Yuzuncu Yil University Journal of the Institute of Natural & Applied Sciences, Volume 27, Issue 3 (December), 667-673, 2022

Yuzuncu Yil University
Journal of the Institute of Natural & Applied Sciences

https://dergipark.org.tr/en/pub/yyufbed

Research Article

Topological Group-Groupoids and Equivalent Categories

Abdullah Fatih ÖZCAN*, İlhan İÇEN

İnönü University, Science and Art Faculty, Mathematics Department, 44280, Malatya, Türkiye
Abdullah Fatih ÖZCAN, ORCID No: 0000-0001-9732-8026, İlhan İÇEN, ORCID No: 0000-0003-3576-0731
* Corresponding author e-mail : abdullah.ozcan@inonu.edu.tr

Article Info

Received: 29.06.2022
Accepted: 25.08.2022
Online December 2022
DOI: 10.53433/yyufbed.1137668

Keywords
Action,
Covering,
Groupoid

Abstract: The concept of groupoid was offered by Brandt (1926). The structure of the topological groupoid was given by Ehresmann (1958). A groupoid action is a significant appliance in algebraic topology offered by Ehresmann. Another algebraic notion is a covering given by Brown (1988). The topological group-groupoids (Γ-groupoid) were first put forward by Icen & Ozcan (2001). The definition of coverings of topological Γ groupoid and actions of topological Γ-groupoid were also presented by Icen et al. (2005). In this paper, we are going to create a category TΓGpdCov(Γ) of covering morphisms of TΓ-groupoid and a category TΓGpdOp(Γ) of actions of TΓ-groupoid. We will then prove that these categories are equivalent.

Topolojik Grup-Grupoidler ve Denk Kategoriler

Makale Bilgileri

Geliş: 29.06.2022
Kabul: 25.08.2022
Online Aralık 2022
DOI: 10.53433/yyufbed.1137668

Anahtar Kelimeler
Etki,
Grupoidler,
Örtü

Özet: Grupoid kavramı Brandt (1926) tarafından tanımlandı. Topolojik grupoidin tanımı Ehresmann (1958) tarafından verilmiştir. Grupoid etki, cebirsel topolojide Ehresmann tarafından sunulan çok önemli bir araçtır. Diğer bir cebirsel kavram ise Brown (1988) tarafından verilen bir örtü kavramıdır. Topolojik grup-grupoidler (Γ-grupoid) ilk olarak Icen & Ozcan (2001) tarafından tanımlanmıştır. Topolojik Γ-grupoidinin örtülerinin ve topolojik Γ-grupoidin etkisinin tanımı Icen ve ark. (2005) tarafından verilmiştir. Bu çalışmada, topolojik Γ-groupoid örtülerinin bir TΓGpdCov(Γ) kategorisi ve topolojik Γ-groupoid etkilerinin bir TΓGpdOp(Γ) kategorisi tanımlandı. Daha sonra bu kategorilerin denk olduğunu ispatlandı.

1. Introduction

Action and covering are very interesting fields in algebraic topology. They are important in the applications of groupoids. Since Ehresmann's description of the topological groupoid a lot of studies have been made in this area. Let Λ be a topological space with a universal covering. So Brown & Danesh-Naruie (1975) showed that $\pi_1\Lambda$ is a topological groupoid. And then Brown et al. (1976) proved that if $p{:}\,\widetilde{\Lambda} \to \Lambda$ is a covering map and $\Lambda, \widetilde{\Lambda}$ have universal coverings, then $\pi_1 p{:}\,\pi_1\widetilde{\Lambda} \to \pi_1\Lambda$ is a covering morphism. In the same paper they defined a category $TGpdCov(\Gamma)$ of coverings of topological groupoid Γ and a category $TGpdOp(\Gamma)$ of actions of topological groupoid Γ. They proved that these categories are equivalent.

Icen & Ozcan (2001) first defined topological group-groupoids ($T\Gamma$ $-$groupoid) and topological crossed modules. In that work, they have obtained a category $T\Gamma Gpd$ of topological group-groupoids and a category TCrsM of topological crossed modules. They proved that these categories are equivalent at the same time. Icen et al. (2005) defined coverings of $T\Gamma$ $-$groupoid and actions of $T\Gamma$ $-$groupoid.

