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The Reflections On Hyperboloidal Model of Hyperbolic Plane H^2

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ABSTRACT

The purpose of the present paper is to define two types of reflections on H^2 and to obtain some geometric results .

ÖZET

Bu dcalışmanın amacı,[13] de verilen H-doğru boyunca, yansıma tanımlamak ve bu yansımanın bazı geometrik zelliklerini eldc etmektir.

1. Introduction. The high point of the prehistory of the model H^2 is Lambert's idea of an "imaginer sphere " in 1766. "Hyperbolic geometry" introduced by Klein in 1871.The parametrization of H^2 is used as Weiestrass coordinates by Killing [8], [9].Meanwhile,in 1881,that parametrization is named "hyperbolic coordinates "
by Poincarè [11]. The comformal disc model of hyperbolic space and the quadratic geometries on arbitrary quadratic surfaces in 3-space are developed by Poincarè[12].Some unpublished works of Poincarè in 1880 related to H^2 are discussed by Gans [3]. H^2 is studied by Minkowski [10], Sommer-

feld [14], Varičak [18] in special relativity without mentioning hyperboloidal model.In [6], hyperboloidal model is studied by Jansen.Recently,H^2 appeared in Reynold's paper [13]. Kinematics of H^2 is studied by Garnier [4], Frank [2], Tôlke [17] and the other many authors.

Tits has proved [15],[16] every finitely generated Coxeter groups can be represented as a group of projective maps generated by reflections and acting discretely in some domain of projective space.An algebraic description of all represantations of this form has been obtained by Vinberg [20], [21]. Among the non-Euclidean Coxeter groups hyperbolic ones has also been studied using projective sphere by Vinberg [22].

The purpose of the present paper is to define two types of reflections on H^2 and to obtain some geometric results . We can summarize some important properties as following:

(i)The product of two reflections along H^1 is a translation along H^1 through twice the H-distance between their reflecting lines.One clearly see that this is an anologue of well-known proposition " The product of two Euclidean reflections is a rotation through twice the angle between their reflecting lines " .

(ii)The product of two reflections along H^1 is a translation through the difference areas of two hyperbolic sectors which are one vertex origin and the other vertices on H^1.

(iii)In H^2,the order of subgroups generated by two reflections along H^1 is infinite.

(iv)Reflection along any H-line is determined uniquely by reflection along H^1

2. Minkowski 3-Space and Hyperboloidal Model In this section, we give some definitions and the lemma due to [13].

Let V be real vector spaces with three dimensional and q be the real valued function on it such that

$$q : V \rightarrow R,$$

$$q(x) = q(\sum_{i=0}^{2} x_i e_i)$$

$$= -x_0^2 + x_1^2 + x_2^2$$

for some basis $\{e_0, e_1, e_2\}$ of V. Then, the Minkowski 3-space is denoted by $M^3 = (V, q)$.

The set $H^2 = \{x \in M^3 \mid q(x) = -1, x_0 > 0\}$ is called hyperboloidal model of hyperbolic plane.The elements of H^2 are called H-points when considered in H^2 .We consider them as either points or vectors when considered in M^3. The H-lines are defined being all nonempty intersections of H^2 with two dimensional subspaces of M^3 .Each pair of distincit H-points A and B lie on a unique H-line,namely the intrsection of H^2 with the plane OAB of M^3 two distinct H-lines have one or zero H-points in common according to the line of intersection of the planes of M^3 in which they lie intersects H^2 or not.The complement of each H-line in H^2 consist of two H-half plane which are called its sides. From now on,we get $M^2 = sp\{e_0, e_1\}$, $H^1 = M^2 \cap H^2$.Let A,B be two elements of H^1 such that $A = x_0e_0 + x_1e_1, B = y_0e_0 + y_1e_1, x_1 < y_1$ and F be one to one smooth mapping of the intervale $x_1 \leq t \leq y_1$, then the distance A and B is defined by

