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Research Article

A Study on the Non-selfadjoint Schrödinger Operator with Negative Density Function

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Abstract: This study focuses on the spectral features of the non-selfadjoint singular operator with an out-of-the-ordinary type weight function. Take into consideration the one-dimensional time-dependent Schrödinger type differential equation

$$-y'' + q(x)y = \mu^2\rho(x)y, x \in [0, \infty),$$

holding the initial condition

$$y(0) = 0,$$

and the density function defined with a completely negative value as

$$\rho(x) = -1.$$

There is an enormous number of the papers considering the positive values of $\rho(x)$ for both continuous and discontinuous cases. The structure of the density function affects the analytical properties and representations of the solutions of the equation. Unlike the classical literature, we use the hyperbolic type representations of the equation's fundamental solutions to obtain the operator's spectrum. Additionally, the requirements for finiteness of eigenvalues and spectral singularities are addressed. Hence, Naimark's and Pavlov's conditions are adopted for the negative density function case.

Negatif Yoğunluk Fonksiyonuna Sahip Kendine Eşlenik Olmayan Schrödinger Operatörü Üzerine Bir Çalışma

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Anahtar Kelimeler
Negatif yoğunluk fonksiyonu,
Spektral analiz,
Spektral tekillikler

Öz: Bu çalışmada kendine eşlenik olmayan, singüler ve standard dışı bir ağırlık fonksiyonuyla birlikte tanımlanmış operatörün spektral özellikleri ele alınacaktır. Bir boyutlu, zamana bağımlı Schrödinger tipli diferansiyel denklem

$$-y'' + q(x)y = \mu^2\rho(x)y, x \in [0, \infty),$$

$$y(0) = 0,$$

başlangıç koşulu ve tamamen negatif olarak tanımlı

$$\rho(x) = -1,$$

yoğunluk fonksiyonuyla birlikte göz önüne alınsın. Pozitif değerli sürekli ve süreksiz yoğunluk fonksiyonuna sahip operatörler için literatürde çok sayıda çalışma bulunmaktadır. Yoğunluk fonksiyonunun yapısı operatörün analitik özelliklerini ve çözümlerin gösterimini etkilemektedir. Klasik literatürden farklı olarak, bu çalışmada hiperbolik tipli temel çözümler operatörün spektrumunu belirlemek için kullanılmıştır. Buna ek olarak, özdeğerlerin ve spektral tekilliklerin sonluluğu için gerekli koşullar elde edilmiştir. Böylece, Naimark ve Pavlov koşulları, negatif yoğunluk fonksiyonuna sahip operatör durumunda çözülmüştür.

1. Introduction

The analysis of differential and discrete equations spectral features have emerged as a topic of curiosity in quantum physics and has become the source of extensive publications. It is clear that studies of spectral and scattering theory help to obtain very important information about nuclear particles and sub-particle physics. For these reasons, mathematical theories modelling the behaviors of the particles in quantum physics remain a popular research area (Naimark, 1954 and 1968; Pavlov, 1962; Chadan & Sabatier, 1977; Marchenko, 1986; Amrein, 2005; Levitan, 1987; Mutlu & Kir, 2020).

Take into account the one-dimensional Schrödinger equation

$$-y'' + q(x)y = \lambda^2 \rho(x)y, x \in [0, \infty), \tag{1}$$

where ρ denotes the density function and λ stands for the eigenparameter. There is excessive number of research papers on the inverse and direct problems for $\rho(x) = 1$ (Bairamov et al., 1999; Adıvar & Akbulut, 2010; Mamedov, 2010; Olgun & Coskun, 2010; Koprubasi & Yokus, 2014; Yokus & Coskun, 2019). The inverse problem of the operator with the equation (1) and discontinuous weight (density) function $\rho(x) = \begin{cases} \alpha^2, 0 \le x < a \\ 1, \ \ x \ge a \end{cases}$ where $0 < \alpha \neq 1$ has been handled by Mamedov (2010). The function's discontinuity heavily influences the structure of the Jost solution. Similarly, the representation of the main equation has been affected by the discontinuous weight, too. As a consequence, discontinuous positive valued weight function case took a prominent attention from various authors (Darwish, 1993; Gasymov & El-Reheem, 1993; Guseinov & Pashaev, 2002; Adıvar & Akbulut, 2010; Mamedov & Cetinkaya, 2015; Nabiev & Mamedov, 2015; Bairamov et. al., 2018).

