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The nontrivial solutions for nonlinear fractional Schrödinger-Poisson system involving new fractional operator

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Abstract

In this paper, we investigate the existence of nontrivial solutions in the Bessel Potential space for nonlinear fractional Schrödinger-Poisson system involving distributional Riesz fractional derivative. By using the mountain pass theorem in combination with the perturbation method, we prove the existence of solutions.

Keywords: Fractional Schrödinger-Poisson system mountain pass theorem Bessel potential space perturbation method.
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1. Introduction

In the last decades, fractional problems have garnered the attention of numerous authors, because of their importance in quantum mechanics, nonlinear optics, semiconductor theory, and plasma physics. For more details on the physics aspects, we refer the interested readers to [5, 6, 9] and their references. In recent years, several results and works have been published for fractional Schrödinger-Poisson systems on the multiplicity of solutions, the existence of ground state solutions, and the existence of nontrivial solutions under various assumptions and conditions, see for instance [3, 7, 12, 13, 17, 19].

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In 1998, Benci and Fortunato [4] firstly proposed the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1}$$

for a bounded domain to describe the interaction of a charged particle for Schrödinger equations with an unknown electrostatic field. Systems similar to (1) has been widely studied via variational methods, see e.g, [1, 2, 18].

For the nonlocal case, by using the symmetric mountain pass theorem, authors [8] obtained the existence of infinitely many solutions for the following fractional system with sign-changing potential and $\lambda > 0$

$$\begin{cases} (-\Delta)^{\alpha} u + V(x)u + \lambda\phi u = g(x,u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\beta} \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{2}$$

We notice that when $\lambda = 1$, the multiplicity of solutions was obtained in [14] via the Fountain theorem without the Palais Smale condition. Moreover, Li in [15] proved the existence of non-trivial solutions when $V(x) \equiv 1$ and $\lambda = 1$ for the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^{\alpha} u + u + \phi u = g(x,u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\beta} \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{3}$$

where $2\beta + 4\alpha > 3$ and $(-\Delta)^s$ is the fractional Laplacian operator for $s = \alpha, \beta \in (0;1]$.

Very recently, an increasing number of researchers focus their attention on the problems related to the distributional Riesz fractional derivative in their works, see e.g [16, 21, 22], this fractional operator satisfies three basic physical invariance requirements as proved in Šilhavý paper [23]. In [21], the authors firstly studied a new class of fractional PDEs related to the distributional fractional gradient, they showed that it has interesting features of interest for the study of fractional problems in PDEs.

In light of the previous cited works, more precisely by [15] and [21], the main purpose of the present paper is to prove the existence of nontrivial solutions for a new class of fractional Schrödinger-Poisson system

$$\begin{cases} -D^{\alpha}.(D^{\alpha} u) + u + \phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -D^{\beta}.(D^{\beta}\phi) = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

where $\alpha, \beta \in (0;1]$; $2\beta + 4\alpha \geqslant 3$, and $-D^{\alpha}.(D^{\alpha}u)$ is the distributional Riesz fractional derivative, and we give its consistency with the usual fractional Laplacian in this work. The starting point of research pursued in [21] for the development of a general theory for fractional PDEs involving this operator, is the distributional Riesz fractional gradient D^{α} of order $\alpha \in (0,1)$ which is called the α-gradient for short. For $u \in L^p(\mathbb{R}^N)$, $1 < p < \infty$ such that $I_{1-\alpha} * u$ is well defined, they set

$$(D^{\alpha}u)_j = \frac{\partial^{\alpha} u}{\partial x_j^{\alpha}} = \frac{\partial}{\partial x_j} I_{1-\alpha} * u \ , \ 0 < \alpha < 1, \ j = 1, ..., N,$$

where $\dfrac{\partial}{\partial x_j}$ is defined in the sense, for every $w \in C_c^{\infty}(\mathbb{R}^N)$,

$$\langle \frac{\partial^{\alpha} u}{\partial x_j^{\alpha}}, w \rangle = (-1)\langle (I_{1-\alpha} * u), \frac{\partial w}{\partial x_j} \rangle = -\int_{\mathbb{R}^N} (I_{1-\alpha} * u) \frac{\partial w}{\partial x_j} dx,$$

where I_{α} is the Riesz potential of order α, $0 < \alpha < 1$ ([24]):

$$(I_{\alpha} * u)(x) = \gamma(N,\alpha) \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-\alpha}} dy \quad where \quad \gamma(N,\alpha) := \pi^{-\frac{N}{2}} 2^{-\alpha} \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}.$$

Thus, the α-gradient (D^α) and the α-divergence ($D^\alpha.$) can be written in finit integral form for sufficiently smoothness functions u and vector ϕ [16, 21, 22, 23], respectively by

$$D^\alpha u(x) = \gamma(N,\alpha)\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\frac{x-y}{|x-y|}dy,$$

and

$$D^\alpha.\phi(x) = \gamma(N,\alpha)\int_{\mathbb{R}^N} \frac{\phi(x)-\phi(y)}{|x-y|^{N+\alpha}}.\frac{x-y}{|x-y|}dy.$$

