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Numerical method to solve generalized nonlinear system of second-order boundary value problems: Galerkin approach

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Abstract

In this study, we consider the system of second-order nonlinear boundary value problems (BVPs). We focus on the numerical solutions of different types of nonlinear BVPs by Galerkin finite element method (GFEM). First of all, we originate the generalized formulation of GFEM for those type of problems. Then we determine the approximate solutions of a couple of second-order nonlinear BVPs by GFEM. The numerical results are unfolded in tabuler form and portrayed graphically along with the exact solutions. Those results demonstrate the applicability, compatibility and accuracy of this scheme.

Keywords: Galerkin finite element method Nonlinear system of equations Higher order nonlinear BVPs Trial function .
2010 MSC: 92D25, 35K57, 35K61, 37N25

1. Introduction

Numerical methods amplify accurate and rapid approximations to problems whose exact solutions are very difficult to obtain because of their perplexity. In real life, we have to face many linear and nonlinear higher-order scientific and engineering problems containing different types of boundary conditions. A considerable number of problems are from the nonlinear system of differential equations. Researchers attempt to solve

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those problems either analytically or numerically with the help of different kinds of powerful and effective methods in order to require higher accuracy. Among them the Finite Difference Method (FDM), Galerkin Method [1], Collocation Method, Least Square Method, Sub-domain Method [2], Adomian Decomposition Method [5], Shooting Method are used frequently in numerical analysis [6]. The FDM has been used widely with some of its limitations like using this method we can get the solutions at particular grid points but it is unable to inflict the solutions at every single point between two grid points. The next one is the computational cost to achieve higher accuracy. To overcome this tenacious situation, during some years Galerkin Finite Element Method (GFEM) plays the most important role in solving problems of engineering and mathematical physics with complicated geometries, loadings, and material properties [1]. The extensively used Galerkin weighted residual method shows the approximate results to any point between any two grid points even in the complex domain of the problem. Moreover, when it is not possible to come up with a nearly form solution of many engineering affairs for complicated geometry and boundary conditions, GFEM, Finite Strip Method, Finite Volume Method, Boundary Element Method, etc were introduced to provide nearby solutions to those unsolved complicated problems by using computer. Among them, GFEM often forms the core of many commercially available engineering analysis software and becomes one of the most popular methods in the case of unusual geometry. Also, it is capable of solving the problems of heat transfer, fluid mechanics, electrostatics, and mechanical systems in a very straightforward manner.

Galerkin finite element method is widely used in solving linear and nonlinear ordinary, partial [3, 4] and fractional order differential equations [8]. Ali and Islam [19] applied GFEM to solve the second-order nonlinear boundary value problems which is limited to single equation. On the other hand, Rupa and Islam [32] solved system of ordinary differential equation by Galerkin method but it is limited on linear problems only. While to solve nonlinear system of second-order boundary value problem, Dehghan and Saadatmandi [33] used sinc-collocation method, and Dehghan and Lakestani [34] used cubic B-spline scalling function Akter *et al.* used the spectral collocation method with Fourier transform to solve partial differential equations [9]. Bhatti and Bracken [11] used the Galerkin method to solve the differential equation with Bernstein polynomial basis that was limited with in first order and only Diritchlet boundary condition. After that Islam and Shirin [18] used the same method for solving both linear and nonlinear BVPs with the help of Bernoulli polynomials for different boundary conditions that was limited to single equation.

The existence of the solution of second-order nonlinear system of differential equations was introduced by many authors [13, 26]. Also the existence of positive solutions for a second-order ordinary differential system was discussed in [7]. Linearisation techniques for solving singular initial value problems were described by Ramos [10]. Lu [12] suggested the variational iteration method for solving a nonlinear system of second-order BVPs.

Investigating the analytical solutions of a two-dimensional nonlinear system of Burger's equations, Shah and Ullah [13] solved two problems for presenting the efficiency of the proposed hybrid techniques formed by coupling Laplace transform with the Adomian polynomials method. Hence introduced a new method named as Laplace transform Adomian decomposition method (LTADM). In [14], the authors studied the global stability of some k-order difference equations. They applied two different techniques. The fixed point tools analyze the asymptotic stability of some k-order difference equations for $k = 1$ and $k = 2$. So this proposed technique can be used for the global stability of more general initial value problems. In order to demonstrate a vast number of biological, chemical, and physical phenomena, fourth-order boundary value problems are practical. The primary purpose of the study as shown in [15], was to analyze the more accurate existence results of positive solution for a nonlinear fourth-order ordinary differential equation (NLFOODE) using four-point boundary value conditions (BVCs). The author applied the upper and lower solution method and Schauder's fixed point theorem to obtain the current results. At first, Green's function was introduced and used to get the numerical solution of the corresponding boundary value problem. Also, for supporting the analytical proof, one example was included in [15]. It has been seen that the uniqueness of solutions for a boundary value problem at resonance, the method of upper and lower solutions were applied in [16]. For the existence of solutions, the shift method was employed. The authors developed a monotone iteration scheme and sequences of approximate solutions which converged monotonically to the unique solution of the boundary value problem at resonance. Moreover, their analysis was proved by including two examples.

