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Research Article

Analysis of asymmetric financial data with directional dependence measures

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Abstract

The increase of the product variety in the financial markets requires a clear understanding of the dependence between such instruments for the decision-makers. For a few decades, such dependence structures were often modeled with symmetric copula families. However, financial data may reveal an asymmetric structure, which can be determined via directional dependence measures in the context of copulas. Previously, some asymmetric copula models were proposed in different ways using Khoudraji's device. But they are merely used for financial time series data in a broader sense. In this study, a new set of asymmetric copulas were defined by using one parameter of Archimedean copula families. For this aim, widely used copula families were studied and the corresponding directional dependence measures were analyzed. To illustrate the efficiency of the parameter estimation method, a small simulation scenario consisting of an asymmetric dependence pattern was carried out. Thereafter, the proposed asymmetric bi-variate copulas with directional dependence coefficients were investigated for two different stock market data. The study's primary findings suggested that the newly generated asymmetric models might be useful for directional dependence. Especially, the estimated directional dependence coefficients can serve as an indicator to explain the variability of one stock in terms of the other.

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1. Introduction

In the past three decades, copulas have gained popularity for modeling dependence structures between random variables, especially in finance and risk management. The main advantage of copulas relies on the fact that the joint distribution of the variables and their distribution functions are considered separately for dependence modeling. Besides, copula functions, which arise from Sklar's theorem, are really flexible modeling tools for multivariate data in various research fields. The widely used normal distribution

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assumption for the multivariate data can be properly relaxed and they allow the identification of possible tail dependencies. For a general overview of the foundations of copula theory, the books of [17] and [30] are referred for interested readers.

With these main advantages, the use of copulas has accelerated with real-life applications in finance, actuarial science, econometrics, bio-statistics, medical research, engineering, and hydrology. Mainly, for the bivariate distribution of continuous random variables, the parametric approach has been investigated over different copula families. On the other hand, most of the existing copulas preserve symmetric dependence patterns, equivalently, the exchangeability of the variables. For that reason, the necessity for the proposed asymmetric copulas appears since classical symmetric copulas are not suitable for all data sets. Regarding this issue, as a new measure between the variables, directional dependence was introduced based on asymmetric copulas.

By the 2000s, many researchers studied the asymmetric class of copulas by introducing different construction methods. Firstly, Rodriguez-Lallena and Ùbeda-Flores [34] introduced a generalization of some widely used bivariate copulas, later that inspires [21] and [27] for deriving new asymmetric models. Thereafter, Alfonsi and Brigo [2] introduced an asymmetric copula based on the periodic functions. Later on, Liebscher [26] proposed new methods, close to the study of [19], for the construction of asymmetric multivariate copulas. Before this study, the asymmetry concept is discussed by [25, 30, 31]. After the study of [26], a new technique for asymmetric copulas is introduced by [7]. The idea is mainly similar to the first approach of [26] so that the products of copulas with powered arguments are used for that construction. Most recently, the mixture of symmetric copulas and the convex combination of asymmetric copulas are proposed by [42] as a new method for the reliability data. As an extension of the study of [7], Mukherjee et al. [29] have investigated various combinations of asymmetric copulas over the car rental data set. As a general understanding of the above constructions, Siburg et al. [37] studied the order of asymmetry of bivariate copulas to fill that theoretical gap. Mainly, the departure point of all the above studies about asymmetric copulas dates back to the earlier study of [19].

On the other side, the directional dependence measure was studied with the help of asymmetric copula families by [6,28,39,40]. Briefly, this measure corresponds to the likely direction of influence between two random variables [18]. In the earlier studies, Sungur [39] mentioned the importance of copula regression for modeling directional dependence. Under the copula regression setting, Sungur [39] described this measure using both marginal and joint behavior of random variables. Additionally, it is possible to detect the existence of directional dependence with a quantified degree. After this contribution, several researchers used that idea by considering asymmetric copulas. To illustrate, Jung et al. [18] and Uhm et al. [41] investigated the asymmetry of financial data using an asymmetric version of the Farlie-Gumbel-Morgenstren (FGM) copula. In addition, Kim et al. [22] and Kim et al. [20] considered both [34] and survival truncated FGM to understand the directional dependence using gene data. In recent work, the previous limitations of the studies for directional dependence were properly addressed and a multi-step procedure is proposed for optimal parameter estimation by [23]. However, the considered asymmetric copula families are limited to the use of independent copula and main Archimedean families.

