Results in Nonlinear Analysis **5** (2022) No. 3, 347–359.
https://doi.org/10.53006/rna.1140743
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Contractions of Kannan-type and of Chatterjea-type on fuzzy quasi-metric spaces

Salvador Romaguera[a]

[a] *Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022 Valencia, Spain.*

Abstract

We characterize the completeness of fuzzy quasi-metric spaces by means of a fixed point theorem of Kannan-type. Thus, we extend the classical characterization of metric completeness due to Subrahmanyam as well as recent results in the literature on the characterization of quasi-metric completeness and fuzzy metric completeness, respectively. We also introduce and discuss contractions of Chatterjea-type in this asymmetric context.

Keywords: Fuzzy quasi-metric space Quasi-metric space Complete Fixed point Kannan contraction Chatterjea contraction.
2020 MSC: 54H25, 54A40, 54E50, 54E70

1. Introduction

The problem of characterizing complete fuzzy metric spaces with the help of fixed point results has been recently discussed in [1, 16, 18, 19, 21, 22] as a natural prolongation of the classical problem of obtaining necessary and sufficient conditions for the metric completeness via fixed point theorems (see e.g. [11, 13, 20, 23, 24, 25]). In particular, Subrahmanyam proved in [23] that both the famous Kannan fixed point theorem [12] and its 'companion' Chatterjea fixed point theorem [4] provide nice characterizations of metric completeness. Quasi-metric and fuzzy metric extensions of Subrahmanyam characterization for the Kannan case were obtained in [2] and [18], respectively. In this paper we investigate the problem of extending that characterizations to the realm of complete fuzzy quasi-metric spaces. In Section 3 we observe that the quasi-metric generalization of Chatterjea's theorem continues to be a good 'companion' of the quasi-metric generalization of Kannan's theorem obtained in [2]. Section 4 is devoted to trying the close of this natural puzzle researching the fuzzy (quasi-)metric case. We will show that while a satisfactory answer is reached in the Kannan setting, the Chatterjea setting presents certain difficulties; despite this, a partial solution to this case is also presented.

Email address: sromague@mat.upv.es (Salvador Romaguera)

Received :July 5, 2022; Accepted: August 5, 2022; Online: August 11, 2022

2. Background

Our main reference for quasi-metric spaces is [6] and for fuzzy quasi-metric spaces they are [10] and [5].

We remind that a quasi-metric on a set X is a function $d : X \times X \to [0,\infty)$ such that for all $x, y, z \in X$: (i) $x = y \Leftrightarrow d(x,y) = d(y,x) = 0$, and (ii) $d(x,z) \leq d(x,y) + d(y,z)$.

By a quasi-metric space we mean a pair (X,d) such that X is a set and d is a quasi-metric on X.

Let d be a quasi-metric on a set X. For each $x \in X$ and $\varepsilon > 0$ set $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$. Then, the family $\{B_d(x,\varepsilon) : x \in X, \varepsilon > 0\}$ is a base of open sets for a T_0 topology τ_d on X, called the topology induced by d.

Given a quasi-metric d on X, the function $d^s : X \times X \to [0,\infty)$ defined by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ for all $x, y \in X$, is a metric on X.

There exist many interesting instances of quasi-metric spaces in the literature, see e.g. [6, 7] (a few examples may also be found at the end of Section 3).

On the other hand, the lack of symmetry yields several different notions of Cauchyness and quasi-metric completeness which coincide with the classical notions when dealing with a metric space. In our context, a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X,d) will be called a Cauchy sequence if it is a Cauchy sequence in the metric space (X,d^s), and we shall use the following very general notion of completeness:

A quasi-metric space (X,d) is complete provided that every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is τ_d-convergent, i.e., if there exists some $x \in X$ such that $\lim_{n\to\infty} d(x,x_n) = 0$.

Let us recall [14] that a binary operation $* : [0,1]\times[0,1] \to [0,1]$ is a continuous t-norm provided that it satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0,1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

It is well known that if $*$ is a continuous t-norm, then $* \leq \wedge$, where $\wedge$ is the continuous t-norm given by $a \wedge b = \min\{a,b\}$.

In [10] (see also [5]) were introduced and discussed the following notions as a natural asymmetric generalization of the classical notions of fuzzy metric space in the senses of Kramosil and Michalek [15] and George and Veeramani [8, 9], respectively.

Definition 2.1. *[5, 10]. A KM-fuzzy quasi-metric on a set X is a pair* $(M,*)$ *such that* $*$ *is a continuous t-norm and M is a fuzzy set in* $X \times X \times [0,\infty)$ *fulfilling the following four conditions for every* $x, y, z \in X$:

(KM1) $M(x,y,0) = 0$;

(KM2) $x = y$ *if and only if* $M(x,y,t) = M(y,x,t) = 1$ *for all* $t > 0$;

(KM3) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$ *for all* $t, s \geq 0$;

(KM4) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ *is left continuous.*

A KM-fuzzy quasi-metric $(M,*)$ on X fulfilling for every $x, y \in X$:

(KM5) $M(x,y,t) = M(y,x,t)$ for all $t > 0$,

is said to be a KM-fuzzy metric on X.

Definition 2.2. *[5, 10]. A KM-fuzzy (quasi-)metric space is a triple* $(X,M,*)$ *such that X is a set and* $(M,*)$ *is a KM-fuzzy (quasi-)metric on X.*

Definition 2.3. *[5, 10]. A GV-fuzzy quasi-metric on a set X is a pair* $(M,*)$ *such that* $*$ *is a continuous t-norm and M is a fuzzy set in* $X \times X \times (0,\infty)$ *fulfilling the following four conditions for every* $x, y, z \in X$:

(GV1) $M(x,y,t) > 0$ *for all* $t > 0$;

(GV2) $x = y$ *if and only if* $M(x,y,t) = M(y,x,t) = 1$ *for some* $t > 0$;

(GV3) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$ *for all* $t, s > 0$;

(GV4) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous.*

A GV-fuzzy quasi-metric $(M, *)$ on X fulfilling for every $x, y \in X$:

(GV5) $M(x, y, t) = M(y, x, t)$ for all $t > 0$

is said to be a GV-fuzzy metric on X.