Very few works have been done so far which are limited within a single equation and/or some less efficient method. Therefore, the main purpose of this paper is to solve the system of second-order nonlinear boundary value problems. For better understanding, we derive the general formulation of Galerkin finite element method for solving nonlinear systems of ordinary differential equations. At the initial stage, we solve a linear system of ODEs. Finally, we formulate the Galerkin finite element method to solve the nonlinear system of second-order boundary value problem. Several problems are solved and the results are compared with the exact solutions to verify the effectiveness of the derived formula.

Since the early 1970s, the most popular weighted residuals method has been the Galerkin finite-element method having piecewise polynomials of low degree. The popularity of this method arises in engineering and mathematical modeling for obtaining the most approximate numerical solution of differential equations. This vigorous method's vast applications areas of interest are as follows [35, 37, 36, 38, 40, 42, 43]:
a. Structural Analysis, b. Heat Transfer, c. Fluid Flow, d. Mass Transport, and, e. Electromagnetic Potential.

The GFEM reduces product development costs of structural analysis and optimization. This one is capable of performing in any given phenomenon that arises in aerospace, mechanical, industrial, and production engineering. Mainly this method concentrates on the following industries like:
a. Energy Industry [40],
b. Heavy Engineering [38],
c. Machine parts and tools [39],
d. Structural and Vibration Analysis of Bracket Design [41], and
e. Bracket structural model [38, 39].

Moreover, this GFEM is more efficient in solving Laplace and Helmholtz equations, which is responsible for carrying the heat diffusion in solids. This method is also used to solve groundwater flow equations and compute the mass balance. Consequently, we have seen that this method is used as a general technique to obtain numerical solutions of linear and nonlinear boundary value problems in many different valuable sectors, representing its importance and acceptance.

In this research article, in section 1, we have represented introduction along with literature review, while in the following section 2, a rigorous formulation of Galerkin finite element method has formulated. Convergence and error analysis has explained in section 3 and some numerical applications are shown in section 4. Finally, the summary and concluding remarks are presented in section 5.

It is now time to derive the Galerkin finite element method for system of linear and Nonlinear ordinary differential equations.

2. Galerkin Finite Element Method and System of second-order BVPs

In the following section, we are going to formulate the GFEM for the system of linear and nonlinear boundary value problems.

2.1. System of Linear BVPs

In order to describe the generalized formulation for system of second-order linear ordinary boundary value problems, initially we need to assume a system of the following form:

$$\begin{cases} u''(x) + p(x)u(x) + q(x)v(x) = f(x),\ a \le x \le b, \\ v''(x) + r(x)v(x) + s(x)u(x) = g(x),\ a \le x \le b, \\ u(a) = u(b) = 0, \\ v(a) = v(b) = 0. \end{cases} \tag{1}$$

Two equations containing in the above system both are second-order linear ODEs, where each of the solution components $u(x)$ and $v(x)$ uniquely satisfy boundary conditions and in the domain $a < x < b$, $u(x)$ and $v(x)$

are pair of functions. The terms $p(x),\ q(x),\ r(x)$ and $s(x)$ are coefficient functions and $f(x)$ and $g(x)$ ensure the non-homogenity of the system (1). Using the linear finite element [20] to the system(1), the Galerkin finite element method reveals the system of equations and defined as

$$\begin{cases} \int_{x_i}^{x_{i+1}} (-W_i'(x)\tilde{u}_i'(x) + p(x)(\tilde{u}_i(x))W_i(x) + q(x)\tilde{v}_i(x)W_i(x))dx = R_1 \\ \int_{x_i}^{x_{i+1}} (-W_i'(x)\tilde{v}_i'(x) + r(x)\tilde{v}_i(x)W_i(x) + s(x)\tilde{u}_i(x)W_i(x))dx = R_2 \end{cases} \tag{2}$$

where the linear approximations for $u(x)$ and $v(x)$ are $\tilde{u}(x) = a_1x + a_2,\ \ \tilde{v}(x) = b_1x + b_2,\ a_1,\ a_2,\ b_1,\ b_2$ are parameters, $W_i(x),\ (i = 0, 1, \ldots, n)$ is the coordinate function and

$$R_1 = \int_{x_i}^{x_{i+1}} f(x)W_i(x)dx - [W_i(x)\tilde{u}_i'(x)]_{x_i}^{x_{i+1}},\ \ R_2 = \int_{x_i}^{x_{i+1}} g(x)W_i(x)dx - [W_i(x)\tilde{v}_i'(x)]_{x_i}^{x_{i+1}}$$

are two parameters.