It is vital to point out at this stage that Naimark was the first who attacked the singular non-selfadjoint problem for $\rho(x) = 1$ (Naimark, 1960 and 1968). Let us remark that since the operator generated by the help of the equation (1) is defined on the unbounded interval, it is said to be a singular operator. Also, complex valued potential function results to a non-selfadjoint (also called non-hermitian) operator. Naimark demonstrated that the operator's spectrum comprises eigenvalues, spectral singularities, and continuous spectrum. Under certain constraints, it is also confirmed by him that all these eigenvalues and spectral singularities must be of finite number and multiplicity. Non-hermitian Sturm-Liouville differential equations with the positive valued discontinuous density function have been researched in various papersemploying Naimark's and Pavlov's methodologies (Pavlov, 1962; Naimark, 1968; Levitan, 1987).

Unlike the known literature, inverse scattering and inverse spectral theory of the Sturm-Liouville type operators with sign-changing density function has been studied by Gasymov and El-Reheem (1993). The interested reader may also consult the papers (El-Raheem and Nasser, 2014; El-Raheem & Salama, 2015) and the references therein for the detailed information about the sign-valued density function case and its application in physics. The most crucial reason distinguishing this problem from the positive-valued weight function case is the new analytical difficulties that arising from the weight function's negative value. As a result of the appearance of the hyperbolic type solutions of the Sturm-Liouville problems, the analytical features of the solutions change entirely. Hence, we need to re-examine the spectral properties of the operators for the potentials including negative values.

Let us also remind that in discrete analogue of the Sturm-Liouville and Dirac operators, the representation of the Jost solution is determined by the eigenparameter transformation. While the trigonometric transformation $\lambda = 2\cos z$ results in analytical solutions in the upper half-plane (Bairamov et al., 2001; Yokus & Coskun, 2016), hyperbolic type transformation $\lambda = 2\cosh z$ gives the Jost solutions which are analytic in the left-half plane (Bairamov et al., 2010; Koprubasi, 2021; Koprubasi & Aygar Küçükevcilioğlu, 2022). Also, in the paper (Bairamov et al., 2010), the eigenparameter of the non-selfadjoint boundary value problem was taken as $\lambda = (iz) - (iz)^{-1}$, $|z| \le 1$. As a result of this transformation, the Jost solution obtained the polynomial type representation which is analytic inthe unit disc.

In addition to that, (Lyantse, 1968), the Jost solution of the difference equation analogue of the Sturm-Liouville operator has been investigated for the eigenparameters $\lambda = \frac{1}{2}(z^{-1} + z)$, $|z| \le 1$. A non-standard representation for the Jost solution has been obtained under this eigenparameter transformation, too.

This manuscript was influenced by the prior researches mentioned above. The spectral features of the non-hermitian singular Sturm-Liouville type equation for $\rho(x) = -1$ will be concenterated on. Compared to the discrete cases (Lyantse, 1968; Bairamov et al., 2001; Bairamov et al., 2010; Yokus & Coskun, 2016; Koprubasi, 2021;Koprubasi & Aygar Küçükevcilioğlu, 2022), it is clear that the problem of under what circumstances one obtains analytical solutions of Sturm-Liouville type differential operators in different regions has not been studied enough. Hence, this paper may fill the gap in the literature. Let us also remark that, while the transformation chosen for the eigenparameter determines the analytical characteristics of the Jost solutions in discrete problems; the structure of the weight function affects the Jost solution in differentialcase. Hence, based on this idea, this paper may also lay the groundwork for new research topics in inverse and direct problems. This paper also has crucial importance since this is one of the first studies considering the negative value of the density function for the singular non-selfadjoint operators.