Definition 2.4. *[5, 10]. A GV-fuzzy (quasi-)metric space is a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a GV-fuzzy (quasi-)metric on X.*

Remark 2.5. *[5, 10]. If $(M, *)$ is a KM-fuzzy quasi-metric on a set X, then, for each $x, y \in X$ the function $M(x, y, \cdot)$ is nondecreasing.*

Remark 2.6. *It easily follows from Remark 2.5 that, given $x, y \in X$, if $M(x, y, t) > 1 - t$ for all $t > 0$, then $M(x, y, t) = 1$ for all $t > 0$.*

Remark 2.7. *Note that the class of KM-fuzzy metric spaces $(X, M, *)$ coincides with the class of fuzzy metric spaces in the sense of Kramosil and Michalek [15], with the exception of the condition $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$, which is required in [15], whereas the class of GV-fuzzy metric spaces is exactly the class of fuzzy metric spaces in the sense of George and Veeramani [8, 9].*

Analogous to the quasi-metric case, if $(M, *)$ is a KM-fuzzy quasi-metric (resp. a GV-fuzzy quasi-metric) on a set X, the pair $(M^{\min}, *)$ is a KM-fuzzy metric (resp. a GV-fuzzy metric) on X, where $M^{\min}$ is the fuzzy set in $X \times X \times [0, \infty)$ (resp. in $X \times X \times (0, \infty)$) given by $M^{\min}(x, y, t) = \min\{M(x, y, t), M(y, x, t)\}$. We shall refer to $(X, M^{\min}, *)$ as the KM-fuzzy metric (resp. the GV-fuzzy metric) space induced by $(X, M, *)$.

On the other hand, and as in the quasi-metric setting, if $(M, *)$ is a KM-fuzzy quasi-metric on a set X and for each $x \in X$, $\varepsilon \in (0, 1)$ and $t > 0$ we put $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$, then, the family $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$ is a base of open sets for a T_0 topology τ_M on X, called the topology induced by $(M, *)$ (see e.g. [10, p. 131]).

From the definition of the topology τ_M we deduce the following well-known and useful fact [10, Proposition 2.8]:

A sequence $(x_n)_{n\in\mathbb{N}}$ in a KM-fuzzy quasi-metric space $(X, M, *)$ τ_M-converges to an $x \in X$ if and only if $\lim_{n\to\infty} M(x, x_n, t) = 1$ for all $t > 0$. (By $\mathbb{N}$ we denote the set of all positive integers).

Before to define the notions of Cauchyness and completeness for KM-fuzzy quasi-metric spaces that we will employ here, remind that a sequence $(x_n)_{n\in\mathbb{N}}$ in a KM-fuzzy metric space $(X, M, *)$ is a Cauchy sequence provided that for each $\varepsilon \in (0, 1)$ and each $t > 0$ there is an $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

Definition 2.8. *A sequence in a KM-fuzzy quasi-metric space $(X, M, *)$ is said to be a Cauchy sequence if it is a Cauchy sequence in the fuzzy metric space $(X, M^{\min}, *)$.*

Definition 2.9. *A KM-fuzzy quasi-metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is τ_M-convergent.*

Remark 2.10. *Each GV-fuzzy (quasi-)metric space $(X, M, *)$ can be considered as a KM-fuzzy (quasi-)metric space simply by defining $M(x, y, 0) = 0$ for all $x, y \in X$. Hence, any GV-fuzzy quasi-metric induces a topology defined as in the KM-case. Furthermore, the notions and properties for KM-fuzzy quasi-metric spaces given above hold for GV-fuzzy quasi-metric spaces.*

The following is an important instance of a KM-fuzzy quasi-metric space which is not a GV-fuzzy quasi-metric space, in general.

Example 2.11. *[3] Let (X, d) be a quasi-metric space. Then $(X, M_{d,01}, *)$ is a KM-fuzzy quasi-metric space for any continuous t-norm $*$, where M is the fuzzy set in $X \times X \times [0, \infty)$ defined as $M_{d,01}(x, y, t) = 0$ if $d(x, y) \geq t$ and $M_{d,01}(x, y, t) = 1$ if $d(x, y) < t$. Furthermore, the topologies τ_d and $\tau_{M_{d,01}}$ agree on X, and $(X, M_{d,01}, *)$ is complete if and only if (X, d) is complete.*

We finish this section with a typical example of a GV-fuzzy quasi-metric space.

Example 2.12. *[10]. Let (X, d) be a quasi-metric space. Then $(X, M_{d,s}, *)$ is a GV-fuzzy quasi-metric space for any continuous t-norm $*$, where $M_{d,s}$ is the fuzzy set in $X \times X \times (0, \infty)$ defined as*

$$M_{d,s}(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $t > 0$. $(X, M_{d,s}, *)$ is said to be the standard GV-fuzzy quasi-metric space of (X, d). Furthermore, the topologies τ_d and $\tau_{M_{d,s}}$ agree on X, and $(X, M_{d,s}, *)$ is complete if and only if (X, d) is complete.

3. The quasi-metric setting

We begin this section by proposing the following natural quasi-metric generalizations of the notions of Kannan contraction and Chatterjea contraction for metric spaces.

Definition 3.1. *We say that a self map T of a quasi-metric space (X, d) is a Kannan contraction (a d-Kannan mapping in [2]) if there is a constant $c \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$.

Definition 3.2. *We say that a self map T of a quasi-metric space (X, d) is a Chatterjea contraction if there is a constant $c \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$.

Remark 3.3. *It is clear that every Kannan contraction (resp. every Chatterjea contraction) on a quasi-metric space (X, d), is a Kannan contraction (resp. a Chatterjea contraction) on the metric space (X, d^s). However, the reverse implications do not hold, in general (see Example 3.6 below).*

The next result will be crucial.

Proposition 3.4. *A quasi-metric space (X, d) is complete if each self map T of X satisfying*

$$d^s(Tx, Ty) \leq \frac{1}{3} \min\{d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)\},$$

for all $x, y \in X$, has a fixed point.