D^α has nice properties for $u \in C_0^\infty(\mathbb{R}^N)$ as shown in [21, 23], it corresponds to the fractional Laplacian as follows:

$$\begin{aligned}(-\Delta)^\alpha u &= -\sum_{j=1}^{N}\frac{\partial^\alpha}{\partial x_j^\alpha}\frac{\partial^\alpha}{\partial x_j^\alpha}u \\ &= -D^\alpha.(D^\alpha u), \end{aligned} \tag{5}$$

where the fractional Laplacian may be given [10], for $\alpha \in (0,1)$ by

$$(-\Delta)^\alpha u(x) = \frac{1}{2}\gamma^2(N,\alpha)\int_{\mathbb{R}^N}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2\alpha}}dy.$$

Furthermore, for $u, w \in C_0^\infty(\mathbb{R}^N)$ equation (5) is to be understood in this sense

$$\int_{\mathbb{R}^N} D^\alpha u.D^\alpha w dx = \int_{\mathbb{R}^N}(-\Delta)^\alpha u.w dx = \int_{\mathbb{R}^N}(-\Delta)^{\frac{\alpha}{2}}u.(-\Delta)^{\frac{\alpha}{2}}w dx,$$

which is particularily helpful for the variational formulation of PDEs involving non-local operators. We refer to [16, 21, 22, 23] for more detailed informations about this fractional operator.

In order to state our main result, we introduce the assumptions on the nonlinearity g and the potential V:

(g_1) : $g \in C(\mathbb{R}^3 \times \mathbb{R};\mathbb{R})$ for every $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, there exists constant $K_1 > 0$, and $p \in]2; 2_\alpha^*[$ such that

$$|g(x,u)| \le K_1(|u| + |u|^{p-1}),$$

where $2_\alpha^* = \frac{6}{3-2\alpha}$ the fractional critical Sobolev exponent .

(g_2) : $g(x,u) = 0, |u| \to 0$ uniformly on $\mathbb{R}^3$,

(g_3) : there exists $\kappa > 4$ such that

$$0 < \kappa G(x,u) \le ug(x,u),$$

holds for every $x \in \mathbb{R}^3$ and $u \in \mathbb{R}\backslash\{0\}$, where $G(x,u) = \int_0^u g(x,s)ds.$

(V): $V \in C\left(\mathbb{R}^3, \mathbb{R}\right)$, $\inf_{x\in\mathbb{R}^3} V(x) \ge V_0 > 0$, where V_0 is a constant, for every $M > 0$ $meas\left\{x \in \mathbb{R}^3\ V(x) \le M\right\} < \infty$, where meas represents the Lesbesgue measure. The main result is the following theorem:

Theorem 1.1. *Suppose that (g_1)-(g_3) and (V) are satisfied. Then, problem (4) possesses at least a nontrivial solution.*

Remark 1.2. *In this paper, we do not need the condition $g \in C^1(\mathbb{R},\mathbb{R})$ since the perturbed method is used. The main idea of this method is to obtain the existence of critical values of the perturbed functional J_λ for sufficiently small $\lambda > 0$, and taking $\lambda \to \infty$ to get solutions of original problems.*

This paper is organized as follows. In section 2, we present some facts about the fractional Sobolev spaces and technical results. In section 3, we use the perturbation method and the mountain pass theorem to prove our main result. In section 4, we give a discussion about our research results.

We next fix the following notations. Let $L^p\left(\mathbb{R}^3\right)$ $(p \in [1,\infty))$ be the usual Lesbesgue space with the norm $\|u\|_p = (\int_{\mathbb{R}^3}|u|^p dx)^{\frac{1}{p}}.$

2. Preliminaries

In this section, we will give some basic definitions on the fractional Sobolev spaces which will be useful along the paper. For any $p \in [1, +\infty)$ and $\alpha \in (0,1)$, the fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^3)$ is defined as

$$W^{\alpha,p}(\mathbb{R}^3) = \left\{ u \in L^p(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^p}{|x-y|^{3+p\alpha}} dxdy < +\infty \right\},$$

endowed with the norm

$$\|u\|_{W^{\alpha,p}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^p}{|x-y|^{3+p\alpha}} dxdy + \int_{\mathbb{R}^3} |u|^p dx \right)^{\frac{1}{p}}.$$