Next part is to form the trial solutions for two equations and they are:

$$\tilde{u} = H_1(x)u_i + H_2(x)u_{i+1},\ \ \tilde{v} = H_1(x)v_i + H_2(x)v_{i+1} \tag{3}$$

as long as

$$H_1(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i},\ \ H_2(x) = \frac{x - x_i}{x_{i+1} - x_i}.$$

Hence the first equation of the system (2) becomes

$$\int_{x_i}^{x_{i+1}} (S_\gamma + T_\gamma + A_\gamma)dx + [B_\gamma]_{x_i}^{x_{i+1}} = \int_{x_i}^{x_{i+1}} L_\gamma dx, \quad \gamma = 1, 2$$

where

$$S_\gamma = -\begin{Bmatrix} H_1' \\ H_2' \end{Bmatrix} \begin{bmatrix} H_1' & H_2' \end{bmatrix} \begin{cases} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}, & \text{if } \gamma = 1 \\ \begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix}, & \text{if } \gamma = 2 \end{cases}$$

$$T_\gamma = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} \begin{cases} p(x)\begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}, & \text{if } \gamma = 1 \\ r(x)\begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix}, & \text{if } \gamma = 2 \end{cases}$$

$$A_\gamma = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} \begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix} \begin{cases} q(x)\begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix}, & \text{if } \gamma = 1 \\ s(x)\begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}, & \text{if } \gamma = 2 \end{cases}$$

$$B_\gamma = \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} \begin{bmatrix} H_1' & H_2' \end{bmatrix} \begin{cases} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}, & \text{if } \gamma = 1 \\ \begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix}, & \text{if } \gamma = 2 \end{cases}$$

and

$$L_\gamma = \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} \begin{cases} f(x), & \text{if } \gamma = 1 \\ g(x), & \text{if } \gamma = 2 \end{cases}$$

Sequentially complete the above integrals of the system and rewritten as $KU = F$. Therefore, $K = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$ and for the first equation L_{11} = coefficient of u and L_{12} = coefficient of v. Again for the second equation L_{21} = coefficient of u and L_{22} = coefficient of v. Also $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and $F = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$. Finally, $U = K^{-1}F$ is our desired solution which is comparable to the exact solution that indicates the accuracy of the result.

2.2. System of Nonlinear BVPs

Let introduce a system of nonlinear ODEs with boundary conditions [1, 2]

$$\begin{cases} u''(x) + p_1(x)(v'(x))^2 + p_2(x)u(x)v(x) = f(x), \\ v''(x) + q_1(x)u'(x) - q_2(x)v(x) = g(x), \\ u(a) = \alpha, \quad u(b) = \beta, \\ v(a) = \delta, \quad v(b) = \nu, \end{cases} \tag{4}$$

with the domain $a < x < b$.

Use the linear approximations for the pair of functions $u(x)$ and $v(x)$ as

$$\tilde{u}(x) = c_1x + c_2, \quad \tilde{v}(x) = d_1x + d_2 \tag{5}$$

where c_1, c_2, d_1, d_2 are parameters and use $W_i(x)$ as a coordinate function. Then the integration of the first terms from both first and second equation of the system by parts gives that

$$\begin{cases} \int_{x_i}^{x_{i+1}} (-W_i'(x)\tilde{u}_i'(x) + p_1(x)(\tilde{v}_i'(x))^2W_i(x) + p_2(x)\tilde{u}_i(x)\tilde{v}_i(x)W_i(x))dx = R_1 \\ \int_{x_i}^{x_{i+1}} (-W_i'(x)\tilde{v}_i'(x) + q_1(x)\tilde{u}_i'(x)W_i(x) - q_2(x)\tilde{v}_i(x)W_i(x))dx = R_2 \end{cases} \tag{6}$$

where

$$R_1 = \int_{x_i}^{x_{i+1}} f(x)W_i(x)dx - [W_i(x)\tilde{u}_i'(x)]_{x_i}^{x_{i+1}}, \quad R_2 = \int_{x_i}^{x_{i+1}} g(x)W_i(x)dx - [W_i(x)\tilde{v}_i'(x)]_{x_i}^{x_{i+1}}.$$