2. Solutions to the Problem

We provide preliminary data for the negative density function case in this section, which could also be derived using similar theorems and methodologies (Marchenko, 1986; Gasymov & El-Reheem, 1993; El-Raheem & Nasser, 2014).

Let us introduce the differential operator T in the Hilbert space $L_\rho^2(\mathbb{R}_+)$ with help of the differential equation

$$-y'' + q(x)y = \mu^2 \rho(x) y, x \in \mathbb{R}_+, \quad (2)$$

and the initial condition

$$y(0) = 0, \quad (3)$$

where

$$\rho(x) = -1, \quad (4)$$

and μ is an eigenparameter. We also assume that the potential function q is complex-valued. Clearly, together with the expressions (2)-(4) and our assumptions, the operator T is a singular and non-selfadjoint operator.

Except otherwise indicated, we presume that $q(x)$ holds

$$\int_0^\infty x|q(x)|dx < \infty. \quad (5)$$

Consider the solutions of (2) as $S(x,\mu)$ and $C(x,\mu)$ which hold the initial conditions

$$\begin{aligned} S(0,\mu) = 0, S'(0,\mu) = 1, \\ C(0,\mu) = 1, C'(0,\mu) = 0. \end{aligned} \quad (6)$$

Take into account the case $q(x) \equiv 0$. Then, (2) takes the form

$$y'' = \mu^2 y, x \in \mathbb{R}_+. \quad (7)$$

Thus, $S(x,\mu)$ and $C(x,\mu)$ can be represented by the hyperbolic type representations

$$\begin{aligned} S(x,\mu) &= \frac{\sinh \mu x}{\mu}, \\ C(x,\mu) &= \cosh \mu x. \end{aligned} \tag{8}$$

Using the results of (Marchenko, 1986) and the constant coefficients method, one can easily verify that the fundamental solutions $S(x,\mu)$ and $C(x,\mu)$ have the Volterra type integral representations as

$$S(x,\mu) = \frac{\sinh \mu x}{\mu} + \int_0^x \mathrm{P}(\mathrm{x},\mathrm{t}) \frac{\sinh \mu(x-t)}{\mu} dt, \tag{9}$$

and

$$C(x,\mu) = \cosh \mu x + \int_0^x \mathrm{Q}(\mathrm{x},\mathrm{t}) \cosh \mu t dt. \tag{10}$$

Moreover, the functions S and C are entire with respect to the variable μ. They are also analytic on the left half-plane $Re\mu \leq 0$. Existence and uniqueness results of the solutions $S(x,\mu)$ and $C(x,\mu)$ can also be proven analogous to (Marchenko, 1986). Also, Wronskian of the solutions S and C might be formulated as $W[S(x,\mu), C(x,\mu)] = -1, \mu \in \mathbb{C}$.

Let us now indicate by $e(x,\mu)$ the solution of (2) that fullfils the asymptotic criteria $\lim_{x\to\infty} e(x,\mu)e^{-\mu x} = 1, Re\mu < 0$.

Under condition (5), (2) has the solution of the form

$$e(x,\mu) = e^{\mu x} + \int_x^\infty K(x,t) e^{\mu t} dt. \tag{11}$$

(11) is referred as the Jost solution of T. The kernel $K(x,.) \in L_1(0,\infty)$ and $K(x,t)$ can be uniquely determined by the potential function q. Moreover, it can be differentiated continuously with respect to its arguments.

