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Research Article

Generalized Fubini transform with two variables

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Abstract

In the present paper, we define the generalized Kwang-Wu Chen matrix. Basic properties of this generalization, such as explicit formulas and generating functions are presented. Moreover, we focus on a new class of generalized Fubini polynomials. Then we discuss their relationship with other polynomials such as Fubini, Bell, Eulerian and Frobenius-Euler polynomials. We have also investigated some basic properties related to the degenerate generalized Fubini polynomials.

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1. Introduction

The nth Bernoulli numbers B_n are defined by the generating function

$$\frac{t}{e^t-1}=\sum_{n\geq 0}B_n\frac{t^n}{n!},\quad |t|<2\pi. \tag{1.1}$$

The rational numbers $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$ and $B_{2n+1}=0$ for $n>0$, have many beautiful properties. The most basic recurrence relation is

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k=0. \tag{1.2}$$

In 2001, Kwang-Wu Chen [5] gave an algorithm for computing Bernoulli numbers, with

$$a_{0,m}=\frac{1}{m+1};\quad a_{n+1,m}=-\left(m+1\right)a_{n,m+1}+ma_{n,m}. \tag{1.3}$$

The primary purpose of this paper is to extend the Fubini transform for generalizing Fubini polynomials and studying its properties. We first generalize (1.3). The idea is

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to construct an infinite matrix $\mathcal{M} := (a_{n,m})_{n,m\geq 0}$ in which the first row $a_{0,m} := \alpha_m$ of the matrix is the initial sequence and the first column $a_{n,0} := \beta_n$ is the final sequence. More precisely, for nonzero complex numbers x and y, we propose to study the following three-term recurrence relation

$$a_{n+1,m}(x,y) = x\,(m+1)\,a_{n,m+1}(x,y) + yma_{n,m}(x,y). \tag{1.4}$$

By setting $x = -1$ and $y = 1$ in (1.4), we get (1.3). More directly, we propose to generalize the Fubini transformation.

The Fubini transform of a sequence $(\alpha_n)_{n\geq 0}$ is the sequence $(\beta_n)_{n\geq 0}$ given by

$$\beta_n = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} t^k \alpha_k$$

and the inverse transform is

$$\alpha_n = \frac{1}{n!t^n} \sum_{k=0}^{n} s\,(n,k)\,\beta_k \quad .$$

2. Definitions and notation

In this section, we introduce some definitions and notations which are useful in the rest of the paper. Following the usual notations [7].

The falling factorial $x^{\underline{n}}$ $(x \in \mathbb{C})$ is defined by

$$x^{\underline{n}} = x\,(x-1)\cdots(x-n+1)\,, x^{\underline{0}} = 1$$

and the rising factorial denoted by $x^{\overline{n}}$, is defined by

$$x^{\overline{n}} = x\,(x+1)\cdots(x+n-1)\,, x^{\overline{0}} = 1.$$

The (signed) Stirling numbers of the first kind denoted $s\,(n,k)$ are the coefficients in the expansion

$$x^{\underline{n}} = \sum_{k=0}^{n} s\,(n,k)\,x^k. \tag{2.1}$$

The exponential generating function is

$$\frac{1}{k!}\left(\ln\left(1+t\right)\right)^k = \sum_{n\geq k} s\,(n,k)\,\frac{t^n}{n!}, \tag{2.2}$$

and $s\,(n,k)$ satisfy the following recurrence relation:

$$s\,(n+1,k) = s\,(n,k-1) - ns\,(n,k) \tag{2.3}$$

and that

$$s\,(n,0) = \delta_{n,0}\ (n \in \mathbb{N}),\ s\,(n,k) = 0\ (k > n \text{ or } k < 0),$$

where $\delta_{n,m}$ denoted Kronecker symbol.

The Stirling numbers of the second kind denoted $\left\{ {n \atop k} \right\}$ count the number of ways to partition a set of n things into k nonempty subsets. Explicitly $\left\{ {n \atop k} \right\}$ are the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}.$$

The r-Stirling numbers [3] denotes $\left\{ {n \atop k} \right\}_r$, for any positive $r \in \mathbb{N}$, the number of partitions of a set of n objects into exactly k nonempty, disjoint subsets, such that the first r elements are in distinct subsets. These numbers obey the recurrence relation

$$\begin{aligned} \left\{ {n \atop k} \right\}_r &= 0, && n < r, \\ \left\{ {n \atop k} \right\}_r &= \delta_{k,r}, && n = r, \\ \left\{ {n \atop k} \right\}_r &= k \left\{ {n-1 \atop k} \right\}_r + \left\{ {n-1 \atop k-1} \right\}_r, && n > r \end{aligned} \tag{2.4}$$

and

$$\left\{ {n \atop k} \right\}_r = \left\{ {n \atop k} \right\}_{r-1} - (r-1) \left\{ {n-1 \atop k} \right\}_{r-1}. \tag{2.5}$$

The exponential generating function is given by

$$\frac{1}{k!} e^{rt} \left(e^t - 1 \right)^k = \sum_{n \geq k} \left\{ {n+r \atop k+r} \right\}_r \frac{t^n}{n!}. \tag{2.6}$$