With the help of linear approximations, first find the parameters value and using them to form the trial solutions for both equations such that

$$\tilde{u} = H_1(x)u_i + H_2(x)u_{i+1}, \quad \tilde{v} = H_1(x)v_i + H_2(x)v_{i+1} \tag{7}$$

where

$$H_1(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i}, \quad H_2(x) = \frac{x - x_i}{x_{i+1} - x_i}.$$

After that the system (6) becomes

$$\int_{x_i}^{x_{i+1}} (S_\gamma + T_\gamma + G_\gamma)dx + [M_\gamma]_{x_i}^{x_{i+1}} = \int_{x_i}^{x_{i+1}} F_\gamma dx, \quad \gamma = 1, 2$$

where

$$S_\gamma = -\begin{Bmatrix} H_1' \\ H_2' \end{Bmatrix} \begin{bmatrix} H_1' & H_2' \end{bmatrix} \begin{cases} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}, & \text{if } \gamma = 1 \\ \begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix}, & \text{if } \gamma = 2 \end{cases}$$

$$T_\gamma = \begin{bmatrix} H_1 & H_2 \end{bmatrix}^2 \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} \begin{cases} p_1(x) \begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix}, & \text{if } \gamma = 1 \\ q_1(x) \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}, & \text{if } \gamma = 2 \end{cases}$$

$$G_\gamma = \begin{cases} p_2(x) \begin{bmatrix} H_1 \ H_2 \end{bmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} \begin{bmatrix} H_1 \ H_2 \end{bmatrix} \begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix} \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix}, & \text{if } \gamma = 1 \\ -q_2(x) \begin{bmatrix} H_1 \ H_2 \end{bmatrix} \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} \begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix}, & \text{if } \gamma = 2 \end{cases}$$

$$M_\gamma = \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} \begin{bmatrix} H_1' & H_2' \end{bmatrix} \begin{cases} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}, & \text{if } \gamma = 1 \\ \begin{Bmatrix} v_i \\ v_{i+1} \end{Bmatrix}, & \text{if } \gamma = 2 \end{cases}$$

and

$$F_\gamma = \begin{Bmatrix} H_1 \\ H_2 \end{Bmatrix} \begin{cases} f(x), & \text{if } \gamma = 1 \\ g(x), & \text{if } \gamma = 2 \end{cases}$$

Compute those integrations and form $KU = F$ where $K = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$ and for the first equation $L_{11} =$ coefficient of u and $L_{12} =$ coefficient of v. Again for the second equation $L_{21} =$ coefficient of u and $L_{22} =$ coefficient of v. Also $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$. At the end of this procedure, find the values of $U = K^{-1}F$.

3. Convergence and Error Analysis

According to [19], to get the approximate solution of a system of second-order boundary value problem, (1) and (4); it is must to form a finite dimensional subspace of $\beta(I)$and choose the trial solutions $\tilde{u}$ and $\tilde{v}$, where the trial solutions will satisfy the boundary conditions. Let the family of functions are denoted by $\alpha(I)$ such that $\alpha(I) \subset \beta(I)$, where $\beta(I)$ is the energy space. Additionally, n is the dimension of $\alpha(I)$, N is the number of elements and l_N is the length of N^{th} element. Hence, in energy norm, we will try to reduce the absolute error and percentage error of approximate solutions. Assume that u_{ex} and v_{ex} are the exact solutions of the system of second-order boundary value problem and u_{ap} and v_{ap} are the GFEM solutions. We need to find $u_{ap}, v_{ap} \epsilon \tilde{\alpha}$ such that $B(u_{ap}, w) = F(w)$ for all $w \epsilon \alpha_0(I)$, where B is a bilinear form on $\tilde{\alpha} \times \alpha_0$.

Theorem 3.1. *[17] The error of approximate solutions $e_1 := u_{ex} - u_{ap}$ and $e_2 := v_{ex} - v_{ap}$ are orthogonal to all test functions in $\alpha_0(I)$ in the following sense:*

$$B(e_1, w) = 0 \ \forall \ w \ \epsilon \ \alpha_0(I)$$

$$B(e_2, w) = 0 \ \forall \ w \ \epsilon \ \alpha_0(I)$$

which indicates a basic property of the error of approximate results, familiar to all as Galerkin orthogonality.