As an extension of cites existing literature, the main contributions of our study are two-fold: (1) to investigate Archimedean family based asymmetric copulas, for completing the missing models not mentioned by [23] and [29], and (2) to discuss the model selection with the interpretation of directional dependence coefficients, not even mentioned by [29] before. Apart from the use of FGM or Ali-Mikhail-Haq (AMH) type families during the construction, widely considered Archimedean families; Clayton, Gumbel, Frank and copulas are incorporated with the independence copula. Previously, in the study of [23], only the combinations with same families are studied for the construction of asymmetric families using the general formula introduced by [7]. On the other hand, even if different constructions are considered using nine main symmetric copula functions by [29], the directional dependence interpretation has not been discussed. In that respect, our study aims to harmonize the concepts of asymmetric copulas and directional dependence by constructing new asymmetric families and investigating their benefits over the financial data set.

In this paper, dissimilar to real life applications performed earlier, the existence of the directional dependence in the financial data was empirically verified rather than focusing on the estimation of directional dependence directly. Besides, after modeling the financial market time series data, the asymmetric behavior is tested for the residuals of the fitted model. In order to show the performance of the asymmetric copula models, the numerical findings of the study were obtained by studying two different stock market datasets, including a comprehensive model selection. The first dataset was taken from the CDVine R package [5] while the second up-to-date data is retrieved taken from the world stock market between 01.01.2019-24.04.2020 including Covid-19 impact (https://tr.investing.com/indices/world-indices). Recently, one of the hot topics of economic studies is to unravel the possible impacts of the pandemic on the economics of countries [1, 3, 4, 8, 15, 32, 35, 43]). Besides, many of the opponents declared that the possible impacts of this worldwide health and economic crisis might outrun the consequences of the 2008 economic crisis. In that respect, the directional dependence measures might be a good subsidiary instrument to understand the influence between the considered stock data pairs. For this motivation, the second stock market data belongs to the beginning period of the worldwide Covid-19 crisis. The most widely applied log-return series are extracted for the first four months of the Covid-19 pandemic to apply time series analysis and asymmetric testing before copula modeling. In that respect, the findings of the proposed asymmetric models can serve as good complementary research for the above mentioned contributions in this field.

This article is organized as follows. In Section 2, with its six subsections, we briefly review the concept of the symmetric and asymmetric copula models with their main properties, asymmetric tests, the directional dependence measure, and primary time series modeling. The same section describes briefly the considered visual and non-visual tools for parameter estimation and model selection. To illustrate the proposed procedure, Section 3 provides the main findings for the parameter estimation of a simulated data set. Thereafter, two empirical financial datasets are studied, one of them including the stock market returns during the initial months of the Covid-19 health crisis. Finally, the main conclusion of the study is summarized with its pros and cons in Section 4.

2. Material and method

2.1. Copulas

Copula is a probabilistic modeling tool, first described by [38]. It is often used to model the dependence structure of multivariate data, especially in the areas such as finance, economics, and actuarial science. Copulas allow us to model the structure of the joint distribution independent of marginal distributions, equivalently saying that a copula is an approach that eliminates the effects of marginal distributions. In this respect, copula modeling has the advantage of using it as an alternative to other joint distributions. Additionally, another important advantage of copulas over joint models is that they can model dependency even if the multivariate normality assumption is violated. Although it is difficult to capture the relationship between variables when they are not normally distributed, the dependence pattern between these variables can be determined by choosing the appropriate copula.

In general, a copula is an n-dimensional joint distribution function, defined over $[0,1]^n$ with uniformly distributed marginals. Specifically, a bivariate copula is a function $C : [0,1]^2 \to [0,1]$ satisfying the following properties:

- $C(u,0) = C(0,v) = 0$,
- $C(u,1) = u$ and $C(1,v) = v$ for all $u, v \in [0,1]$,
- $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$, for all $u_1,u_2,v_1,v_2 \in [0,1]$ where $u_1 \leq u_2$ and $v_1 \leq v_2$.

Due to Sklar's theorem [38], any bivariate continuous distribution function, $F_{XY}(x,y)$, can be represented as a function of its marginal distribution of random variables X and Y, $F_X(x)$ and $F_Y(y)$, by using a two-dimensional copula. There exists a unique bivariate copula $C : [0,1]^2 \to [0,1]$ that can be written as

$$F_{XY}(x,y) = C(F_X(x), F_Y(y) = C(u,v),$$

where u and v show the continuous empirical marginal distribution functions of $F_X(x)$ and $F_Y(y)$, with uniform distribution $U(0,1)$, respectively. Originally, the most widely considered copula families satisfy the ex-changeability, i.e. $C(u,v) = C(v,u)$. However, the asymmetric copula setting emerges from the unsatisfied ex-changeability. This property brings us to the next subsection for the construction of asymmetric copulas using symmetric ones.