3. The Generalized Fubini transform

Theorem 3.1. *Given an initial sequence $(a_{0,m})_{m \geq 0}$, define the matrix $\mathcal{M}$ associated with the initial sequence by (1.4) then*

(1) The entries of the matrix $\mathcal{M}$ are given by

$$a_{n,m}(x,y) = \frac{1}{m!} \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\}_m (k+m)! y^{n-k} x^k a_{0,m+k}. \tag{3.1}$$

(2) Suppose that the initial sequence $a_{0,m+r}$ has the following ordinary generating function $A_r(t) = \sum_{k \geq 0} a_{0,k+r} t^k$. Then, the sequence $(a_{n,r}(x))_{n \geq 0}$ of the rth columns of the matrix $\mathcal{M}$ has an exponential generating function $B_r(t; x, y) = \sum_{n \geq 0} a_{n,r}(x,y) \frac{t^n}{n!}$, given by

$$B_r(t; x, y) = \frac{e^{rty}}{r!} \left(e^{-ty} \frac{d}{dt} \right)^r \left[\left(\frac{e^{ty} - 1}{y} \right)^r A_r \left(\frac{x}{y} (e^{ty} - 1) \right) \right]. \tag{3.2}$$

Proof. (1) We prove the relation (3.1) by induction on n. The result clearly holds for $n = 0$, we now show that the formula for $n+1$ follows from (1.4) and induction hypothesis

$$\begin{aligned} a_{n+1,m}(x,y) = & \frac{1}{m!} \sum_{k=0}^{n-1} \left\{ {n+m+1 \atop k+m+1} \right\}_{m+1} (k+m+1)! x^{k+1} y^{n-k} a_{0,m+k+1} \\ & + m \left\{ {n+m \atop m} \right\}_m y^{n+1} a_{0,m} + \frac{1}{(m-1)!} \sum_{k=1}^{n} \left\{ {n+m \atop k+m} \right\}_m (k+m)! x^k y^{n-k+1} a_{0,m+k} \\ & + \frac{1}{m!} \left\{ {n+m+1 \atop n+m+1} \right\}_{m+1} (n+m+1)! x^{n+1} a_{0,m+n+1}. \end{aligned}$$

After some rearrangements, we get

$$
\begin{aligned}
a_{n+1,m}(x,y) =& \frac{1}{m!}\sum_{k=1}^{n}\begin{Bmatrix} n+m+1 \\ k+m \end{Bmatrix}_{m+1}(k+m)!x^k y^{n-k+1}a_{0,m+k} + m\begin{Bmatrix} n+m \\ m \end{Bmatrix}_{m} y^{n+1}a_{0,m} \\
&+\frac{1}{(m-1)!}\sum_{k=1}^{n}\begin{Bmatrix} n+m \\ k+m \end{Bmatrix}_{m}(k+m)!x^k y^{n-k+1}a_{0,m+k} \\
&+\frac{1}{m!}\begin{Bmatrix} n+m+1 \\ n+m+1 \end{Bmatrix}_{m+1}(n+m+1)!x^{n+1}a_{0,m+n+1}.
\end{aligned}
$$

From (2.4) and (2.5), and after some rearrangements, we get

$$
\begin{aligned}
a_{n+1,m}(x,y) =& \frac{1}{m!}\sum_{k=1}^{n}\left(\begin{Bmatrix} n+m+1 \\ k+m \end{Bmatrix}_{m+1} + m\begin{Bmatrix} n+m \\ k+m \end{Bmatrix}_{m}\right)(k+m)!x^k y^{n-k+1}a_{0,m+k} \\
&+\begin{Bmatrix} n+m+1 \\ m \end{Bmatrix}_{m} y^{n+1}a_{0,m} + \frac{1}{m!}\begin{Bmatrix} n+m+1 \\ n+m+1 \end{Bmatrix}_{m}(n+m+1)!x^{n+1}a_{0,m+n+1} \\
=& \frac{1}{m!}\sum_{k=0}^{n+1}\begin{Bmatrix} n+m+1 \\ k+m \end{Bmatrix}_{m}(k+m)!x^k y^{n-k+1}a_{0,m+k}.
\end{aligned}
$$

(2) The verification of (3.2) follows by induction on n. By using (3.1), we obtain

$$
\begin{aligned}
B_r(t;x,y) &= \sum_{n\geq 0}\left(\frac{1}{r!}\sum_{k=0}^{n}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r}(k+r)!x^k y^{n-k}a_{0,r+k}\right)\frac{t^n}{n!} \\
&= \sum_{k\geq 0}\frac{(k+r)!}{r!}\left(\frac{x}{y}\right)^k a_{0,r+k}\sum_{n\geq k}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r}\frac{(ty)^n}{n!}.
\end{aligned}
$$

From the relation (2.6), we obtain

$$
\begin{aligned}
B_r(t;x,y) &= \sum_{k\geq 0}\frac{(k+r)!}{r!}\left(\frac{x}{y}\right)^k a_{0,r+k}\frac{1}{k!}e^{rty}\left(e^{ty}-1\right)^k \\
&= e^{rty}\sum_{k\geq 0}\binom{k+r}{r}a_{0,r+k}\left(\frac{x}{y}\left(e^{ty}-1\right)\right)^k.
\end{aligned}
$$

Since

$$
\binom{k+r}{r}\left[\frac{x}{y}\left(e^{ty}-1\right)\right]^k = \frac{1}{r!x^r}\left(e^{-ty}\frac{d}{dt}\right)^r\left[\frac{x}{y}\left(e^{ty}-1\right)\right]^{k+r},
$$

we get

$$
B_r(t;x,y) = \frac{e^{rty}}{r!}\left(e^{-ty}\frac{d}{dt}\right)^r\left[\left(\frac{e^{ty}-1}{y}\right)^r A_r\left(\frac{x}{y}(e^{ty}-1)\right)\right].
$$

This evidently completes the proof of Theorem.

