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A New Approach on Some Special Curves

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Abstract. In this paper, we obtain some characterizations for a Frenet curve with the help of an alternative frame different from the Frenet frame. Also, in the present study, we consider weak biharmonic and harmonic 1-type curves by using the mean curvature vector field of the curve. We also study 1-type and biharmonic curves whose mean curvature vector field is in the kernel of Laplacian. We give some theorems for such curves in the Euclidean 3-space. Moreover, we give the classifications of these types of curves.

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1. Introduction

In classical differential geometry, curves are usually characterized with the aid of the well-known Frenet frame. However, in some cases it may not be possible to obtain characterizations using this frame. Therefore, it may be useful to examine the curves with the help of another frame. Uzunoğlu et al., in a study they conducted in 2016, defined an alternative frame on the curve [13]. According to this alternative frame, while the normal vector of the curve and the normal vector according to the Frenet frame are the same along the curve, the tangent and binormal vectors are rotated around the normal vector to obtain C and W vector fields. This newly defined frame is an alternative frame of the curve and is called the frame $\{N, C, W\}$.

On the other hand, special curves are an important field of study of differential geometry. In this study some special curves, namely biharmonic, weak biharmonic, and harmonic 1-type curves, are discussed. 1-type and biharmonic curves were studied by Kocayiğit and Hacısalihoğlu in Euclidean 3-space [8]. Then, the biharmonic curves were moved according to Parallel Transport Frame to E^4 [9]. While Külahcı studied biharmonic curves in isotropic space, Öztürk et al. studied weak biharmonic and harmonic 1-type curves in semi-Euclidean space E_1^4 [10,11]. In addition, the biharmonic curves were studied in the 3-dimensional Heisenberg group [12]. Chen and Ishikawa classified biharmonic curves in pseudo-Euclidean space E_v^n [1]. They showed that every biharmonic curve lies in a 3-dimensional totally geodesic subspace. Further, Inoguchi gave a classification of biharmonic curves in semi-Euclidean 3-space. He pointed out that every biharmonic Frenet curve in Minkowski 3-space E_1^3 is a helix whose curvature and torsion satisfy $\tau^2 = \kappa^2$ [5, 6]. Kılıç examined finite-type curves and surfaces and gave a classification of harmonic and weak biharmonic surfaces [7].

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In this study, we mention the relationship between the alternative frame and the Frenet frame. Then, according to the alternative frame defined in [13], we give some characterizations of 1-type and biharmonic curves using the Laplace Beltrami operator $\triangle$ with the mean curvature vector field H. Then, we give characterizations of the weak biharmonic and the harmonic 1-type using the Levi-Civita Connection $\triangledown$ with the mean curvature vector field H.

2. Preliminaries

In this section, basic definitions and theories about the Serret-Frenet formulas and the alternative frame are given. In addition, basic information about the special curves are presented.

Definition 2.1. Let $\gamma : I \subset R \to E^3$ be a unit speed curve. Each unit speed curve has at least four continuous derivatives one can associate three orthogonal unit vector fields. T, N and B are tangent, the principal normal and the binormal vector fields, respectively. Uzunoğlu et al. [13] defined the alternative moving frame denoted by $\{N, C, W\}$ along the curve γ in Euclidean 3-space as

$$N(s) = N(s), C(s) = \frac{N'(s)}{\|N'(s)\|}, W(s) = N(s)xC(s).$$

For the derivatives of the alternative moving frame, we have

$$\begin{bmatrix} N' \\ C' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix}, \tag{2.1}$$

where f and g are curvatures of the curve γ as

$$\begin{aligned} f &= \sqrt{\kappa^2 + \tau^2}, \\ g &= \frac{\kappa^2}{\kappa^2 + \tau^2}(\frac{\tau}{\kappa})'. \end{aligned}$$

