Results in Nonlinear Analysis **5** (2022) No. 4, 452–458.
https://doi.org/10.53006/rna.1138201
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

On the convergence of the sixth order Homeier like method in Banach spaces

Suma P B[a], Shobha M Erappa[a], Santhosh George[b]

[a]Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India 576104.
[b]Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, India 575025.

Abstract

A sixth order Homeier-like method is introduced for approximating a solution of the non-linear equation in Banach space. Assumptions only on first and second derivatives are used to obtain a sixth order convergence. Our proof does not depend on Taylor series expansions as in the earlier studies for the similar methods.

Keywords: Frechet derivative Homeier method Local convergence Iterative method Banach space.
2010 MSC: 49M15, 47H17, 65J15, 65G99.

1. Introduction

Determining the solutions to the nonlinear equations in the Banach space setting are extensively studied problems in numerical analysis and scientific computing ([1],[2],[8],[16]). Most of the problem arising in the real-life can be mathematically modeled into an equation of the form

$$\mathcal{A}(h) = 0, \tag{1}$$

where $\mathcal{A} : D \subseteq H_1 \to H_2$ is a nonlinear operator between the Banach spaces H_1 and H_2 and D is open convex set in H_1. In recent times, it is observed that to determine the approximations to (1), multistep iterative methods are used. These methods sometimes involves computation of its derivatives at each iteration at a number of values of h. Even though they are not much in practice, one can find few interesting class of formula which are computationally attractive, where the evaluation of $\mathcal{A}'(h)$ is rapid compared with $\mathcal{A}$.

Email addresses: sumapanathale@gmail.com (Suma P B), shobha.me@gmail.com (Shobha M Erappa), sgeorge@nitk.edu.in (Santhosh George)

Received :June 30, 2022; Accepted: September 3, 2022; Online: September 10, 2022

When the function $\mathcal{A}$ is defined by integral, such cases arise. Such methods were proposed by Traub [18], Homeier [6] etc. Iterative formulae given by Traub is of the type,

$$h_{n+1} = h_n - a_1 w_1(s_n) - a_2 w_2(h_n) \tag{2}$$

where,

$$w_1 = \frac{\mathcal{A}(h)}{\mathcal{A}'(h)} \ and \ w_2 = \frac{\mathcal{A}(h)}{\mathcal{A}'(h + \Gamma w_1(h))},$$

on choosing the parameters a_1, a_2 and Γ suitably, a third order processes is obtained costing one evaluation of $\mathcal{A}(h)$ and two of $\mathcal{A}'(h)$ per iteration. On the other hand, Homeier introduced a modified Newton method for vector functions which converges locally cubically, without the neccassity of higher derivative computation. It is a two step iterative scheme given by

$$\begin{aligned} s_n &= h_n - \frac{1}{2}\mathcal{A}'(h_n)^{-1}\mathcal{A}(h_n) \\ h_{n+1} &= h_n - \mathcal{A}'(s_n)^{-1}\mathcal{A}(h_n), n = 0, 1, \cdots . \end{aligned}$$

Per iteration the above method requires one evaluation of the vector function and solving two linear systems with the Jacobian as coefficient matrix, where the Jacobian has to be evaluated twice. This method is suitable if the computation of the derivatives has a similar or lower cost than that of function itself ([2],[4],[7],[14],[15],[17]).