Theorem 3.2. *[17] The coefficients of the basis functions will be considered in such a way that the energy norm of the error $||(e)||_\beta$ will be minimum in GFEM i.e.*

$$||u_{ex} - u_{ap}||_\beta = min||u_{ex} - u||_\beta \ \forall u \ \epsilon \ \tilde{\alpha}(I) \tag{8}$$

and

$$||v_{ex} - v_{ap}||_\beta = min||v_{ex} - v||_\beta \ \forall v \ \epsilon \ \tilde{\alpha}(I) \tag{9}$$

Moreover, the theorem helps us to belief that if we construct a sequence of finite element spaces $\alpha_1 \subset \alpha_2 \subset \alpha_3 \subset ... \subset \alpha_n$ and evaluate the corresponding Galerkin finite element solutions $u_{ap}(1), u_{ap}(2), ..., u_{ap}(3)$ then the calculated error in the energy norm will reduce monotonically with respect to increasing n.

Let us now consider a particular non-homogeneous linear and nonlinear system of differential equations presenting the governing system as described in the previous section and illustrate in the following portion. The illustrated examples verify the result using the derived numerical method.

4. Numerical Examples

This section will consider the solution of some system of linear and nonlinear system of boundary value problems with different boundary conditions. All results are presented graphically and numerically, along with the exact solution. The following formulas are used for computing absolute and percentage errors.

$$\text{Absolute Error} = |\mathcal{E}(x) - \mathcal{A}(x)|$$
$$\text{Percentage Error} = \frac{|\mathcal{E}(x) - \mathcal{A}(x)|}{\mathcal{E}(x)} \times 100,$$

where $\mathcal{A}(x)$ and $\mathcal{E}(x)$ are the approximate and exact solution respectively.

Example 4.1. *Let us consider a second-order linear boundary value problem [12]*

$$\begin{cases} p''(x) + xp(x) + xq(x) = 2, \\ q''(x) + 2xq(x) + 2xp(x) = -2, \\ p(0) = p(1) = 0, \\ q(0) = q(1) = 0, \end{cases} \tag{10}$$

where $0 < x < 1$. *Also the exact solutions of (10) are* $p(x) = x^2 - x$ *and* $q(x) = x - x^2$. *Both equations of this system are second-order linear ODEs.*

So consider that each of the solution components $p(x)$ and $q(x)$ uniquely satisfy two boundary conditions where $p(x)$ and $q(x)$ are pair of functions while $x \in (0, 1)$. Using Galerkin finite element method to the system(10), we get the following residual system of equations:

$$\begin{cases} \int\limits_{x_i}^{x_{i+1}} (-W_i'(x)\tilde{p}_i'(x) + x(\tilde{p}_i(x))W_i(x) + x\tilde{q}_i(x)W_i(x))dx = G_1 \\ \int\limits_{x_i}^{x_{i+1}} (-W_i'(x)\tilde{q}_i'(x) + 2x\tilde{q}_i(x)W_i(x) + 2x\tilde{p}_i(x)W_i(x))dx = G_2 \end{cases} \tag{11}$$

where

$$G_1 = \int\limits_{x_i}^{x_{i+1}} 2W_i(x)dx - [W_i(x)\tilde{p}_i'(x)]_{x_i}^{x_{i+1}}, \quad G_2 = \int\limits_{x_i}^{x_{i+1}} -2W_i(x)dx - [W_i(x)\tilde{q}_i'(x)]_{x_i}^{x_{i+1}}.$$

and i is the number of parameters, $x_i = 0$ & $x_{i+1} = 1$. If the domain is discretized into n equal size of elements which represents the corresponding finite element mesh. We want to express the trial functions in terms of nodal variables [20]. In other words a_1 and a_2 need to be replaced by p_i and p_{i+1}. We evaluate p at $x = x_i$ and $x = x_{i+1}$. Then $p(x_i) = p_i = a_1x_i + a_2$ and $p(x_{i+1}) = p_{i+1} = a_1x_{i+1} + a_2$. Using subtraction find the value of a_1 that is $a_1 = \dfrac{p_i - p_{i+1}}{x_i - x_{i+1}}$. Similarly, we have $a_2 = \dfrac{x_ip_{i+1} - p_ix_{i+1}}{x_i - x_{i+1}}$.

In a similar manner, we want to express the trial functions in terms of nodal variables. In other words, b_1 and b_2 need to be replaced by q_i and q_{i+1}. We evaluate q at $x = x_i$ and $x = x_{i+1}$. Finally, it is found that $b_1 = \dfrac{q_i - q_{i+1}}{x_i - x_{i+1}}$ and $b_2 = \dfrac{x_iq_{i+1} - q_ix_{i+1}}{x_i - x_{i+1}}$.