2.2. Symmetric and asymmetric copulas

According to the theorem defined by [7], for all $\alpha, \beta \in (0,1)$ and for all copulas C_1 and C_2, the function $C_{\alpha,\beta} : [0,1]^2 \to [0,1]$, defined by

$$C_{\alpha,\beta}(u,v) = C_1\left(u^{\overline{\alpha}}, v^{\overline{\beta}}\right) C_2\left(u^{\alpha}, v^{\beta}\right), \tag{2.1}$$

is a copula, where $\overline{\alpha} = 1 - \alpha$ and $\overline{\beta} = 1 - \beta$. Here, an asymmetric copula is formed for $\alpha \neq 1/2, \beta \neq 1/2$. If $\alpha = \beta$ then $C_{\alpha,\beta}$ is symmetric. For two symmetric copulas, $C_1$and C_2, if $C_{\alpha,\beta}(u,v) = C_{\beta,\alpha}(v,u)$ then $C_{\alpha,\beta}$ is a symmetric copula. On the other side, if $C_{\alpha,\beta}(u,v) \neq C_{\beta,\alpha}(v,u)$ then $C_{\alpha,\beta}$ is defined as the asymmetric copula.

According to the Lemma 1 given in [29], if C_1 and C_2 are two symmetric copulas with $\alpha, \beta \in (0,1)$ then $\rho(C_{\alpha,\beta}) = \rho(C_{\beta,\alpha})$ and $\tau(C_{\alpha,\beta}) = \tau(C_{\beta,\alpha})$, where Spearman's rho and Kendall's tau correlation coefficients depend on copula, are defined as $\rho = 12\int_0^1\int_0^1 C(u,v)dudv - 3$ and $\tau = 4\int_0^1\int_0^1 C(u,v)dudv - 1$, respectively. With this proposition, it is said that in asymmetric models, the correlation values calculated depending on the copula for the selected (α,β) parameters are different from the correlation values calculated for (β,α) (i.e. $C_{\alpha,\beta}(u,v) \neq C_{\beta,\alpha}(v,u)$). In order to see this relationship, interested readers referred to the previous study of [29] with reproducible Mathematica codes, (available at `https://goo.gl/plkJ7`).

In order to choose the right model for the data, the symmetry test (or ex-changeability of the variables) must be performed first. For symmetry tests, R_n, S_n^* and T_n measures given by [12] are used in the literature. In this paper, Cramer-von Mises statistics, $S_n^* = \int_0^1\int_0^1 \left\{\hat{C}_n(u,v) - \hat{C}_n(v,u)\right\}^2 d\hat{C}_n(v,u)$ was used for testing symmetry. If the p-value corresponding to this statistic is less than 0.05, the null hypothesis established as $H_0 : \widehat{C}_n(u,v) = \widehat{C}_n(v,u)$, which means the symmetry of the data, is rejected. In other words, it is concluded that the data are asymmetrically dependent. In our study, this ex-changeability test was conducted with the "exchTest" function in the "copula" package [16].

The constructed asymmetric copula models used in this paper are given in Table 1. Here, the C12-C15 copula family formed using Equation (2.1) by choosing the independent copula for C_1 and one of the symmetrical Archimedean copulas for C_2, called the Khoudraji copula family [11,19,23]. Similarly, C22-C33-C44 are previously studied by [23] by setting both C_1 and C_2 as the same Archimedean families. Additionally, new proposed models C23-C25, C34-C35, C45 and C55 are investigated in this study. These added combinations are the new asymmetric copula models, not mentioned in [23] before.

Table 1. Proposed asymmetric copula models (θ, θ_1 and θ_2 show the dependence parameters with α, β are shape parameters).