The order of convergence is an important issue, when one deals with iterative methods. So, let us recall that a sequence $\{h_n\}$ in h with $\lim_{n\to\infty} h_n = h^*$ is said to be convergent of order $R > 1$, if there exist positive reals β_1, β_2, such that for all $n \in \mathbb{N}$, $\|h_n - h^*\| \le \beta_1 e^{-\beta_2 R^n}$ ([9],[11]). The basic tool employed to find the order of convergence is the Taylor expansion which requires existence of higher order derivatives. An alternative approach is to use the computational order of convergence (COC)[19] defined as

$$\overline{\gamma} = \frac{\ln(\|\frac{h_{n+1} - \phi}{h_n - \phi}\|)}{\ln(\|\frac{h_n - \phi}{h_{n-1} - \phi}\|)},$$

where h_{n-1}, h_n, h_{n+1} are three consecutive iterates near root ϕ or the approximate computational order of convergence (ACOC) defined as

$$\overline{\gamma} = \frac{\ln(\|\frac{h_{n+1} - h_n}{h_n - h_{n-1}}\|)}{\ln(\|\frac{h_n - h_{n-1}}{h_{n-1} - h_{n-2}}\|)},$$

where $h_{n-2}, h_{n-1}, h_n, h_{n+1}$ are four consecutive iterates near root ϕ, to obtain the order of convergence. Without increasing the number of derivative evaluations it was not possible to obtain higher order formulae for Traub or Homeier methods. However several modified Traub like methods and modified Homeier like methods are available in the literature (see[3],[5],[9],[10],[11], [13],[19],[20]).

In this paper, we introduce an iteration of order six for solving (1) in Banach space. The proposed Homeier-like method is defined by:

$$\begin{aligned} s_n &= h_n - \frac{2}{3}\mathcal{A}'(h_n)^{-1}\mathcal{A}(h_n) \\ g_n &= h_n - \mathcal{A}'(\frac{3s_n + h_n}{4})^{-1}\mathcal{A}(h_n) \\ h_{n+1} &= g_n - \mathcal{A}'(g_n)^{-1}\mathcal{A}(g_n), \quad n = 0, 1, 2, 3... \end{aligned} \tag{3}$$

Another issue, is the efficiency of the method. In this regard, recall the informational efficiency, introduced by Traub [18], $E.I = a^{\frac{1}{b}}$, where a is the order of the methods and b is the number of function evaluations. Before Traub, Ostrowski [12] introduced a term called efficiency index or computational efficiency defined as

$$E(\Psi) = R^{\frac{1}{\Theta_f}},$$

where R indicates the convergence order of the method and Θ_f gives the number of function evaluations. Thus the informational efficiency and the computational efficiency of the proposed method (3) is $6/5 = 1.2$ and $6^{\frac{1}{5}} = 1.4310$ respectively, wherein for Homeier's method it is $3/3 = 1$ and $3^{\frac{1}{3}} = 1.442$ respectively.

The novelty of our approach is that we obtain convergence of order six without using assumptions on the derivatives of order greater than two. In earlier studies, if the iterative method is of order m, one need assumptions on derivatives of the involved operator upto the order m+1. So using our approach one can increase the applicability of the iterative methods.

Throughout the article, we consider $B(h_0, \gamma) = \{h \in H_1 : \|h - h_0\| < \gamma\}$ and $B[h_0, \gamma] = \{h \in H_1 : \|h - h_0\| \leq \gamma\}$ for $\gamma > 0$.

Following assumptions are required to analyze our results.

Assumption 1.1. *(cf. Assumption 1.1 [4]) There exists $k_1 > 0$, such that for every $h, s \in D(\mathcal{A})$*

$$\|\mathcal{A}'(h)^{-1}(\mathcal{A}'(s) - \mathcal{A}'(h))\| \leq k_1\|s - h\| \tag{4}$$

and there exists $k_2 > 0$, such that for all $h,\ s \in B(h^*, \gamma)$ for some $\gamma > 0$,

$$\|\mathcal{A}'(h)^{-1}\mathcal{A}''(s)\| \leq k_2. \tag{5}$$

Rest of this paper is arranged in the following way. Local convergence, numerical examples and conclusions, respectively are given in Section 2, Section 3 and Section 4.