□

The following corollary represents another expression for the generating function B_r

Corollary 3.2.

$$
B_r(t;x,y) = \frac{1}{r!}\sum_{k=0}^{r}s(r,k)\frac{d^k}{dt^k}\left[\left(\frac{e^{ty}-1}{y}\right)^r A_r\left(\frac{x}{y}(e^{ty}-1)\right)\right]. \tag{3.3}
$$

To prove formula (3.3) using

$$\left(e^{-ty}\frac{d}{dt}\right)^{r}F\left(t\right)=e^{-rty}\sum_{k=0}^{r}s\left(r,k\right)\frac{d^{k}}{dt^{k}}F\left(t\right),$$

with

$$F\left(t\right)=\left(\frac{e^{ty}-1}{y}\right)^{r}A_{r}\left(\frac{x}{y}(e^{ty}-1)\right).$$

Theorem 3.3. *Given final sequence* $\left(a_{n,0}\right)_{n\geq 0}$*, define the matrix* $\mathcal{M}$ *associated with the final sequence by*

$$a_{n,m+1}(x,y)=\frac{1}{x\left(m+1\right)}\left(a_{n+1,m}(x,y)-yma_{n,m}(x,y)\right), \tag{3.4}$$

then

(1) The entries of the matrix $\mathcal{M}$ *are given by*

$$a_{n,m}\left(x,y\right)=\frac{y^{m}}{x^{m}m!}\sum_{k=0}^{m}y^{-k}s\left(m,k\right)a_{n+k,0}\ . \tag{3.5}$$

(2) Suppose that the final sequence $a_{n+r,0}$ *has the following exponential generating function* $\widehat{\mathcal{B}}_{r}\left(t\right)=\sum_{k\geq 0}a_{k+r,0}\frac{t^{k}}{k!}$*. Then, the sequence* $\left(a_{r,m}\left(x\right)\right)_{m\geq 0}$ *of the* r*th row of the matrix* $\mathcal{M}$ *has an ordinary generating function* $\widehat{\mathcal{A}}_{r}\left(t;x,y\right)=\sum_{m\geq 0}a_{r,m}(x,y)t^{m}$*, given by*

$$\widehat{\mathcal{A}}_{r}\left(t;x,y\right)=\widehat{\mathcal{B}}_{r}\left(y^{-1}\ln\left(1+\frac{ty}{x}\right)\right). \tag{3.6}$$

Proof. (1) We prove by induction on m, the result clearly holds for $m=0$. By induction hypothesis and (3.4), we have

$$\begin{aligned}a_{n,m+1}(x,y)&=\frac{y^{m}}{x^{m+1}\left(m+1\right)!}\left(\sum_{k=0}^{m}y^{-k}s\left(m,k\right)a_{n+k+1,0}\ -ym\sum_{k=0}^{m}y^{-k}s\left(m,k\right)a_{n+k,0}\ \right)\\&=\frac{y^{m}}{x^{m+1}\left(m+1\right)!}\left(y^{-m}s\left(m,m\right)a_{n+m+1,0}\ +\sum_{k=0}^{m-1}y^{-k}s\left(m,k\right)a_{n+k+1,0}\ \right)\\&-\frac{y^{m}}{x^{m+1}\left(m+1\right)!}\left(m\sum_{k=1}^{m}y^{-k+1}s\left(m,k\right)a_{n+k,0}\ +mys\left(m,0\right)a_{n,0}\ \right).\end{aligned}$$

After some rearrangements, we get

$$\begin{aligned}a_{n,m+1}(x,y)=&\frac{y^{m}}{x^{m+1}\left(m+1\right)!}\left(\sum_{k=1}^{m}y^{-k+1}s\left(m,k-1\right)a_{n+k,0}\ -m\sum_{k=1}^{m}y^{-k+1}s\left(m,k\right)a_{n+k,0}\ \right)\\&+\frac{y^{m}}{x^{m+1}\left(m+1\right)!}\left(y^{-m}s\left(m,m\right)a_{n+m+1,0}\ -mys\left(m,0\right)a_{n,0}\ \right).\end{aligned}$$

From (2.3) and after some rearrangements, we get

$$\begin{aligned} a_{n,m+1}(x,y) &= \frac{y^{m+1}}{x^{m+1}(m+1)!}\sum_{k=1}^{m} y^{-k}\left(s(m,k-1)-ms(m,k)\right)a_{n+k,0} \\ &+ \frac{y^{m+1}}{x^{m+1}(m+1)!}\left(y^{-m-1}s(m+1,m+1)\,a_{n+m+1,0} \quad + s(m+1,0)\,a_{n,0}\right) \\ &= \frac{y^{m+1}}{x^{m+1}(m+1)!}\sum_{k=0}^{m+1} y^{-k}s(m+1,k)\,a_{n+k,0}\,. \end{aligned}$$

which completes the proof.