At this time, it is necessary to substitute the values of a_1 and a_2 in the trial function which yields

$$\tilde{p} = H_1(x)p_i + H_2(x)p_{i+1} \tag{12}$$

Also using b_1 and b_2, the required result is

$$\tilde{q} = H_1(x)q_i + H_2(x)q_{i+1} \tag{13}$$

where $H_1(x) = \dfrac{x_{i+1} - x}{x_{i+1} - x_i}$ and $H_2(x) = \dfrac{x - x_i}{x_{i+1} - x_i}$. Here, equations (12) and (13) give us the expressions for p and q in terms of nodal variables and H_1, H_2 are called linear shape functions. After that, the test (trial) function for Galerkin's method are $W_1 = H_1(x)$, $W_2 = H_2(x)$. Substituting the values of p, q, W in (11), we found the system of matrix equation which can be solved through the earlier described method.

The results are shown in Table 1 for $p(x)$. The graphical results for both $p(x)$ and $q(x)$ are depicted in Figure 1. The Figure 1 (left) shows that the approximate solution $p(x)$ corresponds with the exact solution

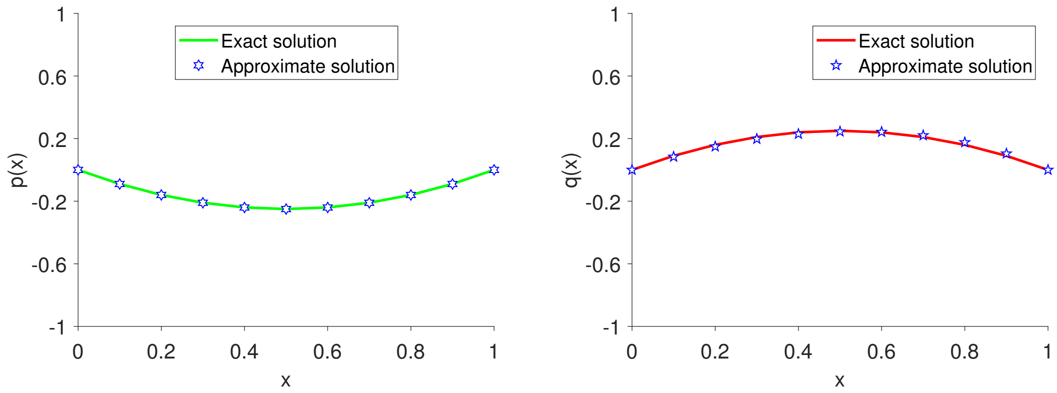

Figure 1: Comparing between approximate solutions and exact solutions (left) $p(x) = x^2 - x$, and (right) $q(x) = x - x^2$.

$p(x) = x^2 - x$ and the percentage error can be omitted after certain decimal points. It is concluded that the Galerkin finite element method is able to deliver results closer to the exact solution.

It is seen that the approximate solution $q(x)$ describes that it almost coincides with the exact solution $q(x) = x - x^2$ (Figure 1 (right)) since the visible absolute error term is very small, similarly the percentage error. Therefore, the characteristics of approximate solution $q(x)$ has good agreement with the exact solution.

Table 1: Numerical solutions of p, comparing with exact solutions $p(x) = x^2 - x$.

x	Numerical $p(x)$	Exact $p(x) = x^2 - x$	Abs. Error	Percentage Error
0.0000	0.0000	0.0000	0.0	
0.1000	-0.0901	-0.0900	1.0×10^{-4}	1.1×10^{-1}
0.2000	-0.1601	-0.1600	1.0×10^{-4}	6.3×10^{-2}
0.3000	-0.2102	-0.2100	2.0×10^{-4}	9.5×10^{-2}
0.4000	-0.2402	-0.2400	2.0×10^{-4}	8.3×10^{-2}
0.5000	-0.2502	-0.2500	2.0×10^{-4}	8.0×10^{-2}
0.6000	-0.2402	-0.2400	2.0×10^{-4}	8.3×10^{-2}
0.7000	-0.2103	-0.2100	3.0×10^{-4}	1.4×10^{-1}
0.8000	-0.1602	-0.1600	2.0×10^{-4}	1.3×10^{-1}
0.9000	-0.0902	-0.0900	2.0×10^{-4}	2.2×10^{-1}
1.0000	0.0000	0.0000	0.0	

Example 4.2. *Let us consider a second-order nonlinear boundary value problem [2, 22]*

$$\begin{cases} u''(x) + (v'(x))^2 + u(x)v(x) = 2 + e^x(e^x + x^2), \\ v''(x) + u'(x) - v(x) = 2x, \\ u(0) = 0, \quad u(1) = 1, \\ v(0) = 1, \quad v(1) = e, \end{cases} \tag{14}$$

where, $0 < x < 1$ *and this proposed system is nonlinear second-order ordinary differential equations.*