Proof. Suppose that (X, d) is not complete. Then, there is a non τ_d-convergent Cauchy sequence $(x_n)_{n\in\omega}$ in (X, d). Following the proof of [2, Theorem 2.8] we can construct a self map T of X without fixed points and such that

$$d^s(Tx, Ty) \leq \frac{1}{5}[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$. (The original proof of [2, Theorem 2.8] is given for $c = 1/4$, but it is also valid, for instance, for $c = 1/5$, without any change). From the triangle inequality it follows that

$$d^s(Tx, Ty) \leq \frac{1}{5}[d(x, Ty) + d^s(Ty, Tx) + d(y, Tx) + d^s(Tx, Ty)],$$

for all $x, y \in X$. Therefore

$$d^s(Tx, Ty) \leq \frac{1}{3}[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$. Consequently

$$d^s(Tx, Ty) \leq \frac{1}{3} \min\{d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)\},$$

for all $x, y \in X$. We have reached a contradiction, which concludes the proof. □

Theorem 3.5. *For a quasi-metric space (X, d) the following conditions are equivalent.*

(1) (X, d) is complete.

(2) Every Kannan contraction on (X,d) has a (unique) fixed point.

(3) Every Chatterjea contraction on (X,d) has a (unique) fixed point.

Proof. (1) ⇔ (2) [2, Theorem 2.8].

(1) ⇒ (3) Let T be a Chatterjea contraction on the complete quasi-metric space (X, d). Then, there is $c \in (0, 1/2)$ such that

$$d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$. It immediately follows that

$$d^s(Tx, Ty) \le c[d^s(x, Ty) + d^s(y, Tx)],$$

for all $x, y \in X$, so T is a Chatterjea contraction on the metric space (X, d^s).

Fix an $x_0 \in X$. Then, the classical proof of Chatterjea's theorem shows that $(T^n x_0)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). By completeness of (X, d) there exists $z \in X$ such that $\lim_{n\to\infty} d(z, T^n x_0) = 0$. We show that Tz is the unique fixed point of T. Indeed, we get

$$\begin{aligned} d(T^{n+1}x_0, Tz) &\le c[d(T^n x_0, Tz) + d(z, T^{n+1}x_0)] \\ &\le c[d(T^n x_0, T^{n+1}x_0) + d(T^{n+1}x_0, Tz) + d(z, T^{n+1}x_0)], \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore

$$d(T^{n+1}x_0, Tz) \le \frac{c}{1-c}[d(T^n x_0, T^{n+1}x_0) + d(z, T^{n+1}x_0)],$$

for all $n \in \mathbb{N}$, so that $\lim_{n\to\infty} d(T^{n+1}x_0, Tz) = 0$. By the triangle inequality $d(z, Tz) = 0$. Thus, we have

$$d(Tz, T^2z) \le c[d(z, T^2z) + d(Tz, Tz)] \le c[d(z, Tz) + d(Tz, T^2z)] = cd(Tz, T^2z),$$

so $d(Tz, T^2z) = 0$, and also

$$d(T^2z, Tz) \le c[d(Tz, Tz) + d(z, T^2z)] \le c[d(z, Tz) + d(Tz, T^2z)] = 0.$$

Hence $Tz = T^2z$.

Finally, let $u \in X$ such that $u = Tu$. Then

$$d^s(u, Tz) = d^s(Tu, T^2z) \le c[d^s(u, T^2z) + d^s(Tz, Tu)] = 2cd^s(u, Tz).$$

Since $2c < 1$, $d^s(u, Tz) = 0$, i.e., $u = Tz$.

(3) ⇒ (1) Let T be a self map of X such that

$$d^s(Tx, Ty) \le \frac{1}{3}\min\{d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)\},$$

for all $x, y \in X$. Then

$$d(Tx, Ty) \le \frac{1}{3}[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$, so T is a Chatterjea contraction on (X, d). By assumption T has a fixed point. Hence (X, d) is complete by Proposition 3.4. □

Regarding our comment at the end of Remark 3.3 we present the following example.

Example 3.6. *Let $X = [0,2]$ and let d be the quasi-metric on X given by $d(x,y) = \max\{y - x, 0\}$ for all $x, y \in X$. Since d^s is the restriction to X of the usual metric on the set $\mathbb{R}$ of all reals, it follows that (X, d^s) is a compact metric space, so (X, d) is a complete quasi-metric space. Now let $T : X \to X$ defined by $Tx = 0$ if $x \in [0,1]$ and $Tx = x/4$ if $x \in (1,2]$. We show that T is both a Kannan contraction and a Chatterjea contraction on the metric space (X, d^s).*

- *If $x, y \in [0,1]$ we have $d^s(Tx, Ty) = 0$.*
- *If $x \in [0,1]$ and $y \in (1,2]$ we get $d^s(Tx, Ty) = y/4$, $d^s(x, Tx) = x$, $d^s(y, Ty) = 3y/4$, $d^s(x, Ty) = |x - y/4|$ and $d^s(y, Tx) = y$, so*
$$d^s(Tx, Ty) = \frac{1}{3}d^s(y, Ty) < \frac{1}{3}d^s(y, Tx).$$
- *If $x, y \in (1,2]$, we get $d^s(Tx, Ty) = |x - y|/4$, $d^s(x, Tx) = 3x/4$, $d^s(y, Ty) = 3y/4$, $d^s(x, Ty) = (4x - y)/4$ and $d^s(y, Tx) = (4.y - x)/4$, so*
$$d^s(Tx, Ty) < \frac{1}{3}d^s(x, Tx) < \frac{1}{3}[d^s(x, Ty) + d^s(y, Tx)].$$

Therefore T is both a Kannan contraction and a Chatterjea contraction on (X, d^s), with $c = 1/3$. However, for $x = 0$ and $y \in (1,2]$ we obtain $d^s(Tx, Ty) = d(x, Ty) = y/4$, and $d(x, Tx) = d(y, Ty) = d(y, Tx) = 0$, so T it is not a Kannan contraction neither a Chatterjea contraction on (X, d).

The following is an example of a self map T on a complete quasi-metric space (X, d) which is a Kannan contraction on (X, d), and thus a Kannan contraction of (X, d^s) but is not a Chatterjea contaction on (X, d^s), and thus not a Chatterjea contraction on (X, d).