(2) According to (3.5), we have

$$\begin{aligned} \widehat{\mathcal{A}}_r(t;x,y) &= \sum_{m\geq 0}\left(\frac{y^m}{x^m m!}\sum_{k=0}^{m} y^{-k}s(m,k)\,a_{r+k,0}\right)t^m \\ &= \sum_{k\geq 0} a_{r+k,0}\ \ y^{-k}\sum_{m\geq k}\frac{y^m}{x^m m!}s(m,k)\,t^m. \end{aligned}$$

From the relation (2.2), we obtain

$$\begin{aligned} \widehat{\mathcal{A}}_r(t;x,y) &= \sum_{k\geq 0} a_{r+k,0}\ \ y^{-k}\frac{1}{k!}\left(\ln\left(1+\frac{ty}{x}\right)\right)^k \\ &= \widehat{\mathcal{B}}_r\left(y^{-1}\ln\left(1+\frac{ty}{x}\right)\right), \end{aligned}$$

which completes the proof.

□

Corollary 3.4. *For $n,m\geq 0$, we have*

$$\sum_{k=0}^{n}\left\{\begin{matrix} n+m \\ k+m \end{matrix}\right\}_m (k+m)!x^k y^{n-k} a_{0,m+k} = \sum_{k=0}^{m} s(m,k)\,x^{-m}y^{m-k}a_{n+k,0}\ . \tag{3.7}$$

The identity (3.7) can be viewed as the generalized Fubini transform which can be reduced, for $m=0$ to the Fubini transform of the sequence α_n, and for $n=0$ to the inverse Fubini transform of the sequence β_m.

4. On generalized Fubini polynomials

Setting the initial sequence $a_{0,m}=1$ in (1.4), we get the following matrix

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ x & y+2x & 2y+3x & 3y+4x & \cdots \\ 2x^2+yx & 6x^2+6yx+y^2 & 12x^2+15xy+4y^2 & \vdots & \\ 6x^3+6x^2y+xy^2 & 24x^3+36x^2y+14xy^2+y^3 & \vdots & & \\ 24x^4+36x^3y+14x^2y^2+xy^3 & \vdots & & & \\ \vdots & & & & \end{pmatrix}.$$

Since $A_r(t) = \frac{1}{1-t}$, it follows from (3.2) and (3.3) that the final sequence has an exponential generating function given by

$$B_r(t,x,y) = \frac{e^{rty}}{r!}\left(e^{-ty}\frac{d}{dt}\right)^r\left[\left(\frac{e^{ty}-1}{y}\right)^r \frac{1}{1-\frac{x}{y}(e^{ty}-1)}\right]$$
$$= \frac{1}{r!}\sum_{k=0}^{r} s(r,k)\frac{d^k}{dt^k}\left[\left(\frac{e^{ty}-1}{y}\right)^r \frac{1}{1-\frac{x}{y}(e^{ty}-1)}\right].$$

In particular for $r=0$, we have

$$B_0(t;x,y) = \frac{1}{1-\frac{x}{y}(e^{ty}-1)}.$$

Definition 4.1. We defined a sequence of polynomials $\mathfrak{F}_n(x,y)$ of two variables x, y, called generalized Fubini polynomials, by means of the generating function

$$\frac{1}{1-\frac{x}{y}(e^{ty}-1)} = \sum_{n\geq 0}\mathfrak{F}_n(x,y)\frac{t^n}{n!}. \tag{4.1}$$

The explicit formula for $\mathfrak{F}_n(x,y)$ is given by

$$\mathfrak{F}_n(x,y) = \sum_{k=0}^{n}\left\{ n \atop k \right\} k! x^k y^{n-k}. \tag{4.2}$$

By setting $y=1$ in (4.1), we get

$$\frac{1}{1-x(e^t-1)} = \sum_{n\geq 0}\mathfrak{F}_n(x,1)\frac{t^n}{n!}$$
$$= \sum_{n\geq 0}\omega_n(x)\frac{t^n}{n!}, \tag{4.3}$$

where $\omega_n(x)$ denotes the Fubini polynomials [1, 8, 11], defined by

$$\omega_n(x) = \sum_{k=0}^{n}\left\{ n \atop k \right\} k! x^k.$$

By (4.1) and (4.3), we can write the relation between $\omega_n(x)$ and $\mathfrak{F}_n(x,y)$, given by the following two formulas

$$\mathfrak{F}_n(x,y) = y^n\omega_n\left(\frac{x}{y}\right) \tag{4.4}$$

and

$$\omega_n(x) = y^{-n}\mathfrak{F}_n(xy,y). \tag{4.5}$$

The Fubini polynomials $\omega_n(x)$ are related to the geometric series in the following way [1, 2]

$$\left(x\frac{d}{dx}\right)^n\frac{1}{1-x} = \sum_{k\geq 0}x^k k^n = \frac{1}{1-x}\omega_n\left(\frac{x}{1-x}\right). \tag{4.6}$$

This relation can be extended to a more general form depending on two variables x and y.