Let Γ be a $T\Gamma$ $-$groupoid. In this study, we are going to create a category $T\Gamma GpdCov(\Gamma)$ of coverings morphism of Γ and a category $T\Gamma GpdOp(\Gamma)$ of actions of Γ. And then we are going to prove that these categories are equivalent.

2. Material and Methods

A groupoid can be defined as a category in which all of the morphisms are an isomorphism.

A topological groupoid (T-groupoid) is a groupoid such that Γ and Γ_0 have a topology. Here, Γ is called the space of morphisms, and Γ_0 is called the space of objects. In addition, the source map $\alpha\colon \Gamma \to \Gamma_0$, the target map $\beta\colon \Gamma \to \Gamma_0$, the object map $\varepsilon\colon \Gamma_0 \to \Gamma$, $x \to 1_x$, and the composition $\circledast\colon \Gamma_\alpha \times_\beta \Gamma \to \Gamma$, $(a,b) \to a \circledast b$ are continuous ($\Gamma_\alpha \times_\beta \Gamma = \{\,(b,a)\colon \alpha(b) = \beta(a)\}$ is a pullback).

Let Γ be a T-groupoid. For $x, y \in \Gamma_0$, $\Gamma(x,y)$ is set such that all morphisms from *x* to *y* are in it. The set $\Gamma_x = \alpha^{-1}(x)$ and the set $\Gamma^x = \beta^{-1}(x)$ are called star and costar at *x*, respectively. The object or vertex group at *x* is shown by $\Gamma(x) = \Gamma(x,x)$.

Let Γ and Γ' be two T-groupoids. A T-groupoids morphism consists of maps $f\colon \Gamma' \to \Gamma$ and $f_0\colon {\Gamma'}_0 \to \Gamma_0$ such that $\alpha_\Gamma \circledast f = f_0 \circledast \alpha_{\Gamma'}$, $\beta_\Gamma \circledast f = f_0 \circledast \beta_{\Gamma'}$ and $f(b \circledast a) = f(b) \circledast f(a)$ for all $(b,a) \in {\Gamma'}_\alpha \times_\beta \Gamma'$.

Let $f\colon \Gamma' \to \Gamma$ be a morphism of T-groupoids. If the restriction map $f_x\colon {\Gamma'}_x \to \Gamma_{f_0(x)}$ is a homeomorphism, then *f* is a topological covering morphism, for all $x \in {\Gamma'}_0$. Let $\Gamma_\alpha \times_{f_0} {\Gamma'}_0 = \{(a,x) \in \Gamma \times {\Gamma'}_0\colon \alpha(a) = f_0(x)\}$ be the pullback. Hence, we have a lifting morphism $s_f\colon \Gamma_\alpha \times_{f_0} {\Gamma'}_0 \to \Gamma'$ assigned to the pair $(a,x) \in \Gamma_\alpha \times_{f_0} {\Gamma'}_0$ the sole element b of ${\Gamma'}_x$ such that $f(b) = a$. It is clear that s_f is the inverse to $(f,\alpha)\colon \Gamma' \to \Gamma_\alpha \times_{f_0} {\Gamma'}_0$. The necessary and sufficient condition for $f\colon \Gamma' \to \Gamma$ to be a covering morphism is that $(f,\alpha)\colon \Gamma' \to \Gamma_\alpha \times_{f_0} {\Gamma'}_0$ is a homeomorphism.

Let $f\colon \Gamma' \to \Gamma$ be a morphism of T-groupoids and $\tilde{x} \in {\Gamma'}_0$ be an object. The characteristic group of f at $\tilde{x}$ is defined by the subgroup $f[\Gamma'(\tilde{x})] \subset \Gamma(f(\tilde{x}))$. If f is a covering morphism then $\Gamma'(\tilde{x})$ and $f[\Gamma'(\tilde{x})]$ are isomorphic. We call that $f\colon \Gamma' \to \Gamma$ is a universal covering if $\Gamma'(\tilde{x}, \tilde{y})$ has only one element.