Theorem 2.2 ([3]). *Let γ be a nonplanar curve with arc length s in E^3. γ is a helix if and only if $g = 0$, where g is second curvature according to the alternative frame of γ.*

Remark 2.3 ([13]). Let γ be a unit speed curve in Euclidean 3-space in terms of the alternative moving frame apparatus $\{N, C, W, f, g\}$. Then, $\frac{g(s)}{f(s)} = c$ is a constant function if and only if γ is a slant helix. Also, the curve γ is a C-slant helix if and only if

$$\frac{(f^2 + g^2)^{\frac{3}{2}}}{f^2(\frac{g}{f})'} = constant.$$

Let (M^3, g) be a Riemannian 3-manifold. Let $\gamma = \gamma(s) : I \subset R \to M$ be an arclengthed curve. Namely the velocity vector field γ' satisfies $g(\gamma', \gamma') = 1$. A unit speed curve γ is said to be a geodesic if $\triangledown_{\gamma'}\gamma' = 0$, where $\triangledown$ is the Levi-Civita connection of the (M^3, g). In particular, an arclengthed curve γ is said to be a geodesic if $\kappa = 0$, where κ is the curvature of γ.

Let us denote the Laplace-Beltrami operator of γ by $\triangle$ and the mean curvature vector field along γ by H. The mean curvature vector field H is given with

$$H = \triangledown_{\gamma'}\gamma'. \tag{2.2}$$

Example 2.4 ([3]). Let us consider the curve α with arc length s in E^3 given by

$$\alpha(s) = (-\frac{1}{12}\cos(4s) - \frac{1}{3}\cos(2s), \frac{1}{12}\sin(4s) + \frac{1}{2}\sin(2s), -\frac{2\sqrt{2}}{3}\cos(s)),$$

(see Figure 1).

The mean curvature vector field H of the curve $\alpha(s)$ is calculated as

$$H = \triangledown_{\alpha'}\alpha' = (\frac{4}{3}\cos(4s) + \frac{4}{3}\cos(2s), -\frac{4}{3}\sin(4s) - 2\sin(2s), \frac{2\sqrt{2}}{3}\cos(s))$$

(see Figure 2).

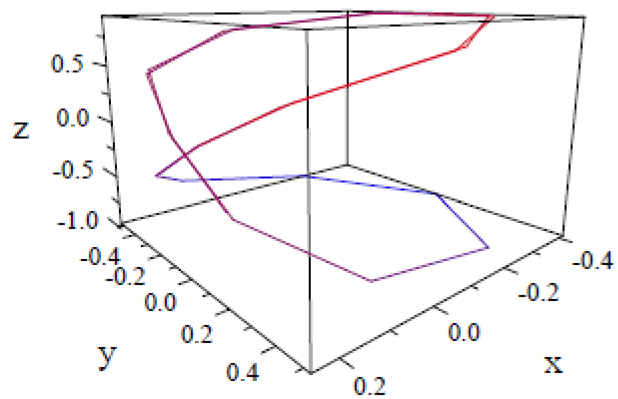

FIGURE 1. The curve $\alpha(s)$

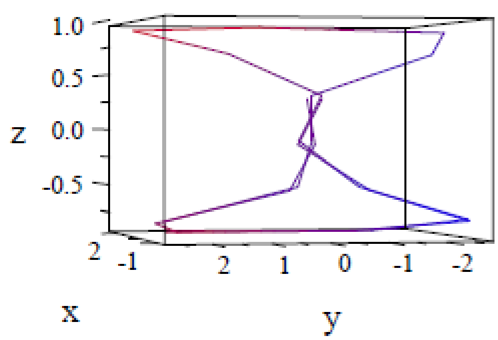

FIGURE 2. The mean curvature vector field H of the curve $\alpha(s)$

Definition 2.5 ([1,4]). The Laplacian operator of γ is defined by

$$\triangle = -\nabla^2_{\gamma'} = -\nabla_{\gamma'}\nabla_{\gamma'}. \tag{2.3}$$

The Laplacian operator along γ associated with the connection in the normal bundle is defined by

$$\triangle^{\perp} H = -\nabla^{\perp}_{\gamma'}\nabla^{\perp}_{\gamma'} H.$$

Definition 2.6. Let (M^3, g) be a Riemannian 3-manifold. Let $\gamma = \gamma(s) : I \subset R \to M$ be an arclengthed curve. The curve γ is a Frenet curve if $g(\gamma'', \gamma'') \neq 0$.