$$d_{H^1}(A, B) = \int_{x_1}^{y_1} \sqrt{q(F\acute{(}t))}dt$$

$$= arcshy_1 - arcshx_1$$

The parametrization of H^1 by H-length is

$$h(s) = \cosh s\ e_0 + \sinh s\ e_1, -\infty < s < +\infty \tag{1}$$

The isometries of M^3 is expressed, in terms of matrices, $O_1(3)$. We denote G, G_1, G_0 the subgroups of $O_1(3)$ such that fix H^2, H^1 and e_0 respectively.Then we see that

$$g = L_s E_1^i E_2^j, 0 \leq i, j \leq 1, g \in G_1 \tag{2}$$

and

$$g = R_\theta E_2^j, 0 \leq j \leq 1, g \in G_0 \tag{3}$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$L_s = \begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

By (2) and (3),we have $G_1 \cap G_0 = \langle E_1, E_2 \rangle$.

If the matrix represantation of the parametrization (1) is $\tilde{h}(r)$, then

$$R_\theta \tilde{h}(r) = \tilde{H}(r, \theta)$$

and

$$H(s, \theta) = \cosh s\ e_0 + \sinh s cos\theta\ e_1 + \sinh s \sin\theta\ e_2, -\infty < s < +\infty \tag{4}$$

(4) is the parametrization of H^2 and

$$L_r\tilde{H}(s,0) = \tilde{H}(s+r,0). \tag{5}$$

The matrix L_r is called the matrix of H-translation along H^1 [1],[13].Since the matrix L_r plays a role of Euclidean rotation in the motions of M^2,it is also called Minkowskian rotation [19]. Since $R_\theta\tilde{H}(s,\phi) = \tilde{H}(s,\phi+\theta)$,$R_\theta$ is considered as a matrix of rotation about e_0. The H-distance between A and B in H^2 is given by

$$d_H(A,B) = arcosh(-p(A,B)) \tag{6}$$

where

$$p(x,y) = \frac{1}{2}[q(x+y) - q(x) - q(y)]$$

The H- angle $< BAC$ is $\vec{AB} \cup \vec{AC}$ where $\vec{AB} \neq \vec{AC}$.Its measure $m(< BAC)$ is defined as follows:Let V,W be the vectors in M^3 tangent to $\overline{AB},\overline{AC}$ respectively at A such that q(V)=q(W)=1, then

$$m(< BAC) = \arccos(p(V,W)). \tag{7}$$

LEMMA 2.1 Let $l_j\vec{AB} \neq \vec{AC}$ be H-line, $A_j\vec{B}_j$ a ray on $l_j, 1 \leq j \leq 2$ and S_j a side of l_j .Then there exists exactly one $T \in G$ such that $T(A_1) = A_2, T(l_1) = l_2, T(A_1\vec{B}_1) = A_2\vec{B}_2$ and $T(S_1) = S_2$.

Proof.It follows from Theorem 8 [13]■

3.Reflections along H^1 Now,we define a reflection along H^1 .Since it is analogue of the half turn in Euclidean Kinematics [1], It may be called half translation along H^1 in H^2.

Let $t, \dot{t}, \ddot{t}$ be lines which are tangent to H^1 at $h(s_0)$ in (1) ,passing through O and parallel to t ,passing through x and parallel to t,respectively.Let ℓ be the line passing through O , $h(s_0)$ and $\mathcal{H}$ be the intersection point of $\ell, \ddot{t}$.If $\tilde{S}_{\dot{h(s_0)}}(x)$ is a point such that it has distance $\sqrt{q(h-x)}$ to x and on different side with x with respect to ℓ in $\mathcal{H}$ which is a plane determined by ℓ, x ,then

$$\tilde{S}_{\dot{h(s_0)}}(x) = x - 2p(x, \dot{h(s_0)})\dot{h(s_0)}$$

or

$$S_{\dot{h(s_0)}} = \begin{pmatrix} \cosh 2s_0 & -\sinh 2s_0 & 0 \\ \sinh 2s_0 & -\cosh 2s_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

The above matrix $S_{\dot{h(s_0)}}$ is called *reflection matrix along* H^1. From now on , we mean dot derivative of .