So it's must that each of the solution components $u(x)$ and $v(x)$ uniquely satisfy two boundary conditions where $u(x)$ and $v(x)$ are pair of functions in $0 < x < 1$. Using the Galerkin finite element method to the system(4.2), we get the following residual system of equations

$$\begin{cases} \int\limits_{x_i}^{x_{i+1}} (-W_i'(x)\tilde{u}_i'(x) + (\tilde{v}_i'(x))^2 W_i(x) + \tilde{u}_i(x)\tilde{v}_i(x)W_i(x))dx + [W_i(x)\tilde{u}_i'(x)]_{x_i}^{x_{i+1}} = A \\ \int\limits_{x_i}^{x_{i+1}} (-W_i'(x)\tilde{v}_i'(x) + \tilde{u}_i'(x)W_i(x) - \tilde{v}_i(x)W_i(x))dx + [W_i(x)\tilde{v}_i'(x)]_{x_i}^{x_{i+1}} = B \end{cases} \tag{15}$$

where $A = \int\limits_{x_i}^{x_{i+1}} (2 + e^x(e^x + x^2))W_i(x)dx$ and $B = \int\limits_{x_i}^{x_{i+1}} 2xW_i(x)dx$.

If the domain $[0, 1]$ is discretized into n equal size of elements which represents the corresponding finite element mesh. We want to express the trial functions in terms of nodal variables. In other words c_1 and c_2 need to be replaced by u_i and u_{i+1}. We evaluate u at $x = x_i$ and $x = x_{i+1}$. Then $u(x_i) = u_i = c_1x_i + c_2$ and $u(x_{i+1}) = u_{i+1} = c_1x_{i+1} + c_2$.

After few steps and simplification we get the values of c_1 and c_2, where $c_1 = \dfrac{u_i - u_{i+1}}{x_i - x_{i+1}}$ and $c_2 = \dfrac{x_iu_{i+1} - u_ix_{i+1}}{x_i - x_{i+1}}$.

Similarly the expressions for d_1 and d_2 can be found and is defined as $d_1 = \dfrac{v_i - v_{i+1}}{x_i - x_{i+1}}$ and $d_2 = \dfrac{x_iv_{i+1} - v_ix_{i+1}}{x_i - x_{i+1}}$. Substituting the values of c_1 and c_2 in the trial function, we obtain

$$\tilde{u} = H_1(x)u_i + H_2(x)u_{i+1} \tag{16}$$

Also using d_1 and d_2, the required result is

$$\tilde{v} = H_1(x)v_i + H_2(x)v_{i+1} \tag{17}$$

where $H_1(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i}$ and $H_2(x) = \frac{x - x_i}{x_{i+1} - x_i}$.

The equations (16) and (17) give us the expressions for u and v in terms of nodal variables and H_1, H_2 are called linear shape functions. Finally it will find the system of equations in a matrix equation and need to solve through Galerkin finite element method.

The results are shown in Tables 2 and 3 for $u(x)$ and $v(x)$, respectively. Their graphical comparisons with the exact solution are displayed in Figure 2.

Figure 2 (left) represents that the approximate solution $u(x)$ corresponds with the exact solution $u(x) = x^2$ and error term is so small which can be neglected as well as the percentage error. Thus it makes sense that the Galerkin finite element method is able to deliver results as similar as exact solution $u(x) = x^2$.

It is noted that the approximate solution $v(x)$ is very close to the exact solution $v(x) = e^x$ and the required percentage error is negligible, see Figure 2 (right). Hence it must says that the characteristics of approximate solution $v(x)$ as harmonious as exact solution e^x. The diagram visually condenses the claims just made: solution curves converge toward the exact solution. Use the sliders above to verify these claims, based on the plots of the various solution curves as shown in different figures. The absolute error graph is shown to visualize the trend of chaning error of $u(x)$ and $v(x)$ with respect to x.

The error maps for u and v are displayed in Figure 4.