Definition 2.7 ([8]). A unit speed curve $\gamma : I \to M$ is said to be a 1-type curve if

$$\triangle H = \lambda H.$$

Definition 2.8 ([8]). A unit speed curve $\gamma : I \to M$ is said to be a biharmonic curve if

$$\triangle H = 0.$$

Lemma 2.9 ([1]). *The mean curvature vector field H is harmonic ($\triangle H = 0$) if and only if*

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = 0.$$

It is clear that in this case, the curve γ will be the biharmonic curve.

Theorem 2.10 ([2]). *If M is the Euclidean space E^m , then along the curve γ, H satisfies $\triangle H = 0$ if and only if γ is biharmonic (i.e., $\triangle(\triangle\gamma) = 0$ since $\triangle H = 0$).*

Definition 2.11 ([8]). A unit speed curve $\gamma : I \to M$ is said to be a harmonic 1-type curve if

$$\triangle^{\perp} H = \lambda H.$$

Definition 2.12 ([8]). A unit speed curve $\gamma : I \to M$ is said to be a weak biharmonic curve if

$$\Delta^{\perp} H = 0.$$

3. The Curves According to the Alternative Frame

3.1. **The Characterizations for the Frenet Curves.** In this section, we give a general characterization for the Frenet curves according to the alternative frame in E^3.

Theorem 3.1. *Let γ be an arclengthed Frenet curve in E^3. Then, γ satisfies the following differential equation according to the alternative frame*

$$\nabla^3_{\gamma'} N + \mu_1 \nabla^2_{\gamma'} N + \mu_2 \nabla_{\gamma'} N + \mu_3 N = 0, \tag{3.1}$$

where f, g are non-zero and

$$\begin{aligned} \mu_1 &= -(\frac{2f'}{f} + \frac{g'}{g}), \\ \mu_2 &= -\frac{f''}{f} + \frac{f'g'}{fg} + 2(\frac{f'}{f})^2 + f^2 + g^2, \\ \mu_3 &= fg(\frac{f}{g})'. \end{aligned}$$

Proof. By the system of equations (2.1) we get $C = \frac{1}{f} \nabla_{\gamma'} N$. From the second equation of the system (2.1), we have

$$W = \frac{1}{fg} \nabla^2_{\gamma'} N + \frac{1}{g}(\frac{1}{f})' \nabla_{\gamma'} N + \frac{f}{g} N. \tag{3.2}$$

Since $\nabla_{\gamma'} W = -gC$, we have

$$\nabla_{\gamma'} W = -\frac{g}{f} \nabla_{\gamma'} N. \tag{3.3}$$

Taking the covariant derivative of (3.2) and considering (3.3) we obtain the equation (3.1). □

Corollary 3.2. *Similar characterizations according to the other two components C, W of the alternative frame can be obtained for an arclengthed Frenet curve in E^3.*

Corollary 3.3. *Let γ be a helix in E^3. Then γ satisfies the following differential equation according to the alternative frame*

$$\frac{1}{f} \nabla^2_{\gamma'} N + (\frac{1}{f})' \nabla_{\gamma'} N + fN = 0. \tag{3.4}$$

Proof. Since γ is a helix for if and only if $g = 0$, by the equation system (2.1) or the equation (3.1) we get the equation (3.4). □