For $p = 2$, the space $W^{\alpha,2}(\mathbb{R}^3)$ is simply denoted by $H^\alpha(\mathbb{R}^3)$. Moreover, The fractional Sobolev space $D^{\alpha,2}(\mathbb{R}^3)$ for $\alpha \in (0,1)$, is defined by

$$D^{\alpha,2}(\mathbb{R}^3) = \left\{ u \in L^{2^*_\alpha}(\mathbb{R}^3) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{3}{2}+\alpha}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the Gagliardo norm

$$\|u\|_{D^{\alpha,2}(\mathbb{R}^3)} = (\int_{\mathbb{R}^3} |D^\alpha u|^2 dx)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2\alpha}} dxdy \right)^{\frac{1}{2}},$$

endowed with the inner product

$$\langle u, w \rangle_{D^{\alpha,2}} = \int_{\mathbb{R}^3} D^\alpha u . D^\alpha w dx.$$

Next, for $\alpha \in (0,1)$. Since the problem (4) involves the distributional Riesz fractional gradient, we will introduce the natural setting for solving (4), which is called the vector space of fractional differentiable functions $X^{\alpha,2}(\mathbb{R}^3)$, and defined for $u \in C_0^\infty(\mathbb{R}^3)$ as the closure of $C_0^\infty(\mathbb{R}^3)$ with the following norm

$$\|u\|^2_{X^{\alpha,2}(\mathbb{R}^3)} = \|u\|^2_{L^2(\mathbb{R}^3)} + \|D^\alpha u\|^2_{L^2(\mathbb{R}^3)}. \tag{6}$$

By Theorem 1.7 in [21], $X^{\alpha,2}(\mathbb{R}^3)$ it is exactly the Bessel potential space $L^{\alpha,2}(\mathbb{R}^3)$ defined for $\alpha \in \mathbb{R}_+$ by ([21, 24])

$$\mathrm{L}^{\alpha,2}(\mathbb{R}^3) = \{u : u = G_\alpha * f \quad for\ some\ f \in L^2(\mathbb{R}^3)\},$$

where the Bessel potentials G_α are defined by ([21, 24])

$$G_\alpha(x) := \frac{1}{(4\pi)^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})} \int_0^{+\infty} e^{\frac{-\pi|x|^2}{t}} e^{\frac{-t}{4\pi}} t^{\frac{\alpha-3}{2}-1} dt.$$

The norm of the Bessel potential space is $\|u\|_{\mathrm{L}^{\alpha,2}(\mathbb{R}^3)} = \|f\|_{\mathrm{L}^2(\mathbb{R}^3)}$ if $u = G_\alpha * f$.
Now, we summarize the main properties of this Bessel space (see [11, 21]).

Theorem 2.1.
1. *If $\alpha \in (0,1)$, then $H^\alpha(\mathbb{R}^3) = W^{\alpha,2}(\mathbb{R}^3) = \mathrm{L}^{\alpha,2}(\mathbb{R}^3) = X^{\alpha,2}(\mathbb{R}^3)$.*
2. *If $\alpha \geq 0$ and $2 \leq q \leq 2^*_\alpha$, then $L^{\alpha,2}(\mathbb{R}^3)$ is continuously embedded in $L^q(\mathbb{R}^3)$, and the embedding is locally compact if $2 \leq q < 2^*_\alpha$.*

Remark 2.2. *From Theorem 2.1, the Bessel potential space $L^{\alpha,2}(\mathbb{R}^3)$ is topologically undistinguishable from the well known fractional Sobolev space $H^\alpha(\mathbb{R}^3)$, and the norms in the two spaces being equivalent given by (6).*

The solvability of the linear fractional PDEs with variable coefficients is established by the following theorem.

Theorem 2.3. *([21]) Let $\Omega \subset \mathbb{R}^3$ is an arbitrary bounded open set. Suppose that $v \in L^{\alpha,2}(\mathbb{R}^3)$ and $h \in L^2(\Omega)$, such that $I_{1-\alpha} * v$ is well defined and $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^{3\times 3}$ with coefficients bounded and measurable such that*

$$a|y|^2 \leq A(x)y.y \quad and \quad A(x)y.y \leq b|y|^2$$

For all $x \in \mathbb{R}^3$, some $a, b > 0$ and all $y \in \mathbb{R}^3$. Then, there exists a unique $u \in L^{\alpha,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^N} A(x)D^\alpha u.D^\alpha w dx = \int_\Omega hw dx,$$

for every $w \in L^{\alpha,2}(\mathbb{R}^3)$ and $u = v$ in $\mathbb{R}^3\backslash\Omega$. In this work A is the identity.

Lemma 2.4. *(See [10]) For every $\alpha \in \left(0, \frac{3}{2}\right)$, $D^{\alpha,2}(\mathbb{R}^3)$ is continuously embedded in $L^{2^*_\alpha}(\mathbb{R}^3)$, i.e there exists $K_\alpha > 0$ such that :*

$$\left(\int\limits_{\mathbb{R}^3} |u|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}} \leq K_\alpha \int\limits_{\mathbb{R}^3} |D^\alpha u|^2 dx,\ u \in D^{\alpha,2}\left(\mathbb{R}^3\right).$$