Define $\alpha(x) = \int_x^\infty |q(s)| ds$. Hence, the inequality

$$|K(x,t)| \leq C\alpha\left(\frac{x+t}{2}\right), \tag{12}$$

yields for $C > 0$ constant. Therefore, the Jost solution $e(x,\mu)$ is analytic with regard to the variable μ in the region $\mathbb{C}_{left} := \{\mu \in \mathbb{C}: Re\mu < 0\}$ and continuous on $\overline{\mathbb{C}}_{left} := \{\mu \in \mathbb{C}: Re\mu \leq 0\}$. For further information about these results, one may consult the books of Marchenko (1986) and Amrein et al. (2005).

The resolvent operator of T is denoted as

$$R_\mu(T)f = \int_0^\infty G(x,t;\mu)\varphi(t) dt, \varphi \in L^2(\mathbb{R}_+), \tag{13}$$

where

$$G(x,t;\mu)=\begin{cases}\dfrac{e(x,\mu)S(t,\mu)}{e(0,\mu)}, 0<t\le x,\\ \dfrac{e(t,\mu)S(x,\mu)}{e(0,\mu)}, x<t<\infty,\end{cases} \tag{14}$$

is the Green's function of T. Thus, the resolvent set $R_\mu(T)$ can be stated in the form

$$R_\mu(T)=\{\lambda:\lambda=\mu^2, Re\mu<0, e(0,\mu)\neq 0\}. \tag{15}$$

3. Spectrum of *T*

Let us define

$$e(\mu):=e(0,\mu), \tag{16}$$

where

$$e(0,\mu)=1+\int_x^\infty K(0,t)e^{\mu t}dt. \tag{17}$$

Define the notation $\sigma_d(T)$ to designate the set of eigenvalues of T. Similarly, use the symbol $\sigma_{ss}(T)$ to show the spectral singularities of the operator T. If we make use of the classical definitions of the spectrum and expressions of the resolvent and Green's function in (14) and (15), we readily obtain

$$\sigma_d(T)=\left\{z: z=\mu^2, \mu\in\mathbb{C}_{left}, e(\mu)=0\right\} \tag{18}$$

$$\sigma_{ss}(T)=\{z: z=\mu^2, z=\xi+i\tau, \xi=0, \tau\in\mathbb{R}, e(\mu)=0\}. \tag{19}$$

Similarly to Naimark (1960 and 1968)'s theorems and using the fundamental concepts of the spectrum from functional analysis, we determine the continuous spectrum of the operator T as the following

$$\sigma_c(T)=\{z: z=\xi+i\tau, \xi=0, \tau\ge 0\}. \tag{20}$$

Definition 1 (Naimark, 1968; Levitan, 1987). The multiplicity of a root of $e(\mu)$ in the region $\overline{\mathbb{C}}_{left}$ is referred to as the multiplicity of the corresponding eigenvalue and spectral singularity of the operator T.

Up to now, using the classical definitions of the spectrum, we obtained the spectrum of the operator T. Hereafter, we will focus on the quantitative properties of the spectrum. For that purpose, it is clear that the zeros of $e(\mu)$ on $\overline{\mathbb{C}}_{left}$ have to be taken into consideration.

Let us define the sets

$$Q_1:=\left\{\mu: \mu\in\mathbb{C}_{left}, e(\mu)=0\right\}, \tag{21}$$

$$Q_2:=\{\mu: \mu=\xi+i\tau, \xi=0, \tau\in\mathbb{R}, e(\mu)=0\}. \tag{22}$$

Define all accumulation points of Q_1 by Q_3. Further, use the notation Q_4 to designate a set of all roots of $e(\mu)$ having infinite multiplicity in $\overline{\mathbb{C}}_{left}$. Obviously, using these set definitions, (18) and (19) can be restated as the following

$$\sigma_d(T)=\{z: z=\mu^2, \mu\in Q_1\}, \tag{23}$$

$$\sigma_{ss}(T) = \{z: z = \mu^2, \mu \in Q_2\}\backslash\{0\}. \tag{24}$$

Lemma 1 If (5) holds, then

(a) Q_1 is bounded set. Q_1 can have at most countably many elements. Furthermore, these elements can only accumulate to the bounded subset of the imaginary axis.

