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ABSOLUTE CESÀRO SUMMABILITY FACTORS OF INFINITE SERIES

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ABSTRACT

In this study a theorem on $|C,1|_k$ summability factors, which generalizes a theorem of Mazhar [3] , has been proved.

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ÖZET

Bu çalışmada Mazhar [3] ın bir teoremini genelleştiren $|C,1|_k$ toplanabilme çarpanlarıyla ilgili bir teorem ispat edilmiştir.

1. INTRODUCTION

Let Σa_n be a given infinite series with partial sums s_n, and $r_n = na_n$. By u_n^α and t_n^α we denote the n-th Cesàro means of order α $(\alpha > -1)$ of the sequences (s_n) and (r_n), respectively. The series Σa_n is said to be summable $|C, \alpha|_k$, $k \geqslant 1$, if (see [1])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty \qquad (1)$$

Since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [2]), condition (1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty \qquad (2)$$

2. Mazhar [3] proved the following theorem.

THEOREM A. If

$$\sum_{v=1}^{n} |s_v| = O(n) \quad \text{as} \quad n \to \infty \qquad (3)$$

and (λ_n) is a bounded sequence such that

$$\sum_{n=1}^{m} |\Delta\lambda_n| = O(1) \tag{4}$$

$$\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1), \text{ and} \tag{5}$$

$$\sum_{n=1}^{m} n \, |\Delta^2 \lambda_n| = O(1) \quad \text{as} \quad m \to \infty \tag{6}$$

then the series $\Sigma a_n \lambda_n$ is summable $|C,1|_1$.

3. Now we shall prove the following theorem.

THEOREM. Let $k \geqslant 1$. If

$$\sum_{v=1}^{n} |s_v|^k = O(n) \quad \text{as} \quad n \to \infty \tag{7}$$

and (λ_n) is a bounded sequence such that satsify the conditions (4)-(6) of the Theorem A, then the series $\Sigma a_n\lambda_n$ is summable $|C,1|_k$.

It should be noted that if we take $k = 1$ in our theorem, then we get Theorem A .

4. PROOF OF THE THEOREM . Let T_n be the n-th Cesâro mean of ored 1 of the sequence $(na_n \lambda_n)$. That is

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n} va_v \lambda_v \ . \tag{8}$$

Now, applying Abel's transformation to the sum (8), we have that

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n-1} v\, s_v\, \Delta \lambda_v - \frac{1}{n+1} \sum_{v=1}^{n-1} s_v\, \lambda_{v+1} + \frac{n s_n \lambda_n}{n+1}$$

$$= T_{n,1} + T_{n,2} + T_{n,3}, \quad \text{say.}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}|^k < \infty \quad , \quad \text{for} \quad r = 1,\ 2,\ 3, \quad \text{by (2)} .$$

Now, applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} \frac{1}{n} | T_{n,1} |^k \leqslant \sum_{n=2}^{m+1} \frac{1}{n^k + 1} \left\{ \sum_{v=1}^{n-1} v\, |s_v|\, |\Delta \lambda_v| \right\}^k$$

$$\leqslant \sum_{n=2}^{m+1} \frac{1}{n^2} \sum_{v=1}^{n-1} v\, |s_v|^k\, |\Delta\lambda_v| \times \left\{ \frac{1}{n} \sum_{v=1}^{n-1} v\, |\Delta \lambda_v| \right\}^{k-1}$$

Since

$$\sum_{v=1}^{n} v\, |\Delta \lambda_v| \leqslant n \sum_{v=1}^{n} |\Delta\lambda_v| \Longrightarrow \frac{1}{n} \sum_{v=1}^{n} v\, |\Delta\lambda_v| \leqslant \sum_{v=1}^{n} |\Delta \lambda_v| = O(1)$$

as $n \to \infty$, by (4), we have that

$$\frac{1}{n} \sum_{v=1}^{n} v\, |\Delta \lambda_v| = O(1) \quad .$$

Hence

$$\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}|^k = O(1) \sum_{v=1}^{m} v |s_v|^k |\Delta\lambda_v| \sum_{n=v+1}^{m+1} \frac{1}{n^2}$$

$$= O(1) \sum_{v=1}^{m} |s_v|^k |\Delta\lambda_v| = O(1) \sum_{v=1}^{m-1} \Delta(|\Delta \lambda_v|) \sum_{r=1}^{v} |s_r|^k$$

$$+ O(1) |\Delta\lambda_m| \sum_{v=1}^{m} |s_v|^k = O(1) \sum_{v=1}^{m-1} v |\Delta^2 \lambda_v| + O(1)\, m\, |\Delta \lambda_m| = O(1)$$

as $m \to \infty$, by (6) and (7) .

Again

$$\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,2}|^k \leq \sum_{n=2}^{m+1} \frac{1}{n^2} \sum_{v=1}^{n-1} |s_v|^k |\lambda_{v+1}|^k \times \left\{\frac{1}{n} \sum_{v=1}^{n-1} 1\right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \frac{|\lambda_v|}{v} |s_v|^k = O(1) \sum_{v=1}^{m-1} |\Delta \lambda_v| + O(1) \sum_{v=1}^{m-1} \frac{|\lambda_{v+1}|}{v}$$

$$+ O(1) |\lambda_m| = O(1) \quad \text{as } m \to \infty$$, by the hypotheses.

Finally, we have

$$\sum_{n=1}^{m} \frac{1}{n} |T_{n,3}|^k = O(1) \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n| |s_n|^k \cdot \frac{1}{n}$$

$$= O(1) \sum_{n=1}^{m} \frac{|\lambda_n|}{n} |s_n|^k .$$

Thus as in $T_{n,2}$ we have that

$$\sum_{n=1}^{m} \frac{1}{n} |T_{n,3}|^k = O(1) \sum_{n=1}^{m} \frac{|\lambda_n|}{n} |s_n|^k = O(1) \quad \text{as } m \to \infty .$$

Therefore, we get

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| T_{n,r} \right|^k < \infty \quad , \text{ for } \quad r = 1, 2, 3.$$

This completes the proof of the theorem.

REFERENCES

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