The linear operator $\mathcal{L}_u : D^{\beta,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined by:

$$\mathcal{L}_u(w) = \int\limits_{\mathbb{R}^3} u^2 w dx.$$

By Hölder inequality, Theorem 2.1 and Lemma 2.4, we derive

$$|\mathcal{L}_u(w)| \leq \|u\|^2_{L^{\frac{12}{3+2\beta}}} \|w\|_{L^{2^*_\beta}} \leq K \|u\|^2_{L^{\alpha,2}} \|w\|_{D^{\beta,2}}. \tag{7}$$

According to the Lax-Milgram theorem, for every $u \in L^{\alpha,2}(\mathbb{R}^3)$, there exists a unique $\phi^\beta_u \in D^{\beta,2}\left(\mathbb{R}^3\right)$ such that

$$\int\limits_{\mathbb{R}^3} D^\beta \phi^\beta_u . D^\beta w dx = \int\limits_{\mathbb{R}^3} u^2 w dx \quad \forall w \in D^{\beta,2}\left(\mathbb{R}^3\right), \tag{8}$$

i.e. ϕ^β_u is a weak solution of $-D^\beta.\left(D^\beta \phi^\beta_u\right) = u^2$. Moreover,

$$\left\|\phi^\beta_u\right\|_{D^{\beta,2}} = \|\mathcal{L}_u\| \leq K \|u\|^2_{L^{\alpha,2}}. \tag{9}$$

Since $2\beta + 4\alpha \geq 3$ and $\beta \in (0, 1]$, then $\frac{12}{3+2\beta} \in (2, 2^*_\alpha)$. By Lemma 2.4, (7) and (8), we have

$$\left\|\phi^\beta_u\right\|^2_{D^{\beta,2}} = \int\limits_{\mathbb{R}^3} \left|D^\beta \phi^\beta_u\right|^2 dx = \int\limits_{\mathbb{R}^3} u^2 \phi^\beta_u dx,$$

and

$$\left\|\phi^\beta_u\right\|^2_{D^{\beta,2}} \leq \|u\|^2_{\frac{12}{3+2\beta}} \left\|\phi^\beta_u\right\|_{L^{2^*_\beta}} \leq K \|u\|^2_{\frac{12}{3+2\beta}} \left\|\phi^\beta_u\right\|_{D^{\beta,2}}. \tag{10}$$

Then

$$\left\|\phi^\beta_u\right\|_{D^{\beta,2}} \leq K \|u\|^2_{\frac{12}{3+2\beta}}. \tag{11}$$

We have, for $x \in \mathbb{R}^3$

$$\phi^\beta_u(x) = c_\beta \int\limits_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2\beta}} dy, \tag{12}$$

which is called β-Riesz potential (see [24]), where

$$c_\beta = \pi^{-\frac{3}{2}} \frac{\Gamma\left(\frac{3-2\beta}{2}\right)}{2^{2\beta}\Gamma(\beta)}.$$

We get the fractional Schrödinger equation by replacing ϕ_u^β in (4)

$$-D^\alpha.(D^\alpha u) + u + \phi_u^\beta u = g(x,u), \quad x \in \mathbb{R}^3. \tag{13}$$

The energy functional $J : L^{\alpha,2}(\mathbb{R}^3) \to \mathbb{R}$ that correspond to (4) is defined as

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|D^\alpha u|^2 + u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^\beta u^2 dx - \int_{\mathbb{R}^3} G(x,u)\,dx.$$

Therefore, J is well defined in $L^{\alpha,2}\left(\mathbb{R}^3\right)$ and $J \in C^1(L^{\alpha,2}\left(\mathbb{R}^3\right), \mathbb{R})$. Moreover, its derivative is

$$< J'(u), w >= \int_{\mathbb{R}^3} (D^\alpha u.D^\alpha w + uw + \phi_u^\beta uw - g(x,u)w)dx, \ w \in L^{\alpha,2}\left(\mathbb{R}^3\right). \tag{14}$$

Theorem 2.5. 1. *$(u,\phi) \in L^{\alpha,2}\left(\mathbb{R}^3\right) \times D^{\beta,2}\left(\mathbb{R}^3\right)$ is a weak solution of (4) if u is a weak solution of (13)* .

2. *u is a weak solution of (13) for any $w \in L^{\alpha,2}\left(\mathbb{R}^3\right)$, if*

$$\int_{\mathbb{R}^3} \left(D^\alpha u.D^\alpha w + uw + \phi_u^\beta uw - g(x,u)\,w\right)dx = 0.$$

Now, we define the work space for (4) *by*

$$H = \left\{ u \in L^{\alpha,2}\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} (|D^\alpha u|^2 + V(x)u^2)dx < +\infty \right\},$$

which is a Hilbert space equipped with the norm $\|u\|_H = \langle u,u\rangle_H^{\frac{1}{2}}$, and the inner product $\langle u,w\rangle_H = \int_{\mathbb{R}^3} (D^\alpha u.D^\alpha w + V(x)\,uw)dx$.