2. Main Results

Lets define some functions and parameters required to establish the convergence analysis. Let $p_1 : [0, \infty) \to [0, \infty)$ be defined by

$$p_1(x) = \Big(\frac{k_1 k_2}{4}\Big)x^2$$

and

$$q_1(x) = p_1(x) - 1.$$

Then $q_1(0) = -1 < 0$ and $q_1\Big(\frac{2\sqrt{2}}{\sqrt{k_1 k_2}}\Big) = 1 > 0$. Using the intermediate value theorem (IVT), q_1 has a minimal zero $r_1 \in \Big(0, \frac{2\sqrt{2}}{\sqrt{k_1 k_2}}\Big)$.

Let $p_2 : \Big[0, \frac{2\sqrt{2}}{\sqrt{k_1 k_2}}\Big) \to [0, \infty)$ be defined by

$$p_2(x) = \Big(\frac{k_1^3 k_2^2}{2^5}\Big)x^5$$

and

$$q_2(x) = p_2(x) - 1.$$

Then $q_2(0) = -1 < 0$ and $q_2\Big(\frac{2\sqrt[5]{2}}{\sqrt[5]{k_1^3k_2^2}}\Big) = 1 > 0$. Hence, again by IVT q_2 has a minimal zero $r_2 \in \Big(0, \frac{2\sqrt[5]{2}}{\sqrt[5]{k_1^3k_2^2}}\Big)$.

Let

$$r = \min\{r_1, r_2\} \tag{6}$$

Then, for all $x \in [0, r)$, we have

$$0 < p_1(x) < 1 \tag{7}$$

and

$$0 < p_2(x) < 1. \tag{8}$$

Next, we furnish the convergence analysis for (3).

Theorem 2.1. *Let Assumption (1.1) holds. Then the sequence $\{h_n\}$ defined by (3) with $h_0 \in B(h^*, r) - \{h^*\}$ converges to h^* with R- order of convergence six, i.e.,*

$$\|h_{n+1} - h^*\| \leq c\|h_n - h^*\|^6 \tag{9}$$

where $c = (\frac{k_1^3k_2^2}{32})$

Proof. Induction methodology is used to prove the following:

$$\|g_n - h^*\| \leq p_1\big(\|h_n - h^*\|\big)\|h_n - h^*\| \tag{10}$$

and

$$\|h_{n+1} - h^*\| \leq p_2\big(\|h_n - h^*\|\big)\|h_n - h^*\|. \tag{11}$$

Let $h_0 \in B(h^*, r)$. By (3)

$$\begin{aligned}
g_0 - h^* &= h_0 - h^* - \mathcal{A}'(\frac{3s_0 + h_0}{4})^{-1}\big[\mathcal{A}(h_0) - \mathcal{A}(h^*)\big] \\
&= \mathcal{A}'(\frac{3s_0 + h_0}{4})^{-1}\Big[\mathcal{A}'(\frac{3s_0 + h_0}{4})(h_0 - h^*) - \int_0^1 \mathcal{A}'(h^* \\
&+ t(h_0 - h^*))(h_0 - h^*)dt\Big] \\
&= \mathcal{A}'(\frac{3s_0 + h_0}{4})^{-1}\Big[\int_0^1 [\mathcal{A}'(\frac{3s_0 + h_0}{4}) - \mathcal{A}'(h^* \\
&+ t(h_0 - h^*))](h_0 - h^*)dt\Big] \\
&= \mathcal{A}'(\frac{3s_0 + h_0}{4})^{-1}\Big[\int_0^1 \int_0^1 \mathcal{A}''\big(h^* + t(h_0 - h^*) + \theta(\frac{3s_0 + h_0}{4} \\
&- h^* - t(h_0 - h^*))\big)d\theta \times (\frac{3s_0 + h_0}{4} - h^* - t(h_0 - h^*))dt\Big](h_0 - h^*).
\end{aligned}$$