Table 2: Comparison between numerical and exact solutions of $u(x)$.

x	Numerical $u(x)$	Exact	Abs. Error	Percentage Error
0.0000	0.0000	0.0000	0.0	
0.1000	0.0103	0.0100	3×10^{-4}	3.0×10^{0}
0.2000	0.0406	0.0400	6.0×10^{-4}	1.5×10^{0}
0.3000	0.0909	0.0900	9.0×10^{-4}	1.0×10^{0}
0.4000	0.1611	0.1600	1.1×10^{-3}	6.9×10^{-1}
0.5000	0.2512	0.2500	1.2×10^{-3}	4.8×10^{-1}
0.6000	0.3612	0.3600	1.2×10^{-3}	3.3×10^{-1}
0.7000	0.4911	0.4900	1.1×10^{-3}	2.2×10^{-1}
0.8000	0.6409	0.6400	9.0×10^{-4}	1.4×10^{-1}
0.9000	0.8106	0.8100	6.0×10^{-4}	7.4×10^{-2}
1.0000	1.0000	1.0000	0.0	0.0

Table 3: Comparison between numerical and exact solutions of $v(x)$.

x	Numerical $v(x)$	Exact	Abs. Error	Percentage Error
0.0000	1.0000	1.0000	0.0000	0.0×10^{-5}
0.1000	1.1052	1.1052	0.0000	0.0×10^{-5}
0.2000	1.2214	1.2214	0.0000	0.0×10^{-5}
0.3000	1.3498	1.3499	0.0001	7.4×10^{-3}
0.4000	1.4917	1.4918	0.0001	6.7×10^{-3}
0.5000	1.6486	1.6487	0.0001	6.0×10^{-3}
0.6000	1.8220	1.8221	0.0002	5.5×10^{-3}
0.7000	2.0136	2.0138	0.0002	9.9×10^{-3}
0.8000	2.2254	2.2255	0.0002	4.5×10^{-3}
0.9000	2.4595	2.4596	0.0001	4.1×10^{-3}
1.0000	2.7183	2.7183	0.0000	0.0×10^{-5}

5. Conclusion

This paper has represented the generalized formulation of Galerkin finite element method for system of nonlinear second-order boundary value problems. The numerical examples related to these formulations are presented. In this study, the main concentration has given not only on the numerical solutions but also on the formulations of Galerkin finite element method to solve higher order systems. All the obtained results are dedicated graphically as well as in tabular form.The computed solutions are compared with the exact solution. Lastly, we include the absolute error in table to give concern in finding the accuracy and efficiency of the Galerkin finite element method.

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Conflict of Interest

The authors declare no conflict of interest exists.

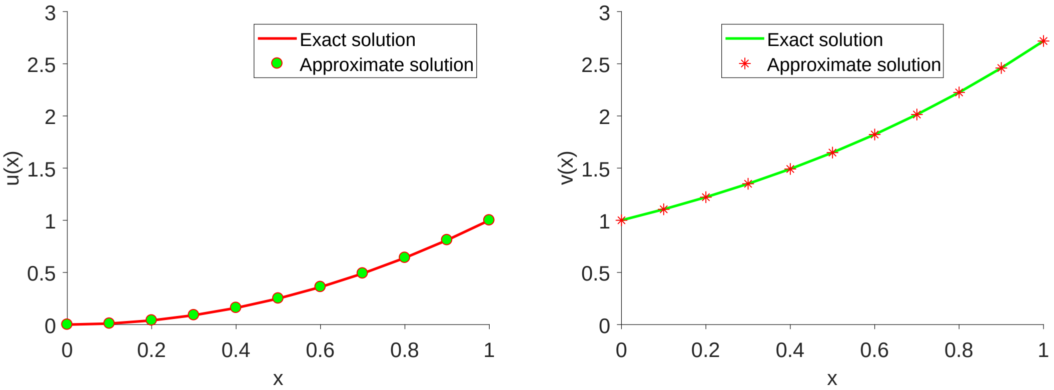

Figure 2: Comparison between exact and approximate solutions (left) $u(x) = x^2$, and (right) $v(x) = e^x$.

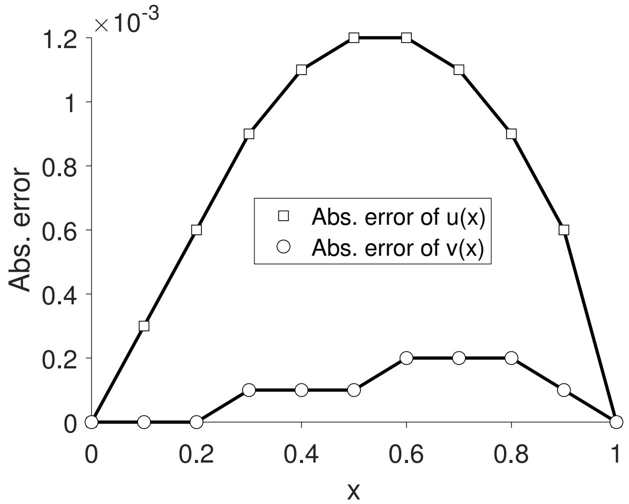

Figure 3: Absolute errors of $u(x)$ and $v(x)$ of example 4.2.

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