Lemma 2.6. *(Theorem 6.5 in [10]) Space H is compactly embedded in $L^q\left(\mathbb{R}^3\right)$ for $q \in [2, 2_\alpha^*)$, and continuously embedded in $L^q\left(\mathbb{R}^3\right)$ for $q \in [2, 2_\alpha^*]$.*

Hence, there exists $K_0 > 0$ such that

$$\|u\|_{L^q} \le K_0 \|u\|_H, \quad \forall q \in [2, 2_\alpha^*].$$

For a constant $\lambda \in (0,1]$, we introduce:

$$\langle u,w\rangle_{H_\lambda} = \int_{\mathbb{R}^3} (D^\alpha u.D^\alpha w + \lambda V(x)uw)\,dx,$$

and the norm $\|u\|_{H_\lambda} = \langle u,u\rangle_{H_\lambda}^{\frac{1}{2}}$. Denote $H_\lambda = \left(H, \|.\|_{H_\lambda}\right)$.
Consider the perturbed functional $J_\lambda : H \to \mathbb{R}$ defined as follows

$$J_\lambda(u) = J(u) + \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)u^2 dx, \ \lambda \in (0,1]. \tag{15}$$

Lemma 2.7. *Let* (g_1) *and* (g_2) *hold. If* $V(x) \geq 0$, *then there exists* $\sigma, \varrho > 0$ *such that for fixed* $\lambda \in (0,1]$,

$$\inf_{u \in H, \|u\|_H = \varrho} J_\lambda(u) > \sigma,$$

where σ *and* ϱ *are independent of* λ.

Proof. From (g_1), (g_2), for any $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$|g(x,u)| \leq \varepsilon |u| + K_\varepsilon |u|^{p-1}, u \in \mathbb{R}.$$

Then

$$|G(x,u)| \leq \frac{\varepsilon}{2}|u|^2 + \frac{K_\varepsilon}{p}|u|^p.$$

For $\varrho > 0$, set

$$S_\varrho = \{u \in H : \|u\|_H \leq \varrho\}.$$

Since $p \in (2, 2_\alpha^*)$, for $u \in \partial S_\varrho$ and from Lemma 2.6

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|_H^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)\, u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^\beta(x)\, u^2 dx - \int_{\mathbb{R}^3} G(x,u)\, dx \\ &\geq \frac{(1-\varepsilon)\varrho^2}{2} - \frac{K_\varepsilon K_0}{p}\varrho^p. \end{aligned}$$

For ϱ small enough and $\varepsilon \in (0,1)$, the conclusion follows. □

Lemma 2.8. *If* (g_3) *holds, there exists* $e \in H$ *with* $\|e\|_H > \varrho$ *such that* $J_\lambda(e) < 0$ *for fixed* $\lambda \in (0,1]$, *where* ϱ *is the same as in Lemma 2.7.*

Proof. According to (g_3), there exists a constant $K > 0$ such that

$$G(x,u) \geq K|u|^\kappa \quad for\ |u|\ \ large. \tag{16}$$

From (9) and (10)

$$\int_{\mathbb{R}^3} \phi_u^\beta u^2 dx = \left\|\phi_u^\beta\right\|_{D^{\beta,2}}^2 \leq K\|u\|_{L^{\alpha,2}}^4. \tag{17}$$

For $\gamma > 0$ and $w \in C_0^\infty(\mathbb{R}^3)$, from (15),(16) and (17), we get

$$\begin{aligned} J_\lambda(\gamma w) &= \frac{\gamma^2}{2}\|w\|_{H_\lambda}^2 + \frac{\gamma^2}{2}\|w\|_{L^2}^2 + \frac{\gamma^2}{4}\int_{\mathbb{R}^3} \phi_u^\beta w^2 dx - \int_{\mathbb{R}^3} G(x,\gamma w) dx \\ &\leq \frac{\gamma^2}{2}\|w\|_H^2 + \frac{\gamma^2}{2}\|w\|_{L^2}^2 + \frac{K\gamma^2}{4}\|w\|_{L^{\alpha,2}}^4 - K\gamma^\kappa\|w\|_{L^\kappa}^\kappa \to -\infty \end{aligned}$$

as $\gamma \to +\infty$. Define a path $f : [0,1] \to H$ by $f(\tau) = \tau\gamma w$. For sufficiently large γ, we get

$$\|f(1)\|_H = \Big(\int_{\mathbb{R}^3} (|D^\alpha f(1)|^2 + V(x) f^2(1)) dx\Big)^{\frac{1}{2}} > \varrho \ \ and \ \ J_\lambda(f(1)) < 0.$$

Choosing $e = f(1) = \gamma w$, we get the conclusion. □

3. Proof of main results

Lemma 3.1. *Suppose that (g_1),(g_3) and (V) hold. Then, J_λ satisfies the Palais-Smale (PS) condition on H for fixed $\lambda \in (0,1]$.*