Let Γ be a T-groupoid and Λ be a topological space. $w\colon \Lambda \to \Gamma_0$ be a continuous map. A left topological action of Γ on Λ via w is a continuous map $\phi\colon \Gamma_\alpha \times_w \Lambda \to \Lambda, (a,x) \mapsto a \cdot x$ satisfying the following conditions: (i) $w(a \cdot x) = \beta(a)$, (ii) $b \cdot (a \cdot x) = (b \circledast a) \cdot x$, (iii) $1_{w(x)} \cdot x = x$, for any $a, b \in \Gamma$ and $x \in \Lambda$. So, if Γ acts on Λ via w, then there is an action groupoid $\Gamma \bowtie \Lambda$. Object set is $\Gamma_0 = \Lambda$, set of morphisms is $\Gamma_\alpha \times_w \Lambda$, source map is defined by $\alpha(a,x) = x$ and target map is defined by $\beta(a,x) = a \cdot x$, and finally the composition is defined by $(b,y) \circledast (a,x) = (b \circledast a, x)$ (Brown et al., 1976).

2.1. Covering of TΓ-Groupoids

Definition 2.1.1: Let Γ be a T-groupoid. If Γ is also a topological group and the following maps are morphism of T-groupoids, then Γ is called a topological group-groupoid (TΓ-groupoid).

i. $m\colon \Gamma \times \Gamma \to \Gamma, (a,b) \mapsto a + b$, group sum,
ii. $u\colon \Gamma \to \Gamma, a \mapsto -a$, inverse map in the group
iii. $e\colon (*) \to \Gamma$, where $(*)$ is a singleton groupoid.

Note that we have the interchange law $(b \circledast a) + (d \circledast c) = (b + d) \circledast (a + c)$, whenever both $b \circledast a$ and $d \circledast c$ are defined.

Example 2.1.2: Let $(\Gamma, \oplus)$ be a topological group. We obtain a TΓ-groupoid $\Gamma \times \Gamma$ with the object set Γ. The set of morphisms is defined by $\Gamma \times \Gamma = \{(b,a)\colon a, b \in \Gamma\}$ (Icen et al., 2005). The composition is given by $(c,b) \circledast (b,a) = (c,a)$. The Group sum is given by $(d,c) \oplus (b,a) =$

$(d \oplus b, c \oplus a)$. Since the Γ topological group, $\Gamma \times \Gamma$ has product topology. So every structure maps of $\Gamma \times \Gamma$ become continuous. Then $\Gamma \times \Gamma$ is a TΓ-groupoid.

We know from Brown & Spencer (1976) that if X is a topological group, then the fundamental groupoid $\pi_1 X$ becomes a Γ-groupoid. The following result is given by Icen & Ozcan (2001).

Proposition 2.1.3: Let Λ be a topological group such that Λ has a universal covering as a topological space. Then $\pi_1 \Lambda$ becomes a TΓ-groupoid.

Proposition 2.1.4: Let Γ be a TΓ-groupoid and e the identity of Γ_0. Then the transitive component of e is a TΓ-groupoid (Icen & Ozcan, 2001).

Let Γ and Γ' be two TΓ-groupoids. A morphism of TΓ-groupoids from Γ' to Γ is a morphism on the underlying topological groupoids that hold group structure.

Definition 2.1.5: Let Γ and Γ' be two $T\Gamma$-groupoids and $f: \Gamma' \to \Gamma$ a morphism of $T\Gamma$-groupoids. f is called a covering morphism of $T\Gamma$ -groupoids if f is a covering morphism on the underlying T-groupoids.

Thus we can obtain a category $T\Gamma GpdCov(\Gamma)$. The set of objects consists of $p: \Gamma' \to \Gamma$ covering morphisms of $T\Gamma$-groupoids and a morphism from $p: \Gamma' \to \Gamma$ to $q: \Gamma'' \to \Gamma$ is a morphism $r: \Gamma' \to \Gamma''$ such that $p = q \circledast r$. This can be given by the following commutative diagram.

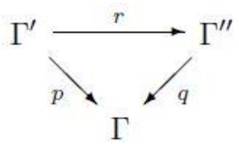

Figure 1. The morphism from p to q.

Structure maps of the category are defined by $\alpha(r) = p$, $\beta(r) = q$ and $1_p: \Gamma' \to \Gamma'$. For $r: \Gamma' \to \Gamma''$ and $r': \Gamma'' \to \Gamma'''$, the composition is defined by the following commutative diagram.

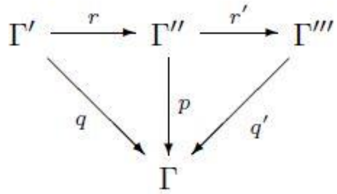

Figure 2. The composition between r and r'.

