Erciyes Üniversitesi
Fen Bilimleri Dergisi
2, 371-377, (1986)

AYRIKLAŞTIRMA YÖNTEMLERİNİN SALINIM VE KARŞILAŞTIRMA TEOREMLERİNDE KULLANIMI ÜZERİNE

Haydar AKÇA

Erciyes Üniversitesi Fen-Edebiyat Fakültesi, Kayseri

ÖZET

Bu çalışmada. ikinci basamaktan lineer diferansiyel denklemler için bilinen salınım ve karşılaştırma teoremlerine ilişkin sonuçlar fark denklemleri için incelenmiştir.

NOTE ON OSCILLATION AND COMPARISON THEOREMS USING DISCRETE METHODS

SUMMARY

In this paper, some well knowing results in the continous case about the oscillation and comparison theorems on second order linear differential equations, have been criticized and try to answer the question when does difference equation have oscillatory properties.

1. INTRODUCTION

The literature on the oscillation of solutions of second-order linear differential equations is voluminous. The purpose of this study is to extend the results of various authors on the second-order ordinary differential equations to the form of difference equations. Consider the second-order non-homogeneous linear differential equation.

$$y'' + p(x)\, y = f(x) \tag{1}$$

We shall suppose that p(x) and f(x) are of classe C^2 on an interval

$I : [a, \infty)\ , \quad a \geqslant 0\ .$

With equation (1) we associate the corresponding homogeneous equation

$$y'' + p(x)\, y = 0 \tag{2}$$

Definition 1 : A solution $y(x)$ of (1) or (2) is said to be oscillatory on $I = [a, \infty)$ if it has an infinite number of zeros on $[x, \infty)$ for every $x \geqslant a$. The equation (1) or (2) will be said to be oscillatory, if one (and therefore all) of its solutions have an infinite number of zeros for $x \geqslant a$.

Definition 2 : A solution $y(x)$ of (1) or (2) is said to be non-oscillatory on I if it has only a finite number of zeros on I for some $x \geqslant a$.

Definition 3 : A solution $y(x)$ of (2) is said to be quickly oscillatory on I if it is oscillatory and the sequence of zeros $\{x_n\}$ is such that $\lim_{n \to \infty} (x_{n+1} - x_n) = 0$.

Definition 4 : A solution $y(x)$ of (1) or (2) is said to be aZ-type solution on I if it has arbitrarily large zeros but is ultimately non-negative or non-positive.

Definition 5 : A solution $y(x)$ of (1) or (2) is said to be disconjugate if it has at most one zero on I.

Definition 6 : A point $x = b$ is colled conjugate to $x = a$ $(b \neq a)$, if there exists a non-null solution of the differential equation that vanishes at both these points.

Many of the properties associated with second-order linear differential equation usually also holds for the difference equation form. An essential property of the majority of computational methods for the solution of differential equations is that of discretization. Thus we seek an appnox-imate solution, not on the continous interval $a \leqslant b \leqslant b$, but on the discrete point set

$$\{x_n \mid n = 0, 1, 2, \ldots, (b - a)/h\}$$

On the set of points, $y_n \approx y(x_n)$, $x_{n+1} = x_n + h$, $x_0 = a$, $x_n = b$, $(n = 0, 1, 2, \ldots, N)$, $h > 0$ is the stepsize and N is sufficiently large integer. Generally speaking a discrete variable method for solving a differential equation consists in an algorithm which corresponding to each lattice point x_n, furnishes a number y_n which is to be regarded as an approximation to the value $y(x_n)$ of the exact solution at the point x_n. Frequently though not always, the points x_n are equidistant.

Let us write equations (1) and (2) in the form of difference equations using the linear multistip method.

$$y_{n+1} + (h^2 p_n - 2)\, y_n + y_{n-1} = h^2 f_n \qquad (3)$$

$$y_{n+1} + (h^2 p_n - 2)\, y_n + y_{n-1} = 0 \qquad (4)$$

respectively where, $p_n \approx p(x_n)$ and $f_n \approx f(x_n)$ on the interval $I : (0 < x_0 < x_1 < x_2 \ldots < x_n < \ldots < x_N)$ such that N is a sufficiently large positive integer.

2. MAIN RESULTS

Now we study the oscillatory behaviour of solutions y_n and u_n of equation (3) and (4) respectively.

THEOREM 1. Assume that the sign of f_n is constant and u_n is solution of equation (4) such that

$$u_0 > 0,\ u_1 < 0,\ \ldots,\ u_n < 0,\ u_{n+1} > 0,\ \ldots \text{ on } I : [\, a, \infty\,)$$

(or vice versa) on the interval I. The sign of solution y_n of equation (3) cannot be constant.

PROOF. Without loss of generality suppose that the sign of solution y_n of equation (3) is positive on I. Multiplying the equation (3) by u_n and equation (4) by $- y_n$ and adding these two equations we have

$$u_n y_{n+1} - u_{n+1} y_n + u_n y_{n-1} - y_n u_{n-1} = h^2 f_n u_n \qquad (5)$$

If the partial sum of the left hand member of (5) is performed.

$$u_1y_2 - u_2y_1 + u_1y_o - u_oy_1$$

$$u_2y_3 - u_3y_2 + u_2y_1 - u_1y_2$$

$$\vdots$$

$$u_{n-1}y_n - u_ny_{n-1} + u_{n-1}y_{n-2} - u_{n-2}y_{n-1}$$

$$u_ny_{n+1} - u_{n+1}y_n + u_ny_{n-1} - u_{n-1}y_n$$

and finally we obtain

$$u_ny_{n+1} - u_{n+1}y_n + u_1y_o - u_oy_1 = \sum_{k=1}^{n} h^2 u_kf_k \qquad (6)$$

While the sign of right hand side of equation (6) will change according to the sign of u_k, the sign of left hand side will remain constant. This contradiction concludes the proof of the theorem.