Proof. Let $\{u_n\}$ be a (PS) sequence in H. We will prove that $\{u_n\}$ has a convergent subsequence in H, then

$$\begin{aligned} K+\|u_n\|_H &\geq J_\lambda(u_n)-\frac{1}{\kappa}\left\langle J'_\lambda(u_n),(u_n)\right\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\kappa}\right)\|u_n\|^2_{H_\lambda}+\left(\frac{1}{2}-\frac{1}{\kappa}\right)\|u_n\|^2_{L^2}+\left(\frac{1}{4}-\frac{1}{\kappa}\right)\int\limits_{\mathbb{R}^3}\phi^\beta_{u_n}u_n^2dx \\ &+ \int\limits_{\mathbb{R}^3}\left(\frac{u_n g(x,u_n)}{\kappa}-G(x,u_n)\right)dx \\ &\geq \left(\frac{1}{2}-\frac{1}{\kappa}\right)\lambda\|u_n\|^2_H, \end{aligned} \tag{18}$$

which means that $\{u_n\}$ is bounded in H. Up to a subsequence, we suppose that $u_n \rightharpoonup u$ in H. From Lemma 2.6, we conclude that $u_n \to u$ in $L^p(\mathbb{R}^3)$ where $2 \leq p < 2^*_\alpha$. Combining (14) and (15), we get

$$\begin{aligned} &\|u_n-u\|^2_{H_\lambda}=\langle J'_\lambda(u_n)-J'_\lambda(u),u_n-u\rangle-\|u_n-u\|^2_{L^2} \\ &-\int\limits_{\mathbb{R}^3}\left(\phi^\beta_{u_n}u_n-\phi^\beta_u u\right)(u_n-u)\,dx+\int\limits_{\mathbb{R}^3}\left(g(x,u_n)-g(x,u)\right)(u_n-u)dx. \end{aligned} \tag{19}$$

Obviously, we have

$$\left\langle J'_\lambda(u_n)-J'_\lambda(u),u_n-u\right\rangle \to 0 \textit{ and } \|u_n-u\|^2_{L^2}\to 0 \textit{ as } n\to\infty. \tag{20}$$

From the generalization of Hölder inequality, Lemma 2.4 and (11), we obtain

$$\begin{aligned} \left|\int\limits_{\mathbb{R}^3}\phi^\beta_{u_n}u_n(u_n-u)\,dx\right| &\leq \left\|\phi^\beta_{u_n}\right\|_{L^{2^*_\beta}}\|u_n\|_{L^{\frac{12}{3+2\beta}}}\|u_n-u\|_{L^{\frac{12}{3+2\beta}}} \\ &\leq K\left\|\phi^\beta_{u_n}\right\|_{D^{\beta,2}}\|u_n\|_{L^{\frac{12}{3+2\beta}}}\|u_n-u\|_{L^{\frac{12}{3+2\beta}}} \\ &\leq K\|u_n\|^3_{L^{\frac{12}{3+2\beta}}}\|u_n-u\|_{L^{\frac{12}{3+2\beta}}} \\ &\leq K\|u_n\|^3_H\|u_n-u\|_{L^{\frac{12}{3+2\beta}}} \end{aligned}$$

Similarly, we obtain

$$\left|\int\limits_{\mathbb{R}^3}\phi^\beta_u u(u_n-u)\,dx\right| \leq K\|u\|^3_H\|u_n-u\|_{L^{\frac{12}{3+2\beta}}}.$$

We have

$$\begin{aligned} &\left|\int\limits_{\mathbb{R}^3}\left(\phi^\beta_{u_n}u_n-\phi^\beta_u u\right)(u_n-u)\,dx\right| \\ &\leq \left|\int\limits_{\mathbb{R}^3}\phi^\beta_{u_n}u_n(u_n-u)\,dx\right|+\left|\int\limits_{\mathbb{R}^3}\phi^\beta_u u(u_n-u)\,dx\right| \to 0 \quad \textit{as } n\to\infty. \end{aligned} \tag{21}$$

From (g_1), Hölder inequality and Minkowski inequality

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \left(g\left(x,u_n\right)-g\left(x,u\right)\right)\left(u_n-u\right)dx\right| \\
&\leq K_1 \int_{\mathbb{R}^3} \left(\left|u_n\right|+\left|u\right|\right)\left|u_n-u\right|dx + K_1 \int_{\mathbb{R}^3} \left(\left|u_n\right|^{p-1}+\left|u\right|^{p-1}\right)\left|u_n-u\right|dx \\
&\leq K_1 \left(\left\|u_n\right\|_{L^2}+\left\|u\right\|_{L^2}\right)\left\|u_n-u\right\|_{L^2} + K_1 \left(\left\|u_n\right\|_{L^p}^{p-1}+\left\|u\right\|_{L^p}^{p-1}\right)\left\|u_n-u\right\|_{L^p} \\
&\leq K \left(\left\|u_n\right\|_{H}+\left\|u\right\|_{H}\right)\left\|u_n-u\right\|_{L^2} + K \left(\left\|u_n\right\|_{H}^{p-1}+\left\|u\right\|_{H}^{p-1}\right)\left\|u_n-u\right\|_{L^p} \to 0
\end{aligned} \tag{22}$$

as $n \to \infty$. From (19)-(22), it follows that $\{u_n\}$ converges strongly in H for fixed $\lambda \in (0,1]$. □