Example 3.7. *Let $X = \{0, 1, 2\}$ and let d be the quasi-metric on X given by $d(x, x) = 0$ for all $x \in X$, $d(0,1) = d(0,2) = d(2,0) = d(2,1) = 1$, $d(1,0) = 2$ and $d(1,2) = 3$. Evidently (X, d) is complete because the Cauchy sequences in (X, d^s) are those that are eventually constant.*

Now let $T : X \to X$ defined by $T0 = T2 = 0$ and $T1 = 2$.

We show that T is a Kannan contraction on (X, d) (and thus on (X, d^s)). To this end it suffices to check that

$$\begin{aligned} d^s(T0, T1) &= d^s(T2, T1) = d^s(0,2) = 1 = \frac{1}{3}[d(0, T0) + d(1, T1)] \\ &< \frac{1}{3}[d(2, T2) + d(1, T1)]. \end{aligned}$$

Note also that T is not a Chatterjea contraction on (X, d^s) (and thus not on (X, d)) because

$$d^s(T2, T1) = 1 = \frac{1}{2}[0 + 2] = \frac{1}{2}[d^s(2, T1) + d^s(1, T2)].$$

We finish this section with an example of a self map of a complete quasi-metric space (X, d) which is a Chatterjea contraction on (X, d), and, consequently, a Chatterjea contraction on the metric space (X, d^s), but not a Kannan contraction on (X, d^s), and, consequently, not a Kannan contraction on (X, d).

Example 3.8. *Let $X = [0,1]$ and let d be the quasi-metric on X given by $d(x,y) = \max\{x - y, 0\}$. It is well known that (X, d) is a complete quasi-metric space (note that d^s is the usual metric on X).*

Let T be the self map of X defined as $T1 = 1/3$, and $Tx = 0$ for all $x \in [0,1)$.

We show that T is not a Kannan contraction on (X, d^s) and thus is not a Kannan contraction on (X, d). Indeed, we get

$$d^s(T1, T0) = d^s(1/3, 0) = \frac{1}{3} = \frac{1}{2}[\frac{2}{3} + 0] = \frac{1}{2}[d^s(1, T1) + d^s(0, T0)].$$

However T is a Chatterjea contraction on (X, d) and thus on (X, d^s). Indeed, for $x = 1$ and $y \in [0,1)$ we get

$$d^s(Tx, Ty) = d^s(1/3, 0) = \frac{1}{3} \leq \frac{1}{3}[d(x, Ty) + d(y, Tx)].$$

4. The fuzzy quasi-metric setting

In this section KM-fuzzy quasi-metric will be simply called fuzzy quasi-metric spaces.

In [18] we introduced the notions of (1)-Kannan contraction and (1/2)-Kannan contraction for fuzzy metric spaces. We generalize that notions in a natural way as follows.

Definition 4.1. *Let $(X, M, *)$ be a fuzzy quasi-metric space. We say that a self map T of X is a (1)-Kannan contraction on $(X, M, *)$ if there is a constant $c \in (0, 1)$ such that for any $x, y \in X$ and $t > 0$,*

$$\min\{M(x, Tx, t), M(y, Ty, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct. \qquad \text{(1Kn)}$$

Analogous to the quasi-metric setting, we say that a self map $T$of X is a (1/2)-Kannan contraction on $(X, M, *)$ if there is a constant $c \in (0, 1/2)$ such that for any $x, y \in X$ and $t > 0$,

$$\min\{M(x, Tx, t), M(y, Ty, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct.$$

Remark 4.2. *If T is a (1)-Kannan contraction on a fuzzy quasi-metric space $(X, M, *)$ and we interchange x and y in condition (1Kn), then that condition can be reformulated as follows:*

$$\min\{M(x, Tx, t), M(y, Ty, t)\} > 1 - t \Rightarrow M^{\min}(Tx, Ty, ct) > 1 - ct. \qquad \text{(1Knm)}$$

Clearly, any (1/2)-Kannan contraction is a (1)-Kannan contraction. However the converse does not hold even for fuzzy metric spaces (see [18, Example 2]).

In the proof of our next theorem we use ideas and methods from [18, Theorem 3]. In particular, we shall apply the following auxiliary result whose demo is almost identical to the first part of the proof of [18, Theorem 3], so it is omitted.

Lemma 4.3. *Let T be a (1)-Kannan contraction on a fuzzy quasi-metric space $(X, M, *)$. Fix $x_0 \in X$. Then $(T^n x_0)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$.*

Theorem 4.4. *Every (1)-Kannan contraction on a complete fuzzy quasi-metric space has a unique fixed point.*

Proof. Let $(X, M, *)$ be a complete fuzzy quasi-metric metric space and let T be a (1)-Kannan contraction on X. Then, there is a constant $c \in (0, 1)$ for which condition (1Kn) is fulfilled. Furthermore T is a (1)-Kannan contraction on the fuzzy metric space $(X, M^{\min}, *)$ with constant of contraction c, by Remark 4.2.

Fix $t_0 > 1$ and $r, s > 0$ such that $c < s < r < 1$. For each $k \in \mathbb{N}$, define

$$A_{k,r,s} := \{\varepsilon \in (0, 1) : \varepsilon + sr^{k-1}t_0 < r^k t_0\}.$$

Now fix $x_0 \in X$. Applying Lemma 4.3 we get that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$, where $x_n := T^n x_0$ for all $n \in \mathbb{N}\cup\{0\}$. So there is $z \in X$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to z in τ_M, i.e., $\lim_{n\to\infty} M(z, x_n, t) = 1$ for all $t > 0$.

Following the proof of [18, Theorem 3], joint with Remark 4.2, we deduce that $M(z, Tz, t) = 1$ for all $t > 0$ (the details are omitted).

In the sequel we show that Tz is a fixed point of T.

To reach it we first check, by mathematical induction, that for each $k \in \mathbb{N}$,

$$M^{\min}(Tz, T^2z, r^k t_0) \geq 1 - r^k t_0. \qquad (\dagger)$$

where we assume, without loss of generality, that $r^k t_0 \leq 1$.

Indeed, since $M(Tz, T^2z, t_0) > 1 - t_0$ and $M(x_n, x_{n+1}, t_0) > 1 - t_0$, we deduce from Remarks 2.5 and 4.2 that

$$M^{\min}(T^2z, x_{n+1}, st_0) \geq M^{\min}(T^2z, x_{n+1}, ct_0) > 1 - ct_0 > 1 - st_0,$$

for all $n \in \mathbb{N} \cup \{0\}$.