2.2. Action of TΓ-Groupoids

Definition 2.2.1: Let Γ be a $T\Gamma$-groupoid over Γ_0 and X be a topological group. Let $w: X \to \Gamma_0$ be a topological group morphism. If there exists a topological group morphism $\varphi: \Gamma_\alpha \times_w X \to X, (a, x) \mapsto a \cdot x$ such that this morphism is satisfied the following conditions, we say that Γ acts on X via w and φ is the left action. This action is shown by (X,p).

i) $w(a \cdot x) = \beta(a)$

ii) $b \cdot (a \cdot x) = (b \circledast a) \cdot x$

iii) $1_{w(x)} \cdot x = x$

Similarly, we can define for right action.

Example 2.2.2: Let Γ be a $T\Gamma$-groupoid over Γ_0 and X be a topological group. Assume that Γ acts over X via $w: X \to \Gamma_0$. So, $\Gamma \ltimes X$ action groupoid is obtained over object set X via this action. This action groupoid's morphisms set is $\Gamma_\alpha \times_w X$. Namely, a morphism is (a,x) such that $a \cdot x = y$ for $x, y \in X$ from x to y. Source map is$\alpha(a, x) = x$, target map is $\beta(a, x) = a \cdot x = y$. Unit map is $x \longmapsto (1_{w(x)}, x)$. Inverse map is $(a, x)^{-1} = (a^{-1}, a \cdot x)$. Composition is defined by $(b, y) \circledast (a, x) = (b \circledast a, x)$. Icen et al. (2005) shown that $\Gamma \ltimes X$ is a $T\Gamma$-groupoid. Now we show that $\Gamma \ltimes X$ is a $T\Gamma$-groupoid. $(\Gamma \ltimes X)_0 = X$ has a topological group structure, since it is defined with X. The group structure on $\Gamma \ltimes X$ is defined as follows.

$+:(\Gamma \ltimes X) \times (\Gamma \ltimes X) \to \Gamma \ltimes X$, $((a,x),(b,y)) \longmapsto (a,x)+(b,y)=(a+b,m+n)$

$u: \Gamma \bowtie X \rightarrow \Gamma \bowtie X$, $(a,x) \longmapsto (-a,-x)$

where + and u are the topological group operations on Γ. Therefore $\Gamma \bowtie X$ is a group-groupoid. $(\Gamma \bowtie X)_0 = X$ is already a topological group. $\Gamma_\alpha \times_w X$ has a relative topology from product topology on $\Gamma \bowtie X$. Since Γ and X are topological groupoid and topological group, respectively, their structure maps and operations are continuous. So $\Gamma \bowtie X$ is a $T\Gamma$-groupoid.

Thus, we obtain a category whose objects are all actions of the $T\Gamma$-groupoid Γ denoted by $T\Gamma GpdOp(\Gamma)$. The objects of this category are actions (X, w) and a morphism from (X, w) to (X', w') is given by the following commutative diagram.

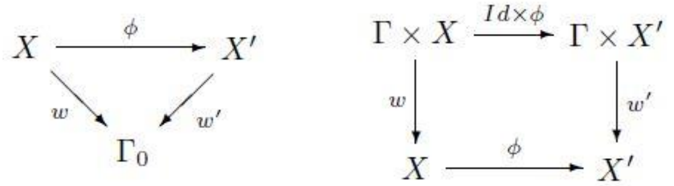

Figure 3. The morphism of actions from (X, w) to (X', w').

The maps α and β maps are given with $\alpha(\phi) = (X, w)$ and $\beta(\phi) = (X', w')$, respectively. A identity map is defined by $1_{(X,w)}: (X, w) \rightarrow (X, w)$. Finally, composition operation is defined by the following commutative diagram.

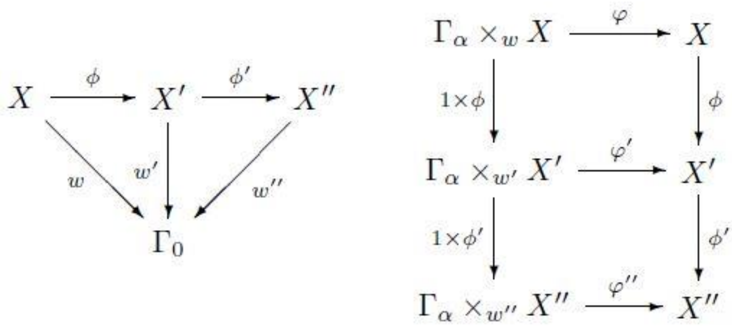

Figure 4. The composition between the morphism of actions.