Theorem 3.2. *Suppose that (g_3) holds. Let $\lambda_n \to 0$ and let $\{u_n\} \subset H$ be a sequence of critical points of J_{λ_n} satisfying $J_{\lambda_n}(u_n) \leq K$ for some K independent of n and $J'_{\lambda_n}(u_n) = 0$. Then, up to a subsequence as $n \to \infty$, $u_n \rightharpoonup u$ in $L^{\alpha,2}(\mathbb{R}^3)$, u is a critical point of J.*

Proof. From $J'_{\lambda_n}(u_n) = 0$ and $J_{\lambda_n}(u_n) \leq K$, we have

$$\begin{aligned}
K &\geq J_\lambda(u_n) - \frac{1}{\kappa}\left\langle J'_\lambda(u_n),(u_n)\right\rangle \\
&= \left(\frac{1}{2}-\frac{1}{\kappa}\right)\left\|u_n\right\|_{H_\lambda}^2 + \left(\frac{1}{2}-\frac{1}{\kappa}\right)\left\|u_n\right\|_{L^2}^2 \\
&+ \left(\frac{1}{4}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3}\phi_{u_n}^\beta u_n^2 dx + \int_{\mathbb{R}^3}\left(\frac{u_n g(x,u_n)}{\kappa} - G(x,u_n)\right)dx \\
&\geq \left(\frac{1}{2}-\frac{1}{\kappa}\right)\left\|u_n\right\|_{L^{\alpha,2}}^2 + \left(\frac{1}{2}-\frac{1}{\kappa}\right)\int_{\mathbb{R}^3}\lambda_n V(x) u_n^2 dx.
\end{aligned} \tag{23}$$

Then, up to a subsequence, we obtain $u_n \rightharpoonup u$ in $L^{\alpha,2}(\mathbb{R}^3)$.
By Lemma 2.3 in [20], $\phi_{u_n}^\beta \to \phi_u^\beta$ in $D^{\beta,2}(\mathbb{R}^3)$, using $w \in C_0^\infty(\mathbb{R}^3)$, we derive

$$\int_{\mathbb{R}^3}\phi_{u_n}^\beta uwdx \to \int_{\mathbb{R}^3}\phi_u^\beta uwdx, \quad \text{as } n \to \infty.$$

By the generalization of Hölder inequality, we obtain

$$\left|\int_{\mathbb{R}^3}\phi_{u_n}^\beta\left(u_n-u\right)wdx\right| \leq \left\|\phi_{u_n}^\beta\right\|_{L^{2^*_\beta}}\left\|u_n-u\right\|_{L^{\frac{12}{3+2\beta}}(\Omega)}\left\|w\right\|_{L^{\frac{12}{3+2\beta}}(\Omega)} \to 0$$

as $n \to \infty$, and Ω is the support of w. Then for all $w \in C_0^\infty(\mathbb{R}^3)$, we have

$$\left|\int_{\mathbb{R}^3}\phi_{u_n}^\beta u_n w - \int_{\mathbb{R}^3}\phi_u^\beta uwdx\right| \leq \left|\int_{\mathbb{R}^3}\left(\phi_{u_n}^\beta - \phi_u^\beta\right)uwdx\right| + \left|\int_{\mathbb{R}^3}\phi_{u_n}^\beta u_n\left(u_n-u\right)wdx\right|$$

$\to 0$ *as* $n \to \infty$. Combining (14) and (15), we derive

$$\begin{aligned}
\left\langle J'_\lambda(u_n),w\right\rangle &= \int_{\mathbb{R}^3}\left(D^\alpha u_n . D^\alpha w + u_n w + \phi_{u_n}^\beta u_n w - g(x,u_n)w\right)dx \\
&+ \lambda_n \int_{\mathbb{R}^3} V(x) u_n w dx,
\end{aligned}$$

where $w \in C_0^\infty(\mathbb{R}^3)$. From (23) and Hölder inequality, we obtain

$$\begin{aligned}\lambda_n \int_{\mathbb{R}^3} V(x)\, u_n w dx &= \lambda_n \int_{\mathbb{R}^3} \left(\sqrt{V(x)} u_n\right)\left(\sqrt{V(x)} w\right) dx \\ &\leq \lambda_n^{\frac{1}{2}} \int_{\mathbb{R}^3} \left(\lambda_n V(x)\, u_n^2 dx\right)^{\frac{1}{2}} \int_{\Omega} \left(V(x)\, w^2 dx\right)^{\frac{1}{2}} \to 0\end{aligned}$$

as $n \to \infty$. Thus, $J'(u)\, w = 0$ for all $w \in C_0^\infty(\mathbb{R}^3)$. Since $C_0^\infty(\mathbb{R}^3)$ is dense in $L^{\alpha,2}(\mathbb{R}^3)$ (see Theorem 2.2 in [21]). Then, $J'(u)\, w = 0$ for all $w \in L^{\alpha,2}(\mathbb{R}^3)$, u is a critical point of J. □

Before we prove **Theorem 1.1**, we need the following vanishing Lemma.

