014; 409; 288–298.
- [5] Babaarslan M, Munteanu M.I. Timelike loxodromes on rotational surfaces in Minkowski 3-space, *Annals of the Alexandru Ioan Cuza University-Mathematics*, 2015; DOI: 10.2478/aicu-2013-0021.
- [6] Babaarslan M, Kayacik M. Differential Equations of the Space-Like Loxodromes on the Helicoidal Surfaces, *Differ Equ Dyn Syst* 2020; 28(2): 495–512.
- [7] Dede M, Ekici C, Goemans W, Ünlütürk Y. Twisted surfaces with vanishing curvature in Galilean 3-space, *International Journal of Geometric Methods in Modern Physics*, 2018; 15(1).
- [8] Goemans W, Van de Woestyne I. Twisted surfaces in Euclidean and Minkowski 3-space, *Pure and Applied Differential Geometry*, Joeri Van der Veken, Ignace Van de Woestyne, Leopold Verstraelen and Luc Vrancken (Editors), Shaker Verlag (Aachen, Germany) 2013; 143-151.
- [9] Goemans W, Van de Woestyne I. Twisted Surfaces with Null Rotation Axis in Minkowski 3-Space, *Results in Mathematics*, (2016; 70: 81–93.
- [10] Kazan A, Karadağ H.B. Twisted Surfaces in the Pseudo-Galilean Space, *NTMSCI* 2017; 5(4): 72-79.
- [11] Kos S, Vranic D, Zec D. Differential Equation of a Loxodrome on a Sphere, *The Journal of Navigation* 1999; 52: 418–420.
- [12] Kos S, Filjar R, Hess M, Differential equation of the loxodrome on a rotational surface, *ION ITM Conference*, 2009; Anaheim, California, USA.
- [13] Petrovic M. Differential Equation of a Loxodrome on the Spheroid, "Naše more" 2007; 54(3-4): 87-89.