Corollary 3.4. *Let γ be a slant helix in E^3. Then, γ satisfies the following differential equation according to the alternative frame*

$$\nabla^3_{\gamma'} N - \frac{3f'}{f} \nabla^2_{\gamma'} N + [-\frac{f''}{f} + 3(\frac{f'}{f})^2 + (1 + c^2)f^2] \nabla_{\gamma'} N = 0. \tag{3.5}$$

Proof. Since γ is a slant helix for if and only if $\frac{g(s)}{f(s)} = c$, by the equation system (2.1) or the equation (3.1) we get the equation (3.5). □

3.2. **1-type and Biharmonic Curves.**

Theorem 3.5. *Let γ be the Frenet curve with arclengthed parametrized in E^3. Then, along the curve γ, $\triangle H = \lambda H$ holds if and only if*

$$\begin{aligned} ff' &= 0, \\ f^3 + fg^2 - f'' &= \lambda f, \\ 2f'g + fg' &= 0. \end{aligned} \tag{3.6}$$

Proof. From the equalities (2.1), (2.2), and (2.3) we get

$$\triangle H = 3ff'N + (f^3 + fg^2 - f'')C - (2f'g + fg')W.$$

By (2.2) and $\triangle H = \lambda H$ we have

$$3ff'N + (f^3 + fg^2 - f'')C - (2f'g + fg')W = \lambda fC.$$

Thus, the equations (3.6) are obtained. Conversely, the equations (3.6) satisfy the equality $\triangle H = \lambda H$. □

Theorem 3.6. *Let γ be the Frenet curve with arclengthed parametrized in E^3. Then, along the curve γ, $\triangle H = \lambda H$ holds if and only if γ is a circular helix according to the alternative frame, where*

$$\lambda = f^2 + g^2. \tag{3.7}$$

Proof. From Theorem (3.5) we have the equations (3.6). If γ is the Frenet curve, $\kappa \neq 0$ and thus $f \neq 0$. In this case, the equalities (3.6) show that f and g are constants. Thus, γ is a circular helix according to the alternative frame and we obtain the equation (3.7). □

Conversely, since γ is a circular helix according to the alternative frame, f and g are non-zero constants, and thus $\lambda = f^2 + g^2$. From that, $\triangle H = \lambda H$ is satisfied.

Corollary 3.7. *An arclengthed parametrized 1-type curve $\gamma : I \to M$ is a circular helix according to the alternative frame, for $\lambda = f^2 + g^2$.*

Theorem 3.8. *Let γ be an arclengthed Frenet curve in E^3. Then, along the curve γ, $\triangle H = 0$ holds if and only if γ is a straight line according to the alternative frame.*

Proof. Let I be an open interval and $\gamma : I \to M$ be an arclengthed curve. By (2.2), direct computation shows that

$$\nabla_{\gamma'} H = -f^2 N + f'C + fgW.$$

Let us compute the Laplacian of H

$$\begin{aligned} \triangle H &= -\nabla_{\gamma'} \nabla_{\gamma'} H \\ &= 3ff'N + (f^3 + fg^2 - f'')C - (2f'g + fg')W. \end{aligned}$$

Thus, along the curve γ, $\Delta H = 0$ holds if and only if $f = 0$. So, γ is a straight line according to the alternative frame in E^3. □

Conversely, every straight line in E^3 satisfies $\Delta H = 0$, since $f = 0$.