Lemma 3.3. *(Lemma 3.4 in [25]) If $\{u_n\}$ be a bounded in $H^\alpha(\mathbb{R}^3)$ and for $q \in [2, 2_\alpha^*)$, we have for some $R > 0$*

$$\sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |u_n|^q \to 0 \quad as \quad n \to \infty,$$

then, $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for every $r \in (2, 2_\alpha^*)$.

Proof of Theorem 1.1 Choosing $e_0 \in C_0^\infty(\mathbb{R}^3)$ and $\tau > 0$.
By Lemma 2.8, we have that for $\kappa > 4$

$$J_\lambda(\tau e_0) \leq \frac{\tau^2}{2} \|e_0\|_H^2 + \frac{\tau^2}{2} \|e_0\|_{L^2}^2 + \frac{K\tau^4}{2} \|e_0\|_{L^{\alpha,2}}^4 - C\tau^\kappa \|e_0\|_\kappa^\kappa \leq -\infty \tag{24}$$

as $n \to +\infty$. Define

$$c_\lambda = \inf_{\gamma \in \Gamma} \sup_{\tau \in [0,1]} J_\lambda(\gamma(\tau)),$$

where $\Gamma = \{\gamma \,|\, \gamma \in C([0,1], H),\ \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}$. By (24), there exists a constant $c > 0$, independent of λ, such that

$$c_\lambda \leq \sup_{\tau \geq 0} J_\lambda(\tau e_0) \leq c.$$

By Lemma 2.7, the mountain pass theorem holds and c_λ is a critical value of J_λ, $c_\lambda > \sigma > 0$, where σ is the same as in Lemma 2.7. As a result, we can choose $\lambda_n \to 0$, and a sequence of critical points $\{u_n\} \subset H$ satisfying $J'_\lambda(u_n) = 0$ and $J_\lambda(u_n) \leq c$. By Theorem 3.2 up to a subsequence $u_n \rightharpoonup u$ in $L^{\alpha,2}(\mathbb{R}^3)$, and u is a critical point of J. It remains to demonstrate that $u \neq 0$, we argue by contradiction. Assume that $u = 0$, From Lemma 3.3, either (i) or (ii) below holds:

(i) $\lim\limits_{n \to +\infty} \int_{\mathbb{R}^3} |u_n|^q dx = 0, \quad$ for $q \in (2, 2_\alpha^*)$.

(ii) There exists a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \mu > 0$ such that

$$\lim_{n \to \infty} \int_{B_R(y_n)} |u_n|^2 dx > \mu > 0.$$

If (i) occurs, by (g_1) and (g_2), we have $\lim\limits_{n \to \infty} \int_{\mathbb{R}^3} g(x, u_n) u_n dx = 0$.

From (12), $\phi_u^\beta(x) \geq 0$. Since

$$\begin{aligned}0 &= J'_\lambda(u_n) u_n \\ &= \int_{\mathbb{R}^3} (|D^\alpha u_n|^2 + u_n^2) dx + \int_{\mathbb{R}^3} \phi_{u_n}^\beta(x) u_n^2 dx - \int_{\mathbb{R}^3} g(x, u_n) u_n dx + \lambda_n \int_{\mathbb{R}^3} V(x) u_n^2 dx,\end{aligned}$$

We have

$$\|u_n\|^2_{L^{\alpha,2}} \leq \|u_n\|^2_{L^{\alpha,2}} + \int\limits_{\mathbb{R}^3} \phi^\beta_{u_n}(x)u_n^2 dx = \int\limits_{\mathbb{R}^3} g(x,u_n)u_n dx + \lambda_n \int\limits_{\mathbb{R}^3} V(x)u_n^2 dx.$$

It follows that $u_n \to 0$ in $L^{\alpha,2}(\mathbb{R}^3)$. Then $J_\lambda(u_n) \to 0$, this contradicts with $J_\lambda(u_n) \to c_\lambda \geq \sigma > 0$. Thus, (ii) is valid. This completes the proof.

4. Conclusion

This paper is devoted to the study of the existence of non-trivial solutions for a new class of fractional Schrödinger-Poisson system. By applying the perturbation method with the mountain pass theorem with (PS) condition we obtained the existence of a critical point for functional J, which in turn proves the existence of a non-trivial solution. As far as we are aware, our attempt is new because we utilize a new fractional Laplacian in this type of system. We wish that the present paper will open wide avenues for further research in the field of the distributional Riesz fractional derivative.

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