Theorem 4.2. *For x different to y, the polynomials $\mathfrak{F}_n(x,y)$ have the following property*

$$\frac{y}{y-x}\mathfrak{F}_n\left(\frac{xy}{y-x},y\right) = \sum_{k\geq 0}\left(\frac{x}{y}\right)^k (yk)^n = y^n\left(x\frac{d}{dx}\right)^n\frac{y}{y-x}. \tag{4.7}$$

Proof. We have

$$\frac{y}{y-x}\sum_{n\geq 0}\mathfrak{F}_n\left(\frac{xy}{y-x},y\right)\frac{t^n}{n!}=\frac{y}{y-x}\left(\frac{1}{1-\frac{x}{y-x}(e^{ty}-1)}\right)$$
$$=\frac{1}{1-\frac{x}{y}e^{ty}}$$
$$=\sum_{k\geq 0}\left(\frac{x}{y}\right)^k\left(e^{ty}\right)^k.$$

Then

$$\frac{y}{y-x}\sum_{n\geq 0}\mathfrak{F}_n\left(\frac{xy}{y-x},y\right)\frac{t^n}{n!}=\sum_{k\geq 0}\left(\frac{x}{y}\right)^k\sum_{n\geq 0}(ky)^n\frac{t^n}{n!}$$
$$=\sum_{n\geq 0}\left(\sum_{k\geq 0}\left(\frac{x}{y}\right)^k(ky)^n\right)\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$, we get

$$\frac{y}{y-x}\mathfrak{F}_n\left(\frac{xy}{y-x},y\right)=\sum_{k\geq 0}\left(\frac{x}{y}\right)^k(ky)^n.$$

On the other hand, we apply the formula (4.1) in [1], we get

$$\sum_{k\geq 0}\left(\frac{x}{y}\right)^k(yk)^n=y^n\sum_{k\geq 0}\left(x\frac{d}{dx}\right)^n\left(\frac{x}{y}\right)^k$$
$$=y^n\left(x\frac{d}{dx}\right)^n\sum_{k\geq 0}\left(\frac{x}{y}\right)^k$$
$$=y^n\left(x\frac{d}{dx}\right)^n\frac{y}{y-x}. \tag{4.8}$$

This evidently completes the proof of the theorem. □

Remark 4.3. By setting $y=1$ in (4.7) we get (4.6).

Now, recall that the exponential generating function for Bell polynomials $\phi_n(x)$, is given by

$$e^{x(e^t-1)}=\sum_{n\geq 0}\phi_n(x)\frac{t^n}{n!} \tag{4.9}$$

and given explicitly by

$$\phi_n(x)=\sum_{k=0}^{n}\left\{ n \atop k \right\}x^k. \tag{4.10}$$

In the following result, we will give the integral representation for $\mathfrak{F}_n(x,y)$ and the link with $\phi_n(x)$.

Theorem 4.4. *For $n\geq 0$, we have*

$$\mathfrak{F}_n(x,y)=y^n\int_0^{+\infty}\phi\left(\frac{x}{y}\lambda\right)e^{-\lambda}d\lambda \tag{4.11}$$

and

$$\sum_{n\geq 0}\mathfrak{F}_n(x,y)\frac{t^n}{n!}=\int_0^{+\infty}e^{-\lambda\left(1-\frac{x}{y}(e^{ty}-1)\right)}d\lambda. \tag{4.12}$$

Proof. Replacing x by $\frac{x}{y}\lambda$ in (4.10) and multiplying both sides by $y^n e^{-\lambda}$ and integrating for λ from zero to infinity, we have

$$\begin{aligned} y^n \int_0^{+\infty} \phi\left(\frac{x}{y}\lambda\right) e^{-\lambda} d\lambda &= y^n \int_0^{+\infty} \left(\sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left(\frac{x}{y}\lambda\right)^k\right) e^{-\lambda} d\lambda \\ &= y^n \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left(\frac{x}{y}\right)^k \int_0^{+\infty} e^{-\lambda} (\lambda)^k \, d\lambda \\ &= y^n \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left(\frac{x}{y}\right)^k k!. \end{aligned}$$

By comparing with (4.2) we get (4.11).

Now to prove (4.12), using (4.11), we have

$$\begin{aligned} \sum_{n\geq 0} \mathfrak{F}_n(x,y) \frac{t^n}{n!} &= \sum_{n\geq 0} \left(y^n \int_0^{+\infty} \phi\left(\frac{x}{y}\lambda\right) e^{-\lambda} d\lambda \right) \frac{t^n}{n!} \\ &= \int_0^{+\infty} \left(e^{-\lambda} \sum_{n\geq 0} \phi\left(\frac{x}{y}\lambda\right) \frac{(ty)^n}{n!} \right) d\lambda \end{aligned}$$

we apply (4.9), we get

$$\begin{aligned} \sum_{n\geq 0} \mathfrak{F}_n(x,y) \frac{t^n}{n!} &= \int_0^{+\infty} \left(e^{-\lambda} e^{\frac{x}{y}\lambda(e^{ty}-1)} \right) d\lambda \\ &= \int_0^{+\infty} e^{-\lambda\left(1-\frac{x}{y}(e^{ty}-1)\right)} d\lambda. \end{aligned}$$

□

Remark 4.5. By setting $y = 1$ in (4.11) and (4.12), respectively, we get (3.11) and (3.13) in [1].