Given $t > 0$ there is $n_t \in \mathbb{N}$ such that $M(x_n, x_{n+1}, t) > 1 - t$ for all $n \geq n_t$. Since $M(z, Tz, t) = 1$, it follows from condition (1Knm) that

$$M^{\min}(Tz, x_{n+1}, ct) > 1 - ct,$$

for all $n \geq n_t$. Hence, for each $\varepsilon \in A_{1,r,s}$ (taking $t = \varepsilon/c$) there is $n_\varepsilon \geq n_t$ such that $M^{\min}(Tz, x_{n_\varepsilon}, \varepsilon) > 1 - \varepsilon$. Therefore

$$\begin{aligned} M^{\min}(Tz, T^2z, rt_0) &\geq M^{\min}(Tz, x_{n_\varepsilon}, \varepsilon) * M^{\min}(x_{n_\varepsilon}, T^2z, st_0) \\ &\geq (1 - \varepsilon) * (1 - st_0) \geq (1 - \varepsilon) * (1 - rt_0). \end{aligned}$$

From the continuity of $*$ we get

$$M^{\min}(Tz, T^2z, rt_0) \geq 1 - rt_0.$$

So, we have proved the inequality (†) for $k = 1$.

Now suppose that the inequality (†) is true for $k = j$, $j \in \mathbb{N}$. We shall check that $M^{\min}(Tz, T^2z, r^{j+1}t_0) \geq 1 - r^{j+1}t_0$.

Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there is $n_j \in \mathbb{N}$ such that

$$M(x_n, x_{n+1}, r^j t_0) > 1 - r^j t_0,$$

for all $n \geq n_j$. This fact along with our induction hypothesis and condition (1Knm) implies that

$$M^{\min}(T^2z, x_{n+1}, cr^j t_0) > 1 - cr^j t_0,$$

for all $n \geq n_j$. Therefore

$$M^{\min}(T^2z, x_{n+1}, sr^j t_0) > 1 - sr^j t_0,$$

for all $n \geq n_j$.

Now let $\varepsilon \in A_{j+1,r,s}$. Then $\varepsilon + sr^j t_0 < r^{j+1}t_0$, and there is $n_\varepsilon > n_j$ for which $M^{\min}(Tz, x_{n_\varepsilon}, \varepsilon) > 1 - \varepsilon$. Hence

$$\begin{aligned} M^{\min}(Tz, T^2z, r^{j+1}t_0) &\geq M^{\min}(Tz, x_{n_\varepsilon}, \varepsilon) * M^{\min}(x_{n_\varepsilon}, T^2z, sr^j t_0) \\ &\geq (1 - \varepsilon) * (1 - sr^j t_0) \geq (1 - \varepsilon) * (1 - r^{j+1}t_0). \end{aligned}$$

Again, from the continuity of $*$ we deduce that

$$M^{\min}(Tz, T^2z, r^{j+1}t_0) \geq 1 - r^{j+1}t_0.$$

We conclude that inequality (†) is true.

Take any $t > 0$. Then, there exists $k \in \mathbb{N}$ such that $r^k t_0 < t$, so

$$M^{\min}(Tz, T^2z, t) \geq M^{\min}(Tz, T^2z, r^k t_0) > 1 - r^k t_0 > 1 - t.$$

From Remark 2.6 it follows that $M^{\min}(Tz, T^2z, t) = 1$ for all $t > 0$. Hence $Tz = T^2z$, so Tz is a fixed point of T.

Finally, let $u \in X$ such that $u = Tu$. So $\min\{M(Tz, T^2z, t), M(u, Tu, t)\} = 1$, for all $t > 0$. By condition (1Knm) we deduce that $M^{\min}(T^2z, Tu, ct) > 1 - ct$ for all $t > 0$. Consequently $T^2z = Tu$, i.e., $Tz = u$, so Tz is the unique fixed point of T. □

Corollary 4.5. *Every (1/2)-Kannan contraction on a complete fuzzy quasi-metric space has a unique fixed point.*

Remark 4.6. *[18, Example 2] shows that Theorem 4.4 is a real generalization of Corollary 4.5.*

The next is an example of a self map of a complete quasi-metric space (X, d) which is not a Kannan contraction on (X, d) but, instead, it is a (1)-Kannan contraction on the complete fuzzy quasi-metric space $(X, M_{d,01}, *)$ for any continuous t-norm $*$.

Example 4.7. *Let (X, d) be the complete quasi-metric space where $X = [0, 1]$ and d is the quasi-metric on X given by $d(x, y) = \max\{x - y, 0\}$. It was shown in Example 3.8 above that the self map T of X defined as $T1 = 1/3$ and $Tx = 0$ for all $x \in [0, 1)$, is not a Kannan contraction on (X, d).*

Let $(X, M_{d,01}, *)$ be the complete fuzzy quasi-metric space as constructed in Example 2.11 above. We are going to prove that T is a (1)-Kannan contraction on $(X, M_{d,01}, *)$ for $c = 1/2$. Thus, it will verify the conditions of Theorem 4.4.

Indeed, let $x, y \in X$ and $t > 0$ such that $\min\{M_{d,01}(x, Tx, t), M_{d,01}(y, Ty, t\} > 1 - t$.

Then, we shall check that $M_{d,01}(Tx, Ty, t/2) > 1 - t/2$.

If $M_{d,01}(Tx, Ty, t/2) = 1$, the conclusion is obvious. Hence, we will assume in the sequel that $M_{d,01}(Tx, Ty, t/2) = 0$. Thus, we get $d(Tx, Ty) \geq t/2$.

By the construction of T it suffices to consider two cases, namely:

Case 1. $x = 1$, $y \in [0, 1)$.

Case 2. $x \in [0, 1)$, $y = 1$.

In Case 1, from $d(Tx, Ty) \geq t/2$ we get $d(1/3, 0) = 1/3 \geq t/2$, so $t \leq 2/3$.
If, in addition, $M_{d,01}(1, T1, t) = 0$, we deduce, by hypothesis, that $t > 1$, a contradiction.
Hence, we will have that $M_{d,01}(1, T1, t) = 1$. This implies that $d(1, 1/3) = 2/3 < t$, which yields again a contradiction.