THEOREM 2. Consider the equation (1) where $p(x) \in C\,[a, \infty)$, $f(x) \in C^2\,[a, \infty)$ and $f(x) \neq 0$ on the interval I. If $f(x)$ is a oscillatory type solution of equation (2) then the solution $y(x)$ of equation (1) is oscillatory.

PROOF. Suppose that the solution of $y(x)$ equation (1) is not oscillatory and solution of $f(x)$ of equation (2) such that

$$f_o > 0,\ f_1 < 0,\ \ldots,\ f_n < 0,\ f_{n+1} > 0,\ \ldots.$$

on the interval $I : [a, \infty)$. Thus

$$f'' + p(x)\, f = 0 \qquad (7)$$

After discretization of the equations (1) and (7) following results can be obtain.

$$f_{n+1} + (h^2 p_n - 2)\, f_n + f_{n-1} = 0$$

$$y_{n+1} + (h^2 p_n - 2)\, y_n + y_{n-1} = h^2 f_n$$

The earlier procedure is readily adapted to the present status and yields the following equation.

$$y_n f_{n+1} - y_{n+1} f_n + y_1 f_o - y_o f_1 = \sum_{k=1}^{n} h^2 f_k^2 \qquad (8)$$

The proof is obvious because of the appearing contradiction.

THEOREM 3. Consider the pair of linear differential equations on I : [a, ∞).

$$y'' + p(x)\, y = f(x) \qquad (1)$$

$$z'' + p(x)\, z = g(x) \qquad (1a)$$

where $p(x)$ is of C-class on I, $f(x) < g(x)$ (or vice versa) are not constant on a common subinterval and their sign are constant on I. If the solution $y(x)$ of equation (1) oscillatory then the solution $z(x)$ of equation (1a) will be so.

PROOF. Using the linear multistep method

$$y_{n+1} + (h^2 p_n - 2)\, y_n + y_{n-1} = h^2 f_n \qquad (3)$$

$$z_{n+1} + (h^2 p_n - 2)\, z_n + z_{n-1} = h^2 g_n \qquad (3a)$$

Assume that y_n is solution of equation (3) such that

$$y_o > 0,\ y_1 < 0,\ \ldots,\ y_n < 0,\ y_{n+1} > 0, \ldots \text{ and } f_n < g_n$$

(or vice versa) and the sign of z_n is constant on the interval I. Using the well-known procedure one can easily obtain the following equation.

$$z_n y_{n+1} - z_{n+1} y_n + z_n y_{n-1} - z_{n-1} y_n = h^2 (z_n f_n - y_n g_n) \qquad (9)$$

Perform the partial sum of the left hand side of (9) we have

$$z_n y_{n+1} - z_{n+1} y_n + z_1 y_0 - z_0 y_1 = \sum_{k=1}^{n} h^2 (z_k f_k - y_k g_k) \qquad (10)$$

The proof follows from usual contradiction. This result leads us to the following theorem.

THEOREM 4. Consider the pair of equations on $I : [a, \infty)$.

$$y'' + p(x)\, y = f(x) \qquad (1)$$

$$z'' + q(x)\, z = g(x) \qquad (11)$$

where $p(x)$, $q(x)$ are of class C and assum that $q(x) > p(x)$ on I. $f(x)$ and $g(x)$ are of class C^1 and not constant on common subinterval and the sign of $f(x)$ and $g(x)$ are constants on the interval I.

If $y(x)$ be a oscillatory solution of (1) then a nonnull solution $z(x)$ of (11) will also be oscillatory on I.

PROOF. Repeating the same procedure we get

$$y_{n+1} + (h^2 p_n - 2)\, y_n + y_{n-1} = h^2 f_n \qquad (3)$$

$$z_{n+1} + (h^2 q_n - 2)\, z_n + z_{n-1} = h^2 g_n \qquad (12)$$

Without loss of generality assume that y_n is solution of (3) such that

$$y_0 > 0,\ y_1 < 0, \ldots,\ y_n < 0,\ y_{n+1} > 0, \ldots \text{ and } f_n < g_n$$

and the sign of solution z_n of (12) is constant on I With the similar procedure we can get

$$z_n y_{n+1} - z_{n+1} y_n + z_n y_{n-1} - z_{n-1} y_n = h^2 \left(y_n z_n (q_n - p_n) + (z_n f_n - g_n y_n)\right) \quad (13)$$

and finally

$$z_n y_{n+1} - z_{n+1} y_n + z_1 y_o - z_o y_1 = \sum_{k=1}^{n} h^2 \left(y_k z_k (q_k - p_k) + (z_k f_k - g_k y_k)\right) \quad (14)$$

The proof of Theorem 4 is analogous to Theorem 3.

REFERENCES

[1]. A. Skidmore and W. Leighton, On the differential equation $y'' + p(x)\, y = f(x)$, J. Math. Anal. Appl. 43 (1973), 46-55.

[2] . M.S. Keener, On the solution of certain non-homogeneous second-order differential equations, Applicable Analysis 1 (1971), 57-63.

[3] . A. A. S. Zaghrout and A.A. Ragab, Oscillatory behaviour solutions of $y'' + p(x)\, y = f(x)$, India J. pure appl. Math., 16 (1985), 853-858.

[4] . H. Akça, An Algorithm for the numerical solution of second order Euler type differential equations, Procedings of numerical analysis symposium, METU, Ankara, 1983, 58-63.

[5] . P. Henrici, Discrete variable Methods in Ordinary Differential Equations, Wiley, New York, 1962.