By Assumption (4), we get

$$\begin{aligned}
\|g_0 - h^*\| &\leq \Big\| \sup_{t\in[01]} \Big\{ \Big\| \int_0^1 \mathcal{A}'(\frac{3s_0+h_0}{4})^{-1}\mathcal{A}''\Big(h^* + t(h_0-h^*) + \theta(\frac{3s_0+h_0}{4} \\
&- h^* - t(h_0-h^*))\Big)d\theta\Big\|\Big\} \times \int_0^1 (\frac{3s_0+h_0}{4} - h^* - t(h_0-h^*))dt(h_0-h^*)\Big\| \\
&\leq \frac{k_2}{4}\Big\| \int_0^1 \big[3s_0 + h_0 - 4h^* - 4t(h_0-h^*)\big]dt(h_0-h^*)\Big\| \\
&\leq \frac{k_2}{4}\Big\| \int_0^1 \big[3(s_0-h^*) + (h_0-h^*) - 4t(h_0-h^*)\big]dt(h_0-h^*)\Big\| \\
&\leq \frac{k_2}{4}\Big\| \int_0^1 \big[3(h_0-h^* - \frac{2}{3}\mathcal{A}'(h_0)^{-1}\mathcal{A}(h_0)) + (h_0-h^*) - 4t(h_0-h^*)\big]dt(h_0-h^*)\Big\| \\
&\leq \frac{k_2}{4}\Big\| \int_0^1 \big[2(h_0-h^* - \mathcal{A}'(h_0)^{-1}\mathcal{A}(h_0)) + (2-4t)(h_0-h^*)\big]dt(h_0-h^*)\Big\| \\
&\leq \frac{k_2}{2}\Big\| \big[h_0-h^* - \mathcal{A}'(h_0)^{-1}\mathcal{A}(h_0)\big](h_0-h^*)\Big\| \\
&\leq \frac{k_2}{2}\Big\| \mathcal{A}'(h_0)^{-1}\big[\mathcal{A}'(h_0)(h_0-h^*) - \int_0^1 \mathcal{A}'(h^* + \theta(h_0-h^*))(h_0-h^*)d\theta\big](h_0-h^*)\Big\| \\
&\leq \frac{k_2}{2}\Big\| \int_0^1 \mathcal{A}'(h_0)^{-1}\big[\mathcal{A}'(h_0) - \mathcal{A}'(h^* + \theta(h_0-h^*))\big]d\theta(h_0-h^*)^2\Big\| \\
&\leq \frac{k_1k_2}{4}\|h_0-h^*\|^3 \qquad (12) \\
&= p_1\big(\|h_0-h^*\|\big)\|h_0-h^*\| \\
&\leq \|h_0-h^*\| < r.
\end{aligned}$$

Thus, $g_0 \in B(h^*, r)$. By the third sub-step of method (3), we have

$$\begin{aligned}
h_1 - h^* &= g_0 - h^* - \mathcal{A}'(g_0)^{-1}[\mathcal{A}(g_0) - \mathcal{A}(h^*)] \\
&= \mathcal{A}'(g_0)^{-1}[\mathcal{A}'(g_0)(g_0-h^*) - \int_0^1 \mathcal{A}'(h^* + t(g_0-h^*))dt(g_0-h^*)] \\
&= -\int_0^1 \mathcal{A}'(g_0)^{-1}[\mathcal{A}'(h^* + t(g_0-h^*)) - \mathcal{A}'(g_0)]dt(g_0-h^*).
\end{aligned}$$

From Assumption 1.1 and (12) we have

$$\begin{aligned}
\|h_1 - h^*\| &\leq k_1\frac{\|g_0-h^*\|^2}{2} \\
&\leq \frac{k_1^3k_2^2}{32}\|h_0-h^*\|^6 \qquad (13) \\
&= p_2\big(\|h_0-h^*\|\big)\|h_0-h^*\| \\
&< r.
\end{aligned}$$

The induction for (10) and (11), will be completed, if we simply replace, h_0, g_0, h_1 by h_n, g_n, h_{n+1} in the preceding arguments. □

Further, uniqueness of the solution is presented below.