Example 2.2.3: Let $p: \tilde{\Gamma} \rightarrow \Gamma$ be a morphism of $T\Gamma$-groupoids. We assume that $X = \tilde{\Gamma}_0$ and $w = p_0: \tilde{\Gamma}_0 \rightarrow \Gamma_0$ then Γ acts $X = \tilde{\Gamma}_0$ via $w = p_0$. The action is defined by $\varphi: \Gamma_\alpha \times_{p_0} \tilde{\Gamma}_0 \rightarrow \tilde{\Gamma}_0$, $(a, \tilde{x}) \longmapsto a \cdot \tilde{x} = \tilde{\beta}(\tilde{a})$. Since p is the covering morphism, there is unique $\tilde{a}$ lifting of a ($\alpha(\tilde{a}) = \tilde{x}$) such that for $\tilde{x} \in X = \tilde{\Gamma}_0$ and $a \in \Gamma_{p_0(\tilde{x})}$, $p(\tilde{a}) = a$ and $p_0(\tilde{x}) = x$. Now we show that action conditions are satisfied. $w(a \cdot \tilde{x}) = p_0(a \cdot \tilde{x}) p_0\left(\tilde{\beta}(\tilde{a})\right) = \beta(a)$. $b \cdot (a \cdot \tilde{x}) = b \cdot \tilde{\beta}(\tilde{a}) = \tilde{\beta}(\tilde{b})$ and $(b \circledast a) \cdot \tilde{x} = \tilde{\beta}(\tilde{b} \circledast \tilde{a}) = \tilde{\beta}(\tilde{b})$ So we have that $b \cdot (a \cdot \tilde{x}) = (b \circledast a) \cdot \tilde{x}$. Finally, $1_{p_0(\tilde{x})} \cdot \tilde{x} = \tilde{\beta}(\tilde{c}) = \tilde{x}$. So, the action conditions are satisfied. Since p is the covering morphism of the $T\Gamma$-groupoid and w is defined by p_0, it is a topological group morphism. Similarly, it is a topological group morphism since φ is defined by $\tilde{\beta}$ target morphism of $\tilde{\Gamma}$.

3. Results

Theorem 3.1: Let Γ be a $T\Gamma$-groupoid. $T\Gamma GpdCov(\Gamma)$ category of coverings of $T\Gamma$-groupoid, and $T\Gamma GpdOp(\Gamma)$ category of actions of $T\Gamma$-groupoid are equivalent.

Proof: A functor $F: T\Gamma GpdOp(\Gamma) \rightarrow T\Gamma GpdCov(\Gamma)$ is defined as follows: Suppose that Γ acts on a topological group X via $w: X \rightarrow \Gamma_0$. This action is given with $\varphi: \Gamma_\alpha \times_w X \rightarrow X, (a, m) \mapsto a \cdot m$. In this case, we have the action $T\Gamma$-groupoid $\Gamma \bowtie X$ from Example.2.2.2. If a morphism $p: \Gamma \bowtie X \rightarrow \Gamma$ is defined by $(a, m) \mapsto a$ and $p_0 = w$ on the morphisms and on the objects, respectively, then p is a covering morphism of $T\Gamma$-groupoids. We have that $p\big((b,n)\circledast(a,m)\big)=p(b\circledast a,m)=b\circledast a=p(b,n)\circledast p(a,m)$, from the definition of p. Since p is defined by w on objects, we have that $p\big(1_{w(m)},m\big)=1_{w(m)}=1_{p(1_{w(m)},m)}$. So, p is a groupoid morphism and a continuous map since p is defined by the projection map. At the same time, it is a topological covering morphism, since (a,m') is only one element of $\Gamma\bowtie X$ with $\alpha(a,m')=m'$ and $p(a,m)=a$, for $a\in\Gamma(x,y)$ and $m'\in w^{-1}(x)$. Thus (p,α) is bijective. Since p, $p_0=w$ and α are continuous, (p,α) is homeomorphism. Furthermore we have that $p((a,m)+(b,n))=p(a+b,m+n)=a+b=p(a,m)+p(b,n)$. Therefore p also $F(X,w)$ is a $T\Gamma$-groupoid morphism.