In Case 2, from $d(Tx, Ty) \geq t/2$ we get $d(0, 1/3) = 0 \geq t/2$, a contradiction.

We conclude that $M_{d,01}(Tx, Ty, t/2) = 1$, and thus $M_{d,01}(Tx, Ty, t/2) > 1 - t/2$. Therefore T is a (1)-Kannan contraction on $(X, M_{d,01}, *)$.

Our main result (Theorem 4.10 below) provides a full quasi-metric extension of [18, Theorem 5]. In its proof we shall use the next quasi-metric generalization of an important result due to Radu [17, Proposition 2.1.1], which can be partially found in [3, Example 2] and in [5, Remark 7.6.1] (let us recall that the famous Łukasiewicz t-norm $*_L$ is the continuous t-norm defined by $a *_L b = \max\{a + b - 1, 0\}$ for all $a, b \in [0, 1]$).

Proposition 4.8. *Let $(X, M, *)$ be a fuzzy quasi-metric space. For each $x, y \in X$ put*

$$d_M(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - t\}.$$

Then d_M satisfies the following condition

$$d_M(x, y) < t \Leftrightarrow M(x, y, t) > 1 - t, \qquad \text{(C1)}$$

for all $t > 0$. Furthermore, if $*_L \leq *$, then d_M is a quasi-metric on X whose induced topology agrees with τ_M, and (X, d_M) is complete if and only if $(X, M, *)$ is complete.

We shall also use the following essentially well-known fact.

Lemma 4.9. *If a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in a fuzzy quasi-metric space $(X, M, *)$ has a subsequence which is τ_M-convergent to some $z \in X$, then $(x_n)_{n\in\mathbb{N}}$ is τ_M-convergent to z.*

Proof. Let $(x_{n(k)})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ for which there is some $z \in X$ such that $(x_{n(k)})$ τ_M-converges to z.
Choose an arbitrary $\varepsilon \in (0, 1)$. By the continuity of $*$ we can find a $\delta \in (0, \varepsilon/2)$ such that $(1 - \delta) * (1 - \delta) > 1 - \varepsilon$. Then, there exists $n_\delta \in \mathbb{N}$ such that $M^{\min}(x_n, x_m, \delta) > 1 - \delta$ for all $n, m \geq n_\delta$. Since for any $m \geq n_\delta$ there exists $k_0 \in \mathbb{N}$ with $n(k_0) \geq m$ and $M(z, x_{n(k_0)}, \delta) > 1 - \delta$, we deduce that, for $m \geq n_\delta$,

$$M(z, x_m, \varepsilon) \geq M(z, x_{n(k_0)}, \delta) * M(x_{n(k_0)}, x_m, \delta) \geq (1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

Consequently, the sequence $(x_n)_{n\in\mathbb{N}}$ τ_M-converges to z. □

Theorem 4.10. *For a fuzzy quasi-metric space $(X, M, *)$ the following conditions are equivalent.*

(1) $(X, M, *)$ is complete.

(2) Every (1)-Kannan contraction on X has a (unique) fixed point.

(3) Every (1/2)-Kannan contraction on X has a (unique) fixed point.

Proof. (1) $\Rightarrow$ (2) Theorem 4.4.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Suppose that $(X, M, *)$ is not complete. Then there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, M, *)$ which is not τ_M-convergent.

We divide the rest of the proof in the following claims.

Claim 1. For each $y \in X$ there is an $n(y) \in \mathbb{N}$ such that $d_M(y, x_n) > 0$ for all $n \geq n(y)$.

Indeed, suppose that there are $y \in X$ and a subsequence $(x_{n(i)})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $d_M(y, x_{n(i)}) = 0$ for all $i \in \mathbb{N}$. By condition (C1) we get $M(y, x_{n(i)}, t) > 1 - t$ for all $t > 0$. Therefore $(x_n)_{n\in\mathbb{N}}$ τ_M-converges to y by Remark 2.6 and Proposition 4.8, a contradiction.

Claim 2. For each $y \in X$ there is a $k(y) \geq n(y)$ such that $d_M(y, x_n) \geq 1/k(y)$ for all $n \geq k(y)$.

Indeed, suppose that there is $y \in X$ such that for any $k \geq n(y)$ there exists $n_k > k$ satisfying $d_M(y, x_{n_k}) < 1/k$. By condition (C1) we get $M(y, x_{n_k}, 1/k) > 1 - 1/k$, which implies that $(x_{n_k})_{k\in\mathbb{N}}$ τ_M-converges to y. By Lemma 4.9, this yields a contradiction.

Now, since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$, for each $y \in X$ there is a $j(y) \in \mathbb{N}$ such that $j(y) \geq k(y)$ and

$$M(x_n, x_m, 1/3j(y)) > 1 - 1/3j(y),$$

whenever $n, m \geq j(y)$. Hence $d_M(x_n, x_m) < 1/3j(y)$, whenever $n, m \geq j(y)$.

Define a self map T of X as follows:

$Ty = x_{j(y)}$ for all $y \in X$.

Claim 3. The self map T has no fixed points in X.

Indeed, put $F := \{x_n : n \in \mathbb{N}\}$. If $y \in X\backslash F$ is obvious that $Ty \neq y$. If $y \in F$ there is $i \in \mathbb{N}$ such that $y = x_i$. Since $j(y) \geq k(y) \geq n(y)$, it follows from Claim 1 that $d_M(y, x_{j(y)}) > 0$, so $y \neq x_{j(y)}$.

Claim 4. T is a (1/2)-Kannan contraction on $(X, M, *)$ (with constant $c = 1/3$).

Indeed, let $y, z \in X$ and $t > 0$ such that $\min\{M(y, Ty, t), M(z, Tz, t)\} > 1 - t$. Then $d_M(y, Ty) < t$ and $d_M(z, Tz) < t$.

If $j(y) < j(z)$ we obtain

$$\begin{aligned} d_M(Ty, Tz) &= d_M(x_{j(y)}, x_{j(z)}) < 1/3j(y) \leq 1/3k(y) \\ &\leq d_M(y, x_{j(y)})/3 = d_M(y, Ty)/3 < t/3. \end{aligned}$$

Therefore $M(Ty, Tz, t/3) > 1 - t/3$.