Theorem 2.2. *Suppose Assumption (4) holds and h^* is a simple solution of the equation $\mathcal{A}(h) = 0$. Then, the only solution of equation $\mathcal{A}(h) = 0$ in the set $A = D \cap B[h^*, \gamma]$ is h^* provided that*

$$k_1\gamma < \frac{2}{3} \qquad (14)$$

Proof. Consider a solution $c \in A$ of equation $\mathcal{A}(h) = 0$. Set $N = \int_0^1 \mathcal{A}'(h^* + t(c - h^*))dt$. Then by Assumption (4) and (14), we have

$$\begin{aligned} \|\mathcal{A}'(h)^{-1}(N - \mathcal{A}'(h))\| &\leq k_1 \int_0^1 \|h^* + t(c - h^*) - h\| dt \\ &\leq k_1 \int_0^1 \|h^* - h\| + t\|h^* - c\| dt \\ &\leq \frac{3}{2} k_1 \gamma < 1. \end{aligned}$$

So, N is invertible and hence we get $c = h^*$ and the identity $0 = \mathcal{A}(c) - \mathcal{A}(h^*) = N(c - h^*)$. □

3. Numerical Example

Example 3.1. *Let $H_1 = H_2 = \mathbb{R}$, $h_0 = 1$, $\Omega = [h_0 - (1 - l), h_0 + (1 - l)]$, $l \in (0, 1)$ and $\mathcal{A} : \Omega \to Y$ be defined by*

$$\mathcal{A}(h) = h^3 - l.$$

Then $\|\mathcal{A}'(h_0)^{-1}\| \leq \frac{1}{3}$, $\|\mathcal{A}'(s) - \mathcal{A}'(h)\| \leq 6(2 - l)\|s - h\|$ and hence for all $h \in \Omega$, we have

$$\|\mathcal{A}'(h)^{-1}\| \leq \frac{\|\mathcal{A}'(h_0)^{-1}\|}{1 - \|\mathcal{A}'(h_0)^{-1}\|\|\mathcal{A}'(h) - \mathcal{A}'(h_0)\|} \leq \frac{1}{3(1 - 2(2 - l)(1 - l))}.$$

Hence,

$$\|\mathcal{A}'(h)^{-1}(\mathcal{A}'(s) - \mathcal{A}'(h))\| \leq \|\mathcal{A}'(h)^{-1}\| L \|s - h\| \leq \frac{2(2 - l)}{1 - 2(2 - l)(1 - l)} \|s - h\|$$

Then, we have $k_1 = \frac{2(2 - l)}{1 - 2(2 - l)(1 - l)}$. Now $\|\mathcal{A}''(s)\| \leq 6(2 - l)$. Therefore $\|\mathcal{A}'(h)^{-1}\mathcal{A}''(s)\| \leq \frac{2(2 - l)}{1 - 2(2 - l)(1 - l)}$. Hence $k_2 = \frac{2(2 - l)}{1 - 2(2 - l)(1 - l)}$. For $l = 0.65$, $r_1 = 0.0407 = r_2$, so r=0.0407.

Example 3.2. *Let $H_1 = H_2 = \mathbb{R}^3$, $h_0 = (0, 0, 0)^T$, $D = B[0, 1]$. Define function $\mathcal{A}$ on D for $w = (h, s, g)^T$ by*

$$\mathcal{A}(w) = \left(e^h - 1, \frac{e - 1}{2} s^2 + s, g\right)^T.$$

Then, we get

$$\mathcal{A}'(w) = \begin{bmatrix} e^h & 0 & 0 \\ 0 & (e - 1)s + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $k_1 = (e - 1)$ and $k_2 = e$. Thus, $r_1 = 0.9254$, $r_2 = 0.9688$. So, r=0.9254.

