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CERTAIN MATRIX TRANSFORMATIONS INTO ALMOST BOUNDED SEQUENCE SPACE

İhsan SOLAK
Department of Mathematics İnönü University, 44069, Malatya-TURKEY

ABSTRACT

The concept of almost boundedness was introduced and discussed by Nanda, [3]. The object of this paper is to obtain necessary and sufficient conditions to characterize the matrices of the classes $(m(p):\widehat{ms})$, $(ms(p):\widehat{m})$ and $(ms(p):\widehat{ms})$. Those sequence spaces are described below.

HEMEN HEMEN SINIRLI DİZİ UZAYI İÇİNE BAZI MATRİS TRANSFORMASYONLAR

ÖZET

Hemen hemen sınırlılık kavramı Nanda tarafından tanıtıldı ve tartışıldı, [3]. Bu çalışmanın amacı $(m(p):\widehat{ms})$, $(ms(p):\widehat{m})$ ve $(ms(p):\widehat{ms})$ matris sınıflarını karakterize etmek için gerek ve yeter şartları elde etmektir. Bu dizi uzayları aşağıda belirtilmiştir.

1. INTRODUCTION, DEFINITIONS AND NOTATIONS

In this paper, by m and ms we respectively denote the linear spaces of all real bounded sequences and series. The shift operator D is defined on m by

$Dx=(x_k)^{\infty}{}_{k=1}$, $D^2x=(x_k)^{\infty}{}_{k=2}$ and so on.

The following sequence space was defined by Simons [5] and Maddox,[2];

$$m(p)=\{x=(x_k):\sup_k |x_k|^{p_k}<\infty\},$$

where $p=(p_k)$ denotes a sequence of strictly positive numbers such that $\sup_k p_k<\infty$ (This assumption is made throught). The boundedness of (p_k) is not necessary, in general, but it is sufficient for the space m(p) to be linear.

Υ, the space of entire sequences introduced by Ganapathy Iyer, [1] and its dual Υ^* become particular cases respectively of $c_0(p)$ and m(p), the generalized sequence spaces introduced by Maddox, [2];

$$\Upsilon=\{x=(x_k):|x_k|^{1/k}\rightarrow 0 \text{ as } k\rightarrow\infty\},$$

$$\Upsilon^*=\{x=(x_k):\sup_k |x_k|^{1/k}<\infty\}.$$

The space $\hat{m}(p)$ of almost bounded sequences was introduced by Nanda, [3];

$$\hat{m}(p)=\{x=(x_k):\sup_{m,n}|t_{mn}(x)|^{p_m}<\infty\}.$$

where

$$t_{mn}(x)=\frac{1}{m+1}\sum_{i=0}^{m} D^i x_n, \ (D^0=1).$$

It is proved by Nanda [3] that $m\subset\hat{m}$.

It is natural to expect that the space ms can be extended to ms (p) just as m was extended to m(p). Then we define ms (p) as follows;

$$ms(p)=\{x=(x_k):(\sum_{n=0}^{k} x_n)\varepsilon m(p)\}.$$

By using similar argument, we also define

$$\hat{m}s(p)=\{x=(x_k):(\sum_{n=0}^{k} x_n)\varepsilon\hat{m}(p)\}.$$

When $p_k=p$ for all k, we have $m(p)=m$, $\hat{m}(p)=\hat{m}$, $ms(p)=ms$ and $\hat{m}s(p)=\hat{m}s$. By $\hat{m}s$, we denote the space of almost bounded series. If $p_k=1/k$ for all k, we also have $m(p)=\Upsilon^*$ and $ms(p)=\Upsilon^*s$, where

$$\Upsilon^*s=\{x=(x_k):(\sum_{n=0}^{k} x_n)\varepsilon\Upsilon^*\}.$$

Let $A=(a_{nk})$ be an infinite matrix of real numbers a_{nk} $(n,k=0,1,\ldots..)$ and λ,μ two non-empty subsets of the space s of real sequences. We say that the matrix A defines a transformation from λ, into μ, if for every sequence $x=(x_k)\ \varepsilon\ \lambda$, the sequence $Ax=((Ax)_n)$-which is called the A-transform of the sequence x-exists and is in μ, where $(Ax)_n=\sum_k a_{nk}x_k$. For simplicity in notations, here and after we write $\sum_k$ instead of $\sum_{k=0}^{\infty}$. By (λ,μ) we denote the class of all such matrices.

Throught the paper, we shall use the notation a(n,k) instead of a_{nk} and by s=(s_k) we donete the sequence of partial sums of the series Σx_n. Thus, it is clear that sεm (or $\hat{m}$) whenever xεms (or $\widehat{ms}$).

2. WE ESTABLISH THE FOLLOWING THEOREMS

In this section, we examine the classes (m(p):$\widehat{ms}$), (ms(p):$\hat{m}$) and (ms(p):$\widehat{ms}$). We start with the following lemma due to Nanda [3] which requires in the proof of Theorem 2.1.