If $j(y) > j(z)$ we obtain

$$\begin{aligned} d_M(Ty, Tz) &= d_M(x_{j(y)}, x_{j(z)}) < 1/3j(z) \leq 1/3k(z) \\ &\leq d_M(z, x_{j(z)})/3 = d_M(z, Tz)/3 < t/3. \end{aligned}$$

Therefore $M(Ty, Tz, t/3) > 1 - t/3$.

We have constructed a (1/2)-Kannan contraction on $(X, M, *)$ without fixed points. This contradiction concludes the proof. □

The last part of this paper is devoted to discuss the extension of Theorems 4.4 and 4.10 above to contractions of Chatterjea-type on complete fuzzy quasi-metric spaces. Although we have not been able to obtain results as resounding as such theorems, some partial results can found in Propositions 4.13, 4.14 and 4.16 below.

Definition 4.11. *Let $(X, M, *)$ be a fuzzy quasi-metric space. We say that a self map T of X is a (1)-Chatterjea contraction on $(X, M, *)$ if there is a constant $c \in (0, 1)$ such that for any $x, y \in X$ and $t > 0$,*

$$\min\{M(x, Ty, t), M(y, Tx, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct. \qquad \text{(1Ch)}$$

By analogy with the quasi-metric setting, we say that a self map $T$of X is a (1/2)-Chatterjea contraction on $(X, M, *)$ if there is a constant $c \in (0, 1/2)$ such that for any $x, y \in X$ and $t > 0$,

$$\min\{M(x, Ty, t), M(y, Tx, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct.$$

Remark 4.12. *If T is a (1)-Chatterjea contraction on a fuzzy quasi-metric space $(X, M, *)$ and we interchange x and y in condition (1Ch), then that condition can be reformulated as follows:*

$$\min\{M(x, Ty, t), M(y, Tx, t)\} > 1 - t \Rightarrow M^{\min}(Tx, Ty, ct) > 1 - ct. \qquad \text{(1Chm)}$$

Clearly, any (1/2)-Chatterjea contraction is a (1)-Chatterjea contraction.

If T is a (1)-Chatterjea contraction on a fuzzy (quasi-)metric space $(X, M, *)$ it is possible to show that for any $x_0 \in X$, $(T^n x_0)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$ (see Proposition 4.13 below). Thus, if $(X, M, *)$ is complete, the sequence $(x_n)_{n\in\mathbb{N}}$ τ_M-converges to some $z \in X$. Unfortunately, and in contrast to the Kannan case, we don't know if $M(z, Tz, t) = 1$ for all $t > 0$ (compare with Theorem 4.4 above and [18, Theorem 3]). Nevertheless, and as we pointed out above, a partial result is provided in Proposition 4.14 below.

Proposition 4.13. *Let T be a (1)-Chatterjea contraction on a fuzzy quasi-metric space $(X, M, *)$. Fix $x_0 \in X$. Then $(T^n x_0)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$.*

Proof. Let $c \in (0, 1)$ for which condition (1Ch) in Definition 4.11 is fulfilled. Fix $t_0 > 1$. For any $x, y \in X$ we get $M(x, Ty, t_0) > 1 - t_0$ and $M(y, Tx, t_0) > 1 - t_0$.

Thus, by the contraction condition (1Ch), $M(Tx, Ty, ct_0) > 1 - ct_0$.

In particular, from $M(x, T^2y, t_0) > 1 - t_0$ and $M(Ty, Tx, t_0) > 1 - t_0$, it follows that

$$M(Tx, T^2y, ct_0) > 1 - ct_0.$$

Analogously,

$$M(Ty, T^2x, ct_0) > 1 - ct_0.$$

Then, it follows from Remark 4.12 that

$$M^{\min}(T^2x, T^2y, c^2t_0) > 1 - c^2t_0.$$

Repeating this process we obtain, for each $n \in \mathbb{N}$,

$$M^{\min}(T^n x, T^n y, c^n t_0) > 1 - c^n t_0.$$

Put $x_n := T^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$. We see that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$.

Indeed, given $t > 0$ and $\varepsilon \in (0, 1)$, there is $n_\varepsilon \in \mathbb{N}$ such that $c^n t_0 < \min\{\varepsilon, t\}$ for all $n \geq n_\varepsilon$. Let $m, n \geq n_\varepsilon$. Assume that $m > n$. Then $m = n + k$ for some $k \in \mathbb{N}$, and thus

$$\begin{aligned} M^{\min}(x_n, x_m, t) &= M^{\min}(T^n x_0, T^n T^k x_0, t) \geq M^{\min}(T^n x_0, T^n T^k x_0, c^n t_0) \\ &> 1 - c^n t_0 > 1 - \varepsilon. \end{aligned}$$

We conclude that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$. □

Proposition 4.14. *Every (1/2)-Chatterjea contraction on a complete fuzzy quasi-metric space $(X, M, *)$ such that $*_L \leq *$ has a unique fixed point.*

Proof. Let T be a (1/2)-Chatterjea contraction (with constant $c \in (0, 1/2)$) on $(X, M, *)$. We shall show that T is a Chatterjea contraction on the complete quasi-metric space (X, d_M) as constructed in Proposition 4.8.

Indeed, let $x, y \in X$ and let $t \geq 0$ such that $M(Tx, Ty, t) \leq 1 - t$. Then

$$\min\{M(x, Ty, t/c), M(y, Tx, t/c)\} \leq 1 - t/c,$$

so, by condition (C1) in Proposition 4.8,

$$\max\{d_M(x, Ty), d_M(y, Tx)\} \geq t/c.$$

Hence $t \leq c \max\{d_M(x, Ty), d_M(y, Tx)\}$. Thus

$$\begin{aligned} d_M(Tx, Ty) &= \sup\{t \geq 0 : M(Tx, Ty, t) \leq 1 - t\} \\ &\leq c \max\{d_M(x, Ty), d_M(y, Tx)\} \leq c[d_M(x, Ty) + d_M(y, Tx)]. \end{aligned}$$

We have proved that T is a Chatterjea contraction on (X, d_M), so, by Theorem 3.5, it has a unique fixed point. □

The following example illustrates Proposition 4.14.