If (X,w) and (X',w') are actions of $T\Gamma$-grupoid Γ, then $F(X,w)$ and $F(X',w')$ are topological coverings of $T\Gamma$-groupoid Γ. Let these coverings be $p:\Gamma\bowtie X\to\Gamma$ and $q:\Gamma\bowtie X'\to\Gamma$. If $\phi:X\to X'$ is a morphism of topological actions, then $F(\phi)=r$ is a morphism of topological covering morphisms with $r_0=\phi$ and $r=1\times\phi$. This situation is given in the following commutative diagram

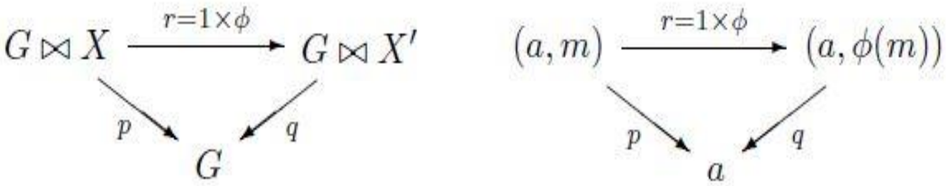

Figure 5. The morphism from p to q.

In addition, if $\phi:X\to X'$ and $\phi':X'\to X''$ are morphisms of $T\Gamma$-groupoid actions then we have that $F(\phi'\circledast\phi)=F(\phi')\circledast F(\phi)$. For $F(X,w)=\Gamma\bowtie X$, $F(X',w')=\Gamma\bowtie X'$, $F(X'',w'')=\Gamma\bowtie X''$, $F(\phi)=r$ and $F(\phi')=r'$, we have that $\phi'\circledast\phi:X\to X''$ and $F(\phi'\circledast\phi)=r'\circledast r=F(\phi')\circledast F(\phi)$. This situation is given in the following commutative diagram.

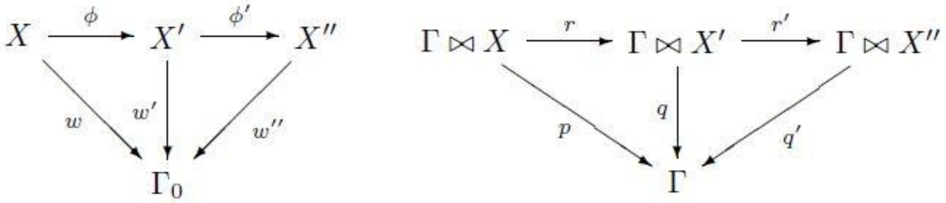

Figure 6. The composition between r and r'.

Therefore, F is a functor.

Define a functor $G:T\Gamma GpdCov(\Gamma)\to T\Gamma GpdOp(\Gamma)$ as follows:

Let $p:\tilde{\Gamma}\to\Gamma$ be a topological covering morphism of $T\Gamma$-groupoids. If we suppose that $X=\tilde{\Gamma}_0$ and $w=p_0:\tilde{\Gamma}_0\to\Gamma_0$ then (X,w) is an action of $T\Gamma$-groupoid over X from example.2.2.3. Namely φ is an topological action of $T\Gamma$-groupoid Γ over X. Thus if $p:\tilde{\Gamma}\to\Gamma$ and $q:\tilde{\Gamma}'\to\Gamma$ are topological covering morphisms of $T\Gamma$-groupoids then $G(p)$ and $G(q)$ are topological action of $T\Gamma$-groupoid Γ over $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_0{}'$ topological groups via p_0 and q_0, repectively. Let these actions be denoted by $(\tilde{\Gamma}_0,p_0)$ and $(\tilde{\Gamma}_0{}',q_0)$. We known that if p and q are topological covering morphisms of $T\Gamma$-groupoids then $r:\tilde{\Gamma}\to\tilde{\Gamma}'$ is the topological covering morphisms of $T\Gamma$-groupoids with $p=q\circledast r$. Therefore $r_0=\phi$ and $G(r)=\phi$ are morphisms of the topological group actions. This situation is given the following commutative diagram.

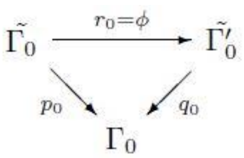

Figure 7. The morphism from p_0 to q_0.

We see that the action is protected from the following commutative diagram since we have that $p = q \circledast r$ and $p_0 = q_0 \circledast r_0$.