Lemma. Aε(m(p):m) if and only if, for every integer N>1

$$\sup_{n,m} \sum_k |a(n,k,m)| N^{1/p_k} < \infty \tag{2.1}$$

where

$$a(n,k,m)=\frac{1}{m+1}\sum_{i=0}^{m} a(n+i,k).$$

we now have,

Theorem 2.1. Aε(m(p):$\widehat{ms}$) if and only if, for every integer N>1

$$\sup_{n,m}\sum_k |\sum_{j=0}^{n} a(j,k,m)| N^{1/p_k} < \infty \tag{2.2}$$

Proof. Let xεm(p). Consider the following equality obtained from the n,q^{th} partial sums of $(Ax)_j$;

$$\sum_{j=0}^{n}\sum_{k=0}^{q} a(j,k)x_k = \sum_{k=0}^{q}(\sum_{j=0}^{n} a(j,k))x_k \quad ; q,n=0,1,\ldots \tag{2.3}$$

which yields by letting $q \to \infty$ that

$$\sum_{j=0}^{n}\sum_{k} a(j,k)x_k = \sum_{k}(\sum_{j=0}^{n} a(j,k))x_k \quad ; n=0,1,\ldots \tag{2.4}$$

Thus, it is seen in (2,4) that B = (b(n, k)) ε (m(p):$\hat{m}$) if and only if Aε(m(p):$\widehat{ms}$), where $b(n,k)=\sum_{j=0}^{n} a(j,k)$. This completes the proof.

By Theorem 2.1, we have

Corollary 2.2. $A \varepsilon (\Upsilon^* : \widehat{ms})$ if and only if the condition (2.2) holds with $p_k = 1/k$ for all k.

Theorem 2.3. $A \varepsilon (ms(p):\widehat{m})$ if and only if for every integer N>1

$$\sup_{n,m} \sum_k |\Delta a(n,k,m)| N^{1/p_k} < \infty \tag{2.5}$$

$$\lim_k a(n,k) = 0 \text{ for each } n, \tag{2,6}$$

where $\Delta a(n,k,m)=a(n,k,m)-a(n,k+1,m)$.

Proof. Necessity. Let $A\varepsilon(ms(p):\widehat{m})$ and $x\varepsilon ms(p)$. To show the necessity of (2.6), we assume that (2.6) is not satisfied for some n and obtain a contradiction as in Theorem 2.1 of Öztürk, [4]. Indeed, under this assumption we can find some $x\varepsilon ms(p)$ such that Ax does not belong to $\widehat{m}$. For example, if we choose $x=((-1)^n)\varepsilon ms(p)$, then $(Ax)_n = \sum_k a(n,k)(-1)^k$. However, that series $\sum_k a(n,k)(-1)^k$ does not converge for each n. That is to say that A-transform of the series $\Sigma(-1)^n$, which belongs to ms(p), does not even exists. But this contradicts the fact that $A\varepsilon(ms(p):\widehat{m})$. Hence (2.6) is necessary.

Let us consider the equality

$$\sum_{k=0}^{m} a(n,k)x_k = \sum_{k=0}^{m-1} \Delta a(n,k)s_k + a(n,m)s_m \ ; \ m,n=0,1,\ldots \tag{2.7}$$

obtained by applying the Abel's partial summation on the m^{th} partial sums of Ax. From (2.6), it is obtained by leting $m \to \infty$ in (2.7) that

$$\sum_k a(n,k)x_k = \sum_k \Delta a(n,k)s_k \ ; \ n=0,1,\ldots \tag{2.8}$$

Thus, it is seen that $C=(c(n,k))\varepsilon(m(p):\widehat{m})$ satisfies (2.1) which is equivalent to (2.5), where $c(n,k)= \Delta a(n,k)$ for all n and k.

Sufficieny. Suppose that the conditions hold and $x\varepsilon ms(p)$. Now, reconsider $C=(\Delta a(n,k))$ in (2.8). Therefore "C satisfies (2.1) if and only if A satisfies (2.5)" is true . So, $C\varepsilon(m(p):\widehat{m})$. This implies by (2.8) that $A\varepsilon(ms(p):\widehat{m})$ and the proof of Theorem is completed.

By Theorem 2.3, we have

Corollary 2.4. $A\varepsilon(\Upsilon^{*}s:\hat{m})$ if and only if (2.6) holds, (2.5) also holds with $p_k=1/k$ for all k.

Theorem 2.5. $A\varepsilon(ms(p):\hat{m}s)$ if and only if (2.6) holds and

$$\mathrm{Sup}_{m,n} \sum_k \left| \sum_{j=0}^{n} \Delta a(j,k,m) \right| N^{1/p_k} < \infty \tag{2.9}$$

for every integer N>1.

Proof. Let $A\varepsilon(ms(p):\hat{m}s)$ and $x\varepsilon(ms(p)$. Since $(ms(p):\hat{m}s) \subset (ms(p):\hat{m})$, the necessity of (2.6) is obvious by Theorem 2.3.

Now, consider the equality which is obtained in a similar way of (2.7);

$$\sum_{j=0}^{n} \sum_{k=0}^{m} a(j,k)x_k = \sum_{k=0}^{m-1} \sum_{j=0}^{n} \Delta a(j,k)s_k + \sum_{j=0}^{n} a(j,m)s_m \; ; m,n=0,1,\ldots \tag{2.10}$$

Therefore we get by considering (2.6) and letting $m \to \infty$ in (2.10) that

$$\sum_{j=0}^{n} \sum_k a(j,k)x_k = \sum_k \left(\sum_{j=0}^{n} \Delta a(j,k)s_k \right. \; ; \; n=0,1,\ldots \tag{2.11}$$

Thus, it is seen that $B = (b(n,k)) \; \varepsilon \; (m(p):\hat{m})$ satisfies (2.1) which is equivelant to (2.9), where $b(n,k)= \sum_{j=0}^{n} \Delta a(j,k)$ for all n and k.

The sufficiency is trivial.

Finally, we have

Corollary 2.6. $A \; \varepsilon \; (\Upsilon^{*} \; s: \; \hat{m}s)$ if and only if (2.6) holds, (2.9) also holds with $p_k=1/k$ for all k.

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