The Fubini polynomials of two variables $\omega_n(x,y)$ are defined in [8,9,11] by the following generating function

$$\frac{e^{ty}}{1-x(e^t-1)} = \sum_{n\geq 0} \omega_n(x,y) \frac{t^n}{n!}. \tag{4.13}$$

The next result represents the relation between $\mathfrak{F}_n(x,y)$ and $\omega_n(x,y)$.

Theorem 4.6. *For $n \geq 0$, we have*

$$\mathfrak{F}_n(x,y) = y^n \sum_{k=0}^{n} y^k \binom{n}{k} (-1)^k \omega_{n-k}\left(\frac{x}{y}, y\right). \tag{4.14}$$

Proof. From (4.1) and (4.13), we have

$$\begin{aligned}
\sum_{n\geq 0}\mathfrak{F}_n(x,y)\frac{t^n}{n!} &= e^{-ty^2}\frac{e^{ty^2}}{1-\frac{x}{y}(e^{ty}-1)}\\
&= \left(\sum_{n\geq 0}\frac{(-ty^2)^n}{n!}\right)\left(\sum_{n\geq 0}\omega_n\left(\frac{x}{y},y\right)\frac{(ty)^n}{n!}\right)\\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}(-y^2)^k\omega_{n-k}\left(\frac{x}{y},y\right)y^{n-k}\right)\frac{t^n}{n!}\\
&= \sum_{n\geq 0}\left(y^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k y^k\omega_{n-k}\left(\frac{x}{y},y\right)\right)\frac{t^n}{n!},
\end{aligned}$$

that is to say

$$\mathfrak{F}_n(x,y) = y^n\sum_{k=0}^{n}y^k\binom{n}{k}(-1)^k\omega_{n-k}\left(\frac{x}{y},y\right).$$

□

In addition to the above properties of $\mathfrak{F}_n(x,y)$ polynomials, we now present some recurrence relations. The following lemma will be useful for the proof of the next theorem.

Lemma 4.7. *For nonzero complex numbers x and y, we have*

$$\frac{e^{ty}}{1-\frac{x}{y}(e^{ty}-1)} = \left(\frac{1}{x}-\frac{1}{y}(e^{ty}-1)\right)\frac{d}{dt}\left(\frac{1}{1-\frac{x}{y}(e^{ty}-1)}\right). \tag{4.15}$$

Theorem 4.8. *For $n\geq 0$, we have*

$$\mathfrak{F}_{n+1}(x,y) = \left(\frac{xy}{x+y}\right)\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\left(\mathfrak{F}_k(x,y)+\frac{1}{y}\mathfrak{F}_{k+1}(x,y)\right). \tag{4.16}$$

Proof. Using the above lemma, then (4.15) is equivalent to

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\mathfrak{F}_k(x,y)\right)\frac{t^n}{n!} = \left(\frac{1}{x}-\frac{1}{y}\sum_{n\geq 0}\frac{(ty)^n}{n!}+\frac{1}{y}\right)\sum_{n\geq 0}\mathfrak{F}_{n+1}(x,y)\frac{t^n}{n!}.$$

Then,

$$\begin{aligned}
\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\mathfrak{F}_k(x,y)\right)\frac{t^n}{n!} &= \frac{1}{x}\sum_{n\geq 0}\mathfrak{F}_{n+1}(x,y)\frac{t^n}{n!}-\frac{1}{y}\sum_{n\geq 0}\frac{(ty)^n}{n!}\sum_{n\geq 0}\mathfrak{F}_{n+1}(x,y)\frac{t^n}{n!}\\
&\quad+\frac{1}{y}\sum_{n\geq 0}\mathfrak{F}_{n+1}(x,y)\frac{t^n}{n!}\\
&= \frac{1}{x}\sum_{n\geq 0}\mathfrak{F}_{n+1}(x,y)\frac{t^n}{n!}-\frac{1}{y}\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\mathfrak{F}_{k+1}(x,y)\right)\frac{t^n}{n!}\\
&\quad+\frac{1}{y}\sum_{n\geq 0}\mathfrak{F}_{n+1}(x,y)\frac{t^n}{n!}\\
&= \sum_{n\geq 0}\left(\frac{1}{x}\mathfrak{F}_{n+1}(x,y)-\frac{1}{y}\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\mathfrak{F}_{k+1}(x,y)+\frac{1}{y}\mathfrak{F}_{n+1}(x,y)\right)\frac{t^n}{n!}.
\end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$, we get

$$\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\mathfrak{F}_k(x,y) = \frac{1}{x}\mathfrak{F}_{n+1}(x,y)-\frac{1}{y}\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\mathfrak{F}_{k+1}(x,y)+\frac{1}{y}\mathfrak{F}_{n+1}(x,y)$$

and after some rearrangements, we obtain the result. □

Remark 4.9. As a special case, we get the formula (24) in [6] by setting $y = 1$ in (4.16).