(b) Q_2 is a compact set. Moreover, its Lebesgue measure is zero.

Proof. Using the inequality (12) and the expression of $e(\mu)$, it can be easily seen that $e(\mu)$ is an analytic function with regard to the variable μ in $\mathbb{C}_{left}$. Also, it is continuous on the imaginary axis. Further, it yields the asymptotic

$$e(\mu) = 1 + o(1), \mu \in \overline{\mathbb{C}}_{left}, |\mu| \to \infty. \tag{25}$$

The boundedness of the sets Q_1 and Q_2 follows from (25). Hence, the proof of part (a) follows from analyticity of $e(\mu)$ in $\mathbb{C}_{left}$ and continuity on the imaginary axis. For part (b), we shall consider the boundary uniqueness theorems of analytic functions (Dolzhenko, 1979). Using these theorems, we get that Q_2 is a closed set and $\mu(Q_2) = 0$, where μ stands for the linear Lebesgue measure in the imaginary axis.

The following theorem can be stated easily using (23), (24) and Lemma 1:

Theorem 2 Suppose the condition (5) yields. In this case,

(i) The set of eigenvalues of T is bounded. Further, it can have at most countably many elements. Also, these elements can only accumulate to a bounded subinterval of the imaginary axis.

(ii) $\sigma_{ss}(T)$ is bounded set. $\mu\big(\sigma_{ss}(T)\big) = 0$.

Note that we obtained some quantitative properties of the spectrum of T under the condition (5). From now on, we will consider more strict conditions on the potential.

Theorem 3 Assume that

$$\int_0^\infty e^{\varepsilon x}|q(x)|dx < \infty, \tag{26}$$

for some $\varepsilon > 0$. If the condition (26) is fulfilled, then the operator T has finitely many eigenvalues and spectral singularities. Moreover, each of these eigenvalues and spectral singularities is of finite multiplicity.

Proof. Using (12) and (26), we can write

$$|K(x,t)| \leq C\exp\left(-\varepsilon\frac{x+t}{2}\right), \tag{27}$$

for arbitrary positive constant C. Considering the expression of $e(\mu)$ and (27), it is clear that $e(\mu)$ continues analytically from the complex left-half plane to the right half-plane $Re\mu < \frac{\varepsilon}{4}$. Consequently, the accumulation points of the roots of $e(\mu)$ in $\mathbb{C}_{left}$ cannot lie in the imaginary axis. From Lemma 1, we can see that the bounded sets Q_1 and Q_2 have a finite number of elements. Also, taking into account the analyticity of $e(\mu)$ for $Re\mu < \frac{\varepsilon}{4}$, we deduce that the zeros of $e(\mu)$ in $\mathbb{C}_{left}$ are of finite number, and they are of finite multiplicity. As a result, the operator T has a finite number of eigenvalues and spectral singularities with finite multiplicities.

The condition (26) is recognized as Naimark's condition in the discipline, which enables us to utilise the Jost function's analytic continuation characteristics for the proof. However, there is a more

strict condition for the potential called Pavlov's condition, which pushes us to use new methods to prove the finiteness of the sets $\sigma_d(T)$ and $\sigma_{ss}(T)$.
Let

$$\int_0^\infty e^{\varepsilon\sqrt{x}}|q(x)|dx < \infty, \varepsilon > 0. \tag{28}$$

Clearly, $e(\mu)$ is the analytic in the complex left-half plane $\mathbb{C}_{left}$ and continuous on the imaginary axis. Nevertheless, analytic continuation property does not hold from the left-half plane to the right-half plane. We will also benefit from the following relations between the sets Q_1, Q_2, Q_3 and Q_4 for the proof of the following theorem, which can be inferred directly from the boundary uniqueness theorems of the analytic functions (Dolzhenko, 1979):

$$Q_1 \cap Q_4 = \emptyset, Q_3 \subset Q_2, Q_4 \subset Q_2, Q_3 \subset Q_4, \tag{29}$$

and

$$\mu(Q_3) = \mu(Q_4) = 0. \tag{30}$$

Theorem 4 If the condition for the potential (28) holds to be accurate, then $Q_4 = \emptyset$.