4. Conclusion

This article studied a Homeier like method in a Banach space using assumptions on Frechet derivative of the operator upto order 2. Our approach does not involve Taylor series expansion. Also this approach can be used for obtaining converegence order of other methods without using assumptions on the derivatives of order more than two.

References

[1] I.K. Argyros, S. Hailout, Computational methods in nonlinear analysis: efficient algorithms, fixed point theory and applications, World Scientific (2013).

[2] A. Cordero, A. Franques, J.R. Torregrosa, Chaos and convergence of a family generalizing homeier's method with damping parameters, Nonlinear Dynamics 85(3) (2016) 1939-1954.

[3] A. Cordero, M.A. Hernandez-Veron, N. Romero, J.R. Torregrosa, Semilocal Convergence by using recurrence relations for a fifth-order method in banach spaces, Journal of Computational and Applied Mathematics 273 (2015) 205-213.

[4] S. George, I.K. Argyros, K. Senapati, K. Kanagaraj, Local convergence analysis of two iterative methods, The Journal of Analysis (2022) 1-12.

[5] M. Grau-Sanchez, A. Grau, M. Noguera, On the computational efficiency index and some iterative methods for solving systems of nonlinear equations, Journal of Computational and Applied Mathematics 236(6) (2011) 1259-1266.

[6] H.H.H. Homeier, A modified newton method with cubic convergence: the multiverse case, Journal of Computational and Applied Mathematics 168(1) (2004) 161-169.

[7] J.P. Jaiswal, Semilocal convergence analysis and comparison of alternative computational efficiency of the sixth-order method in Banach spaces, Novi Sad Journal of Mathathematics 49(2) (2019) 1-16.

[8] P. Jarratt, Some fourth order multipoint iterative methods for solving equations, Mathematics of computation 20(95) (1966) 434-437.

[9] C.T. Kelley, Iterative methods for linear and nonlinear equations, SIAM (1995).

[10] E. Martínez, S. Singh, J.L. Hueso, D.K. Gupta, Enlarging the convergence domain in local convergence studies for iterative methods in Banach spaces, Applied Mathematics and Computation 281 (2016) 252-265.

[11] J.M. Ortega, W.C. Rheinboldt, Iterative solution of nonlinear equations in several variables, SIAM (2000).

[12] A.M. Ostrowski, Solution of equations and system of equations, Pure and Applied Mathematics: A series of monographs and textbooks volume 9 (2016).

[13] B. Panday, J.P. Jaiswal, On the local Convergence of Modified Homeier-like Method in Banach Spaces, Numerical Analysis and Application 11(4) (2018) 332-345.

[14] L.I. Piscoran, D. Miclaus, A new steffensen homeier iterative method for solving nonlinear equations, Investigacion Operacional 40(1) (2019) 74-80.

[15] J.R. Sharma, P. Gupta, An efficient fifth order method for solving systems of nonlinear equations, Computers and Mathematics with Applications 67(3) (2014) 591-601.

[16] S. Singh, D.K. Gupta, E. Martinez, J.L Hueso, Semilocal convergence analysis of an iteration of order five using recurrence relations in banach spaces, Mediterranean Journal of Mathematics 13(6) (2016) 4219-4235.

[17] O.S. Solaiman, I. Hashim, An iterative scheme of arbitrary odd order and its basins of attraction for nonlinear systems, Computers, Materials and Continua Computers 66 (2021) 1427-1444.

[18] J.F. Traub, Iterative methods for the solution of equations, American Mathematical Society volume 312 (2013).

[19] S. Weerakoon, T.G.I. Fernando, A variant of newton's method with accelerated third-order convergence, Applied Mathematics Letters 13(8) (2000) 87-93.

[20] X. Wang, J. Kou, C. Gu, Semilocal convergence of a sixth-order Jarratt method in Banach spaces, Numerical Algorithms, 57 (4) (2011) 441-456.