Theorem 4.10. *For $n \geq 0$, we have*

$$\mathfrak{F}_{n+1}(x,y) + y\mathfrak{F}_n(x,y) = (x+y)\sum_{k=0}^{n}\binom{n}{k}\mathfrak{F}_k(x,y)\,\mathfrak{F}_{n-k}(x,y). \tag{4.17}$$

Proof. Considering the derivative of the generating function of the polynomials $\mathfrak{F}_n(x,y)$ (4.1), we have

$$\begin{aligned}\sum_{n\geq 0}\mathfrak{F}_{n+1}(x,y)\frac{t^n}{n!} &= \frac{xe^{ty}}{\left(1-\frac{x}{y}(e^{ty}-1)\right)^2}\\ &= \left(\frac{x+y}{1-\frac{x}{y}(e^{ty}-1)} - y\right)\frac{1}{1-\frac{x}{y}(e^{ty}-1)}\\ &= (x+y)\sum_{n\geq 0}\mathfrak{F}_n(x,y)\frac{t^n}{n!}\sum_{n\geq 0}\mathfrak{F}_n(x,y)\frac{t^n}{n!} - y\sum_{n\geq 0}\mathfrak{F}_n(x,y)\frac{t^n}{n!}\\ &= (x+y)\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}\mathfrak{F}_k(x,y)\,\mathfrak{F}_{n-k}(x,y) - y\mathfrak{F}_n(x,y)\right)\frac{t^n}{n!}.\end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$, and after some rearrangements, we obtain the result. □

For $y = 1$, we get the result of the Theorem 1 in [8].

Theorem 4.11. *For $n \geq 0$ and for x_1 different to x_2, we have*

$$\sum_{k=0}^{n}\binom{n}{k}\mathfrak{F}_k(x_1,y)\,\mathfrak{F}_{n-k}(x_2,y) = \frac{x_2\mathfrak{F}_n(x_2,y) - x_1\mathfrak{F}_n(x_1,y)}{x_2-x_1}. \tag{4.18}$$

Proof. The proof of (4.18) becomes as follows

$$\begin{aligned}\frac{1}{1-\frac{x_1}{y}(e^{ty}-1)}\frac{1}{1-\frac{x_2}{y}(e^{ty}-1)} &= \frac{x_2}{x_2-x_1}\frac{1}{1-\frac{x_2}{y}(e^{ty}-1)}\\ &\quad - \frac{x_1}{x_2-x_1}\frac{1}{1-\frac{x_1}{y}(e^{ty}-1)}.\end{aligned}$$

□

Now, in this part of the paper, we will connect the polynomials $\mathfrak{F}_n(x,y)$ with Eulerian polynomials and Frobenius-Euler polynomials. It is known that for $x \neq 1$ and $n \geq 0$, the Eulerian polynomials $A_n(x)$ and the Frobenius-Euler polynomials $H_n(x;y)$ are defined respectively by the following generating functions [12, 13]

$$\frac{1-x}{e^{t(x-1)}-x} = \sum_{n\geq 0}A_n(x)\frac{t^n}{n!}, \tag{4.19}$$

$$\frac{1-x}{e^t-x}e^{ty} = \sum_{n\geq 0}H_n(x;y)\frac{t^n}{n!}. \tag{4.20}$$

Theorem 4.12. *For $n \geq 0$, and for nonzero complex numbers x and y,we have*

$$\mathfrak{F}_n(x,y) = x^n A_n\left(1+\frac{y}{x}\right) \tag{4.21}$$

and for $t \neq 1$, we have

$$A_n(t) = \left(\frac{t-1}{y}\right)^n \mathfrak{F}_n\left(\frac{y}{t-1}, y\right) = \left(\frac{1}{x}\right)^n \mathfrak{F}_n(x, x(t-1)). \tag{4.22}$$

Proof. The generating functions (4.1) and (4.19) can be rewritten as

$$\frac{1}{e^{ty}-(1+\frac{y}{x})}=-\frac{x}{y}\sum_{n\geq 0}\mathfrak{F}_n\left(x,y\right)\frac{t^n}{n!} \tag{4.23}$$

and for $x\neq 1$

$$\frac{1}{e^{t}-x}=-\sum_{n\geq 0}\frac{A_n\left(x\right)}{(x-1)^{n+1}}\frac{t^n}{n!}. \tag{4.24}$$

Then,

$$\begin{aligned}\frac{x}{y}\mathfrak{F}_n\left(x,y\right)&=y^n\frac{A_n\left(1+\frac{y}{x}\right)}{(\frac{y}{x})^{n+1}}.\\&=x^n\frac{A_n\left(1+\frac{y}{x}\right)}{(\frac{y}{x})}.\end{aligned}$$

Which is equivalent to (4.21).

Now, for $t=1+\frac{y}{x}$ in (4.21), we obtain (4.22).