Proof. Using Lemma 1, we obtain that

$$\left|\int_{-\infty}^{-M} \frac{\ln|e(\mu)|}{1+\mu^2} d\mu\right| < \infty, \left|\int_M^\infty \frac{\ln|e(\mu)|}{1+\mu^2} d\mu\right| < \infty, \tag{31}$$

for sufficiently large values of $M > 0$. Moreover, $e(\mu)$ is analytic in $\mathbb{C}_{left}$, all its derivatives are continuous up to the imaginary axis and

$$\left|e^{(r)}(\mu)\right| \leq C_r, \mu \in \overline{\mathbb{C}}_{left}, r = 1,2,\dots, |\mu| < 2M, \tag{32}$$

where

$$C_r := c\int_0^\infty t^r|K(0,t)|dt. \tag{33}$$

If we make use of (31), (32) and Pavlov's theorem, we get

$$\int_0^\omega \ln t(s) d\mu(Q_{4,s}) > -\infty, \tag{34}$$

where $t(s) = \inf\limits_r \frac{C_r s^r}{r!}, C_r$ is defined by (33), $\mu(Q_{4,s})$ is the linear Lebesgue measure of the s-neighborhood of Q_4, and $\omega > 0$ is a constant (Bairamov et al., 1999; Adıvar & Akbulut, 2010). We can also write the following estimations

$$C_r = c\int_0^\infty t^r|K(0,t)|dt \leq c\int_0^\infty t^r \exp\left(-\frac{\varepsilon}{4}t\right) dt \leq Bb^r r^r r!, \tag{35}$$

for B and b are constants depending on c and ε. If we substitute the estimation (35) in the definition of $t(s)$, we get

$$t(s) = \inf_r \frac{C_r s^r}{r!} \leq B\inf_r\{b^r s^r r^r\} \leq B\exp\{-s^{-1}e^{-1}b^{-1}\}, \tag{36}$$

by (34). It follows from (35) and (36) that

$$\int_0^\omega s^{-\frac{\delta}{1-\delta}} d\mu(Q_{4,s}) < \infty. \tag{37}$$

Clearly, $\frac{\delta}{1-\delta} \geq 1$. Therefore, if we consider the convergent integral in (37), this might be true if and only if, for arbitrary s, $\mu(Q_{4,s}) = 0$ or $Q_4 = \emptyset$.

□

Theorem 5 In case the condition (28) is true, then the operator T does have a finitely many eigenvalues and spectral singularities with a finite multiplicity.

Proof. To verify the theorem, we shall demonstrate that $e(\mu)$ has a finite number of zeros with finite multiplicities in $\overline{\mathbb{C}}_{left}$. Using the relation (29) between the sets and the former theorem, it may well be observed that $Q_3 = \emptyset$. That is, the accumulation points of the bounded sets Q_1 and Q_2 can not exist. Therefore, $e(\mu)$ has only a finite number of roots in $\overline{\mathbb{C}}_{left}$. Because $Q_4 = \emptyset$, we can see that these roots are of finite multiplicity.

4. Discussion and Conclusion

In this study, we investigated the spectrum and spectral properties of the non-selfadjoint Sturm-Liouville type operator with the negative density function. We used hyperbolic type representations of the fundamental solutions of the operator to obtain the spectrum. We obtained the Jost function which is analytic on the left-half complex plane. We also adopted Naimark's and Pavlov's conditions for the potential function to be met for the finiteness of the eigenvalues and spectral singularities.

The exciting feature of this study is that we present the relation between the discrete operator case and differential operator case from a different perspective. In particular, this study is analogous to the hyperbolic eigenparameter-dependent case in discrete operators.

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