Corollary 3.9. *An arclengthed parametrized biharmonic curve $\gamma : I \to M$ is a straight line in E^3.*

3.3. **Harmonic 1-type and Weak Biharmonic Curves.** Let denote the normal bundle of the curve $\gamma : I \subset R \to M$ with $\chi^{\perp}(\gamma(I))$, where $\chi^{\perp}(\gamma(I)) = S_P\{C(s), W(s)\}$. For $\forall X \in \chi^{\perp}(\gamma(I))$, the normal connection $\nabla^{\perp}$ and the normal Laplace-Beltrami operator $\Delta^{\perp}$ are defined as

$$\begin{aligned} \nabla^{\perp}_N X &= \nabla_N X + \langle \nabla_N X, N \rangle N, \\ \Delta^{\perp}_N X &= -\nabla^{\perp}_N \nabla^{\perp}_N X. \end{aligned} \tag{3.8}$$

Theorem 3.10. *Let γ be an arclengthed parametrized Frenet curve in E^3. The curve γ is a harmonic 1-type curve according to the alternative frame if and only if*

$$\begin{aligned} ff' &= 0, \\ 2f^3 - f'' + fg^2 &= \lambda f \\ 2f'g + fg' &= 0. \end{aligned} \tag{3.9}$$

Proof. Since the curve γ is a harmonic 1-type curve, along the curve γ, the equality $\triangle^{\perp} H = \lambda H$ is provided**.** From (3.8),

$$\begin{aligned} \nabla^{\perp}_{\gamma'} H &= -2f^2 N + f'C + fgW, \\ \Delta^{\perp} H &= 5ff'N + (2f^3 - f'' + fg^2)C - (2f'g + fg')W. \end{aligned}$$

Thus, the equalities (3.9) are obtained. □

Theorem 3.11. *Let γ be a harmonic 1-type curve according to the alternative frame in E^3 and let s be its arclength function. Then,*
i) γ is a straight line according to the alternative frame,
ii) γ is a circular helix according to the alternative frame with the curvatures $f = c = constant$, $g = \sqrt{\lambda - 2c^2}$.

Proof. If we assume that $f = 0$, then the equations (3.9) are satisfied. So, γ is a straight line according to the alternative frame in E^3. If the $f = c$ is a constant function, then from the equations (3.9), we find $2f^2 + g^2 = \lambda$ and $g = \sqrt{\lambda - 2c^2}$ is a constant function. Thus, γ is a circular helix according to the alternative frame and we obtain the equation (3.9). □

Conversely, since γ is a circular helix according to the alternative frame, f and g are non-zero constants, and thus $\lambda = 2f^2 + g^2$. From that, $\triangle^{\perp} H = \lambda H$ is satisfied, then γ is the harmonic 1-type curve.

Theorem 3.12. *Let γ be an arclengthed parametrized curve in E^3. The curve γ is a weak biharmonic curve according to the alternative frame if and only if*

$$\begin{aligned} ff' &= 0, \\ 2f^3 - f'' + fg^2 &= 0, \\ 2f'g + fg' &= 0. \end{aligned} \tag{3.10}$$

Proof. The proof is provided with the equalities (3.10) and $\triangle^{\perp} H = 0$. □

Corollary 3.13. *Every straight line in E^3 is a weak biharmonic curve according to the alternative frame.*

Theorem 3.14. *Let γ be a weak biharmonic curve according to the alternative frame in E^3 and let s be its arclength function. Then, γ is a straight line.*

Proof. If we assume that $f = 0$, then the equations (3.10) are satisfied. So, the weak biharmonic curve γ according to the alternative frame is a straight line. □

Conversely, if γ is a straight line according to the alternative frame in E^3, since the equations (3.10) are satisfied for $f = 0$, then γ is a weak biharmonic curve.

4. Conclusion

In this paper we gave the characterizations of 1-type, biharmonic, weak biharmonic, and harmonic 1-type curves according to the alternative frame in Euclidean 3-Space. At the same time, we showed that every 1-type and harmonic 1-type curve is a circular helix according to the alternative frame,for $\lambda = f^2 + g^2$ and $2f^2 + g^2 = \lambda$, respectively. Moreover, every biharmonic and weak biharmonic curve is a straight line according to the alternative frame in Euclidean 3-Space. Also, we gave some characterizations for the curves according to the alternative frame.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors contributed equally and they have read and agreed to the published version of the manuscript.

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