THEOREM 3.1 Let $S_{\dot{h(s)}}, S_{\dot{h(t)}}$ be two reflection along H^1 and $\lambda \in R, g \in O_1(3)$.Then

(i)$S_{\dot{h(s)}}{}^2 = I_3$,

(ii)$S_{\dot{h(s)}} S_{\dot{h(t)}} S_{\dot{h(s)}} = S_{S_{\dot{h(s)}}\dot{h(t)}}$,

(iii)$det S_{\dot{h(s)}} = -1$,

(iv)$S_{\lambda \dot{h(s)}} = S_{\dot{h(s)}}$,

(v)$S_{\dot{h(s)}} \in G_1$,

(vi)$g S_{\dot{h(s)}} g^-1 = S_{g\dot{h(s)}}$, (**vii**)$S_{\dot{h(s)}}(h(t)) = h(2s-t)$,

(**viii**)$S_{\dot{h(s)}}(\dot{h(t)}) = -\dot{h(2s-t)}$

Proof. (i),(ii),(iii),(iv),(vi) are the same as Semi-Euclidean reflections and may be seen in [7].(v) is evident by (3) and $S_{\dot{h(s)}} = L_{2s}E_1$.(vii),(viii) are obtained by matrix calculation ■

THEOREM 3.2 The product of two reflections along H^1 is a translation along H^1 through twice the H-distance between their reflecting lines.

Proof.It is obvious by definition of reflection alongH^1 and translation along H^1 ■

Then we have the following result.

COROLLARY.3.3 Let A_1, A_2 be areas of hyperbolic sectors in M^3 which are one vertex origin and the other vertices on H^1,then the product of two reflections along H^1 is a translation through the $A_2 - A_1$.

THEOREM 3.4 Let $S_{\dot{h(s)}}, S_{\dot{h(t)}}$ be two reflection along H^1 .Then

(**i**)$S_{\dot{h(s)}}L_{2t} = S_{\dot{\alpha}(s-t)}$

(**ii**)$L_{2t}S_{\dot{h(s)}} = S_{\dot{h(s+t)}}$

(**iii**)$S_{\dot{h(s)}}L_tS_{\dot{h(s)}} = L_{-t}$

(**iv**)$(S_{\dot{h(s)}}S_{\dot{h(t)}})^p = S_{\dot{h(ps)}}S_{\dot{h(pt)}}, p \in R$

Proof.It follows from Theorem 3.2 and $L_tL_s = L_{s+t}$ ■

Now we can say that the product of a translation and a reflection along H^1 is another reflection along H^1 .

COROLLARY 3.5 Let $S_{\dot{h(s)}}, S_{\dot{h(t)}}$ be two reflection along H^1 .Then

$$(S_{\dot{h(s)}} S_{\dot{h(t)}})^m = L_{2m(s-t)}, m \in R$$

Proof.It can be seen from Theorem 3.2 and the definition of translation along H^1.

THEOREM 3.6 If we take $l_1 = H^1$ in Lemma 2.1,then reflection along any H-line is determined uniquely by reflection along H^1.

Proof. By Lemma 2.1,there is a unique $g \in G$ such that $g(H^1) = H$.By (iv) of Theorem 3.1, $gS_{\dot{h(s)}}g^-1 = S_{g\dot{h(s)}}$. The proof is complete ■

4.Reflections Along the s- Curves In this section we give another reflections in H^2.This kind of reflections in H^2 are obtained by the curve s=constant in (4). Therefore,we call such reflections as *reflections along the s- curves* .

If we take $\alpha(\theta) = \cosh s\ e_0 + \sinh s cos\theta\ e_1 + \sinh s \sin\theta\ e_2$ for s=constant ,then, we have

$$\overline{S}_{\dot{\alpha}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta & -\cos 2\theta \end{pmatrix} \qquad (9)$$

We call the reflection matrix along the s- curves to (9).