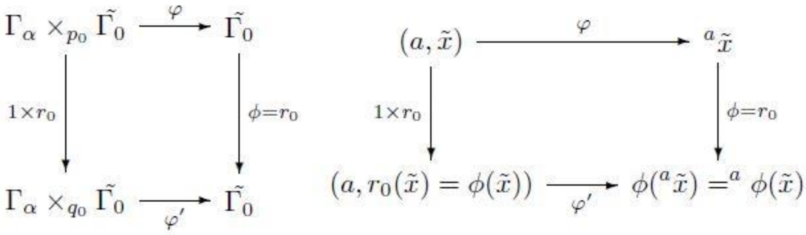

Figure 8. The morphism of the actions.

Let $r: \tilde{\Gamma} \to \tilde{\Gamma}'$ be a morphism from $p: \tilde{\Gamma} \to \Gamma$ to $q: \tilde{\Gamma}' \to \Gamma$ and let $r': \tilde{\Gamma}' \to \tilde{\Gamma}''$ be a morphism from $q: \tilde{\Gamma}' \to \Gamma$ to $p': \tilde{\Gamma}'' \to \Gamma$. Then we have that $G(r' \circledast r) = G(r') \circledast G(r)$. For $G(p) = (\tilde{\Gamma}_0, p_0)$, $G(q) = (\tilde{\Gamma}_0{}', q_0)$, $G(p') = (\tilde{\Gamma}_0{}'', p'_0)$, $G(r) = \phi$ and $G(r') = \phi'$, $r' \circledast r: \tilde{\Gamma} \to \tilde{\Gamma}''$ a topological covering morphism of $T\Gamma$-groupoids. So, we find $G(r' \circledast r) = \phi \circledast \phi' = G(r') \circledast G(r)$ This is seen from the following commutative diagram.

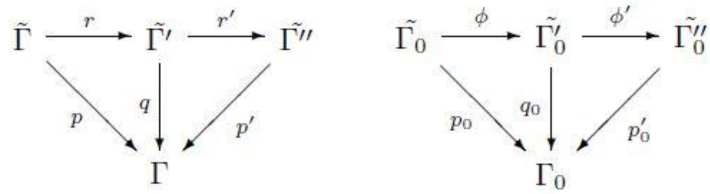

Figure 9. The composition between r and r'.

Therefore, G is a functor.

It is obvious that $FG \cong 1_{T\Gamma GpdCov(\Gamma)}$ and $GF \cong 1_{T\Gamma GpdOp(\Gamma)}$.

4. Discussion and Conclusion

Topological group-groupoids are internal categories in the category of topological groups. Coverings and actions of groupoids are also important in algebraic topology. So, it would be interesting to develop these results in terms of topological groupoids with operations and internal categories rather than special categories.

References

Brandt, H. (1926). Uber ein verallgemeinerung des Gruppen begriffes. *Mathematische Annalen*, 96, 360-366. doi: 10.1007/BF01209171

Brown, R., & Danesh-Naruie, G. (1975). The fundamental groupoid as a topological groupoid. *Proceedings of the Edinburgh Mathematical Society*, 19-2(3), 237-244. doi: 10.1017/S0013091500015509

Brown, R., Danesh-Naruie, G., & Hardy, J. P. L. (1976). Topological groupoids II: Covering morphism and G-space. *Mathematische Nachrichten*, 74, 143-145. doi: 10.1002/mana.3210740110

Brown, R., & Spencer, C. B. (1976). G-groupoids, crossed modules and the fundamental groupoid of a topological group. *Proceedings of the Koninklijke Nedderlandse Akademie van Wetenschappen*, 79, 196-302. doi: 10.1016/1385-7258(76)90068-8

Brown, R. (1988). *Topology: A Geometric Account of General Topology, Homotopy Types and the Fundamental Groupoid*. Chichester, England: Ellis Horwood.

Ehresmann, C. (1958). Categories topologiques et categories differentiables. *Colloque de Geometrie Differentielle Globale, Cenrtre Belge de Recherches Mathematiques, Bruxelles*, 137-150.

Icen, I., & Ozcan, A. F. (2001). Topological crossed modules and G groupoids. *Algebras, Groups and Geometries*, 18, 401-410.

Icen, I., Ozcan, A. F., & Gursoy, M. H. (2005). Topological group-groupoids and their coverings. *Indian Journal of Pure and Applied Mathematics*, 36(9), 493-502.