□

Theorem 4.13. *For $n\geq 0$, we have*

$$\mathfrak{F}_n\left(x,y\right)=y^n\sum_{k=0}^{n}y^k\binom{n}{k}(-1)^kH_{n-k}\left(1+\frac{y}{x};y\right).$$

Proof. From the generating functions (4.1) and (4.20), we have,

$$\begin{aligned}\sum_{n\geq 0}\mathfrak{F}_n\left(x,y\right)\frac{t^n}{n!}&=e^{-ty^2}\frac{(1-(1+\frac{y}{x}))}{e^{ty}-(1+\frac{y}{x})}e^{ty^2}\\&=e^{-ty^2}\sum_{n\geq 0}H_n\left(1+\frac{x}{y};y\right)\frac{(ty)^n}{n!}.\end{aligned}$$

In the same way as the proof of Theorem 4.6. we get the result. □

5. Probabilistic representation

We consider a geometric distributed random variable X. The probability density function, for $k\in\mathbb{N}^*$ and two parameters p and q, such that $q=1-p$, as follows:

$$P(X=k)=pq^{k-1}.$$

The higher moment of X is given by

$$E(X^n)=\sum_{k\geq 1}k^np(1-p)^{k-1}. \tag{5.1}$$

In the next paragraph, we show that $\mathfrak{F}_n\left(x,y\right)$ can be viewed as the nth moment of a random variable $X-1$ where X follows the geometric law.

Theorem 5.1. *Let X be a random variable follows the geometric law and for $p=\frac{y}{x+y}>0$, we have*

$$\begin{aligned}\mathfrak{F}_n\left(x,y\right)=&\frac{y}{x+y}\sum_{k\geq 0}\left(\frac{x}{x+y}\right)^k(yk)^n &(5.2)\\=&y^nE((X-1)^n). &(5.3)\end{aligned}$$

Proof. From (4.1), we have

$$\begin{aligned}\sum_{n\geq 0}\mathfrak{F}_n(x,y)\frac{t^n}{n!} &= \frac{y}{x+y}\frac{1}{\left(1-\frac{x}{x+y}e^{ty}\right)}\\ &= \frac{y}{x+y}\sum_{k\geq 0}\left(\frac{x}{x+y}\right)^k\left(e^{ty}\right)^k\\ &= \frac{y}{x+y}\sum_{k\geq 0}\left(\frac{x}{x+y}\right)^k\sum_{n\geq 0}(ky)^n\frac{t^n}{n!}\\ &= \frac{y}{x+y}\sum_{n\geq 0}\left(\sum_{k\geq 0}\left(\frac{x}{x+y}\right)^k(ky)^n\right)\frac{t^n}{n!}.\end{aligned}$$

Equating $\frac{t^n}{n!}$ and by comparing with (5.1), we obtain the result. □

6. Degenerate generalized Fubini polynomials

For any nonzero real number λ, we define the degenerate generalized Fubini polynomials as

$$\frac{1}{1-\frac{x}{y}((1+\lambda ty)^{\frac{1}{\lambda}}-1)} = \sum_{n\geq 0}\mathfrak{F}_{n,\lambda}(x,y)\frac{t^n}{n!}. \tag{6.1}$$

It is clear that $\lim_{\lambda\to 0}(1+\lambda ty)^{\frac{1}{\lambda}} = e^{ty}$ and therefore $\lim_{\lambda\to 0}\mathfrak{F}_{n,\lambda}(x,y) = \mathfrak{F}_n(x,y)$.

Now, recall that the degenerate Stirling numbers of the second kind $\left\{{n \atop k}\right\}_\lambda$, are defined by the following generating function [4]

$$\frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^k = \sum_{n\geq k}\left\{{n \atop k}\right\}_\lambda\frac{t^n}{n!}. \tag{6.2}$$

In the next result, we will give the explicit formula for $\mathfrak{F}_{n,\lambda}(x,y)$.

Theorem 6.1. *For $n\geq 0$, we have*

$$\mathfrak{F}_{n,\lambda}(x,y) = \sum_{k=0}^{n}\left\{{n \atop k}\right\}_\lambda k!x^k y^{n-k}. \tag{6.3}$$

Proof. From (6.1), we note that

$$\begin{aligned}\sum_{n\geq 0}\mathfrak{F}_{n,\lambda}(x,y)\frac{t^n}{n!} &= \frac{1}{1-\frac{x}{y}((1+\lambda ty)^{\frac{1}{\lambda}}-1)}\\ &= \sum_{k\geq 0}\left(\frac{x}{y}\right)^k\left((1+\lambda ty)^{\frac{1}{\lambda}}-1\right)^k\\ &= \sum_{k\geq 0}\left(\frac{x}{y}\right)^k k!\sum_{n\geq k}\left\{{n \atop k}\right\}_\lambda\frac{(ty)^n}{n!}\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\left\{{n \atop k}\right\}_\lambda k!x^k y^{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{6.4}$$

Equating $\frac{t^n}{n!}$, we obtain the result. □

Remark 6.2. Now, by setting $y = 1$ in (6.1), we get

$$\begin{aligned}\frac{1}{1 - x((1+\lambda t)^{\frac{1}{\lambda}} - 1)} &= \sum_{n\geq 0} \mathfrak{F}_{n,\lambda}(x,1)\frac{t^n}{n!} \\ &= \sum_{n\geq 0} \omega_{n,\lambda}(x)\frac{t^n}{n!},\end{aligned}$$

where $\omega_{n,\lambda}(x)$ denotes the degenerate Fubini polynomials [10], defined by

$$\omega_{n,\lambda}(x) := \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda} k! x^k.$$

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