THEOREM 4.1 Let $\overline{S}_{\dot{\alpha}(\theta_1)}, \overline{S}_{\dot{\alpha}(\theta_2)}$ be two reflections along the s-curves .Then

$$\text{(i)}\overline{S}_{\dot{\alpha}(\theta)}^{\ 2} = I_3,$$

$$\text{(ii)}\overline{S}_{\dot{\alpha}(\theta_1)}\overline{S}_{\dot{\alpha}(\theta_2)}\overline{S}_{\dot{\alpha}(\theta_1)} = \overline{S}_{\overline{S}_{\dot{\alpha}(\theta_1)}}(\dot{\alpha}(\theta_2)),$$

$$\text{(iii)}det\overline{S}_{\dot{\alpha}(\theta)} = -1,$$

$(\mathbf{iv})\overline{S}_{\lambda\dot{\alpha}(\theta)} = \overline{S}_{\dot{\alpha}(\theta)}, \lambda \in R,$

$(\mathbf{v})\overline{S}_{\dot{\alpha}(\theta)} \in G_0$

$(\mathbf{vi})g\overline{S}_{\dot{\alpha}(\theta)}g^{-}1 = \overline{S}_{g\dot{\alpha}(\theta)}, g \in O_1(3),$

$(\mathbf{vii})\overline{S}_{\dot{\alpha}(\theta_1)}\dot{\alpha}(\theta_2) = -\dot{\alpha}(2\theta_1 - \theta_2)$

$(\mathbf{viii})\overline{S}_{\dot{\alpha}(\theta_1)}(\alpha(\theta_2)) = \alpha(2\theta_1 - \theta_2)$

Proof. (i),(ii),(iii),(iv),(vi) is same with Euclidean reflections which is seen in [5].(v) is evident by (9) and $\overline{S}_{\dot{\alpha}(\theta)} = R_{2\theta}E_2$.the others are followed by matrix calculations ■

THEOREM 4.2Let $\overline{S}_{\dot{\alpha}(\theta_1)}, \overline{S}_{\dot{\alpha}(\theta_2)}$ be two reflections along the s- curves . Then

$$\overline{S}_{\dot{\alpha}(\theta_1)}\overline{S}_{\dot{\alpha}(\theta_2)} = R_{2(\theta_1-\theta_2)}$$

Proof.It follows from (9) ■

THEOREM 4.3

$(S_{\dot{h(s)}}\overline{S}_{\dot{\alpha}(\theta)})^m = I_3$ if and only if $s = 0$ and $\theta = \frac{m-2}{2m}\pi$.

Proof.If $(S_{\dot{h(s)}}\overline{S}_{\dot{\alpha}(\theta)})^m = I_3$, using matrix calculations ,we see that s must be zero. Since

$$S_{\dot{h(0)}}\overline{S}_{\dot{\alpha}(\theta)} = R_{\pi-2\theta}$$

we find

$$(S_{\dot{h(0)}}\overline{S}_{\dot{\alpha}(\theta)})^m = R_{m(\pi-2\theta)} \quad (10)$$

By (10),we see that θ must be $\frac{m-2}{2m}\pi$. Reverse of theorem is evident.

COROLLARY 4.4 **(i)** If m is even integer ,then,

$$(S_{h\dot(0)}\overline{S}_{\dot\alpha(\pi/m)})^m = I_3$$

(ii) If m is odd integer,then

$$(S_{h\dot(0)}\overline{S}_{\dot\alpha(\pi/2m)})^m = I_3$$

Proof. It is evident by Theorem 4.3 ■

COROLLARY 4.5

(i) The order of the product any two reflections along H^1 is infinite

(ii) The order of the product any reflection along the s- curves $s \neq 0$ with $S_{h\dot(0)}$ is an integer if $\theta = \pi/m$ or $\theta = \pi/2m$ where m is order of product.

(iii) The order of the product any two reflections along the s- curves is an integer if $\theta_2 = \theta_1 + \pi/m$ where m is order of product .

Proof. It follows from Theorem.3.3,Corollary 4.4 and Theorem 4.2 ■

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