Example 4.15. *Let (X, d) the complete quasi-metric space where $X = [0, 1]$ and d is given by $d(x, y) = \max\{x - y, 0\}$ for all $x, y \in X$. Consider the complete fuzzy quasi-metric space $(X, M_{d,01}, *)$ and fix an $x_0 \in (0, 1/2)$. Then, we define the self map T of X given by $T1 = x_0$ and $Tx = 0$ otherwise.*

We shall check that T is a (1/2)-Chatterjea contraction on $(X, M_{d,01}, *)$ with constant $c = x_0$.

Indeed, let $x, y \in X$ and $t > 0$ such that $\min\{M_{d,01}(x, Ty, t), M_{d,01}(y, Tx, t)\} > 1 - t$.

Suppose that $M_{d,01}(Tx, Ty, ct) = 0$. Then $d(Tx, Ty) \geq ct = x_0 t$.

It suffices to consider two cases.

Case 1. $x = 1, y \in [0, 1)$.

From $d(Tx, Ty) \geq x_0 t$ we deduce that $x_0 \geq x_0 t$, so $t \leq 1$. Since, by hypothesis, $M_{d,01}(x, Ty, t) > 1 - t$ we deduce that $M_{d,01}(x, Ty, t) = 1$, thus $d(1, 0) = 1 < t$, a contradiction.

Case 2. $x \in [0, 1)$ and $y = 1$. From $d(Tx, Ty) \geq ct$ we deduce that $d(0, x_0) = 0 \geq ct$, a contradiction.

We conclude that $M_{d,01}(Tx, Ty, ct) = 1 > 1 - ct$, so T is a (1/2)-Chatterjea contraction on $(X, M_{d,01}, *)$, and all conditions of Proposition 4.14 are satisfied.

We finish with the following partial converse of Proposition 4.14.

Proposition 4.16. *A fuzzy quasi-metric space $(X, M, *)$ such that $*_L \leq *$ is complete if every (1)-Chatterjea contraction has a fixed point.*

Proof. Let T be a Chatterjea contraction (with constant $c \in (0, 1/2)$) on the quasi-metric space (X, d_M). Then, for any $x, y \in X$ we have

$$d_M(Tx, Ty) \leq c[d_M(x, Ty) + d_M(y, Tx)].$$

We check that T is a (1)-Chatterjea contraction on $(X, M, *)$ (with constant $2c$).

Indeed, let $x, y \in X$ and $t > 0$ such that $\min\{M(x, Ty, t), M(y, Tx, t)\} > 1 - t$. By condition (C1), we get $\max\{d_M(x, Ty), d(y, Tx)\} < t$. Then

$$d_M(Tx, Ty) < 2ct,$$

and thus $M(Tx, Ty, 2ct) > 1 - 2ct$. By our assumption T has a fixed point. It follows from Theorem 3.5 that (X, d_M) is a complete quasi-metric space, so $(X, M, *)$ is complete by Proposition 4.8. □

References

[1] M. Abbas, B. Ali, S. Romaguera, Multivalued Caristi's type mappings in fuzzy metric spaces and a characterization of fuzzy metric completeness, Filomat 29 (2015) 1217-1222.

[2] C. Alegre, H. Dăg, S. Romaguera, P. Tirado, Characterizations of quasi-metric completeness in terms of Kannan-type fixed point theorems, Hacettepe J. Math. Stat. 46 (2017) 67-76.

[3] F. Castro-Company, S. Romaguera, P. Tirado, The bicompletion of fuzzy quasi-metric spaces, Fuzzy Sets Syst. 166 (2011) 56-64.

[4] S.K. Chatterjea, Fixed point theorems. C. R. Acad. Bulgare Sci. 25 (1972) 727-730.

[5] Y.J. Cho, M. Grabiec, V. Radu, On non Symmetric Topological and Probabilistic Structures, Nova Science Publisher, Inc. New York, 2006.

[6] S. Cobzaş, Functional Analysis in Asymmetric Normed spaces, Frontiers in Mathematics, Birkhaŭser/Springer Basel AG, Basel, Switzerland, 2013.

[7] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, Marcel Dekker, New York, 1982.

[8] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994) 395-399.

[9] A. George, P. Veeramani, On some results of analysis of fuzzy metric spaces, Fuzzy Sets Syst. 90 (1997) 365-368.

[10] V. Gregori, S. Romaguera, Fuzzy quasi-metric spaces, Appl. Gen. Topol. 5 (2004) 129-136.

[11] T.K. Hu, On a fixed point theorem for metric spaces, Amer. Math. Monthly 74 (1967) 436-437.

[12] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968) 71-76.

[13] A.W. Kirk, Caristi's fixed point theorem and metric convexity, Colloq. Math. 36 (1976) 81-86.

[14] E. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic, Dordrecht, 2000.

[15] I. Kramosil, J. Michalek, Fuzzy metrics and statistical metric spaces, Kybernetika 11 (1975) 326-334.

[16] R.P. Pant, A. Pant, R.M. Nikolić, S.N. Jěsić, A characterization of completeness of Menger PM-spaces, J. Fixed Point Theory Appl. (2019) 21:90.

[17] V. Radu, Some suitable metrics on fuzzy metric spaces, Fixed Point Theory 5 (2004) 323-347.

[18] S. Romaguera, A fixed point theorem of Kannan type that characterizes fuzzy metric completeness, Filomat 34 (2020) 4811-4819.

[19] S. Romaguera, w-distances on fuzzy metic spaces and fixed points, Mathematics 2020, **8**, 1909.

[20] S. Romaguera, P. Tirado, A characterization of quasi-metric completeness in terms of $\alpha - \psi$-contractive mappings having fixed points, Mathematics 2020, **8**, 16.

[21] S. Romaguera, P. Tirado, Characterizing complete fuzzy metric spaces via fixed point results, Mathematics 2020, **8**, 273.

[22] S. Romaguera, P. Tirado, Contractive self maps of $\alpha - \psi$-type on fuzzy metric spaces, Dyn. Syst. Appl. 30 (2021) 359-370.

[23] P.V. Subrahmanyam, Completeness and fixed-points, Mh. Math. 80 (1975) 325-330.

[24] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008) 1861-1869.

[25] T. Suzuki, W. Takahashi, Fixed point theorems and characterizations of metric completeness, Top. Methods Nonlinear Anal. 8 (1996) 371-382.