$C_{\alpha,\beta}(u,v)$	$C_1(u^{\alpha}, v^{\beta})$	$C_2(u^{\overline{\alpha}}, v^{\overline{\beta}})$
C12	Independent	Clayton (θ)
C13	Independent	Frank (θ)
C14	Independent	Gumbel (θ)
C15	Independent	Joe (θ)
C22	Clayton (θ_1)	Clayton (θ_2)
C23	Clayton (θ_1)	Frank (θ_2)
C24	Clayton (θ_1)	Gumbel (θ_2)
C25	Clayton (θ_1)	Joe (θ_2)
C33	Gumbel (θ_1)	Gumbel (θ_2)
C34	Gumbel (θ_1)	Frank (θ_2)
C35	Gumbel (θ_1)	Joe (θ_2)
C44	Frank (θ_1)	Frank (θ_2)
C45	Frank (θ_1)	Joe (θ_2)
C55	Joe (θ_1)	Joe (θ_2)

For the construction of asymmetric families in Table 1, the widely known Archimedean one-parameter copula families (2-5) can be defined as follows:

(1) Independent: $C(u,v) = uv$,

(2) Clayton: $C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}$, $\theta \in (0,\infty)$,

(3) Gumbel: $C(u,v) = \exp\{-\left[(-\log(u))^{\theta} + (-\log(v))^{\theta}\right]^{\frac{1}{\theta}}\}$, $\theta \in [1,\infty)$,

(4) Frank: $C(u,v) = -\frac{1}{\theta}\log\left\{1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{(e^{-\theta}-1)}\right\}$, $\theta \in R\{0\}$,

(5) Joe: $C(u,v) = 1 - \left[(1-u)^{\theta} + (1-v)^{\theta} - (1-u)^{\theta}(1-v)^{\theta}\right]^{\frac{1}{\theta}}$, $\theta \in [1,\infty)$,

whereas the independent family shows no dependence pattern among the variables so that the parameter is directly equal to zero [30]. Here, Clayton, Gumbel, Frank, and Joe are well-known families having different dependence structures. For example, Frank copula is preferable to examine the symmetric dependence structures. On the other hand, Clayton and Gumbel are useful to identify the tail dependencies at lower and upper quantiles, respectively. Even if the formulation is a bit different, Joe family behaves similarly to Gumbel with the same dependence parameter space.

2.3. The measurement of directional dependence

Suitable dependence modeling of the bivariate dataset is a vogue but not a trivial study. Besides, real datasets such as currency or stock data may not always be symmetrical. For this reason, asymmetric copula models are more useful than symmetrical copula models in such financial applications. In addition, the examination of directional dependence among the considered variables can explain the variability of one variable with respect to another. For this purpose, directional copula models created with asymmetric copulas are used in different research fields such as economics [29], finance [18,37,41], medicine [23], reliability and life models [42].

The existing asymmetry, in the non-linear type copula functions, allows us to describe the direction of dependence [39]. More clearly, in terms of the copula regression functions, when the dependency between the (U, V) variable pair is not symmetric, the forms of the regression functions for U and V will not be the same. Therefore, the direction of dependence from U to V or V to U will be different for asymmetric structures. For that purpose, we recalled the general measures for the directional dependence in joint behaviour proposed by [39], in which two types of definitions are discussed. While one of them is affected by the marginal behaviour of the variables, the other one is affected by their joint behaviour in terms of their copula. In practice, whenever the marginal distributions are not known, normalized ranks of observed data are used as a starting point for the directional dependence analysis. For the interested reader, the detailed properties and examples of directional dependence measures can be obtained from [39] and [40]. In addition, Jung et al. [18] and Kim and Kim [23] summarized some features related to directional dependence. Recently, there is another theoretical study about the directional dependency, which is renamed as copula correlation ratio, over the generalized FGM copula family [36].

The copula approach to directional dependence eliminates the influence of marginals. If it is desired to create a directional dependence model in the case of joint behaviour, it is recommended to use the asymmetric copula model. In [0,1], U and V marginals are the same for the uniformly distributed, transformed $(U, V) = (F_X(X), F_Y(Y))$ pair, whereas in copula regression functions, from a given value of U $(U = u)$ to the V direction and from a given value of V $(V = v)$ to the U direction, there could be a difference as stated by [39, 40]. That is, while the marginals of the variables are the same, the results in the joint behaviour may be different.

In this setting, the directional dependence within the joint behaviour can be examined with copula regression functions. For this purpose, as it was mentioned before it would be more appropriate to choose asymmetric copula as the candidate copula for directional dependence models by [18, 23, 39, 40]. In order to calculate the directional dependence coefficients, firstly, for a uniform marginal (U, V) random vector defined on [0,1], with a copula function C, conditional distributions are defined as follows:

The conditional distribution function for V given $U = u$ is denoted by $C_u(v)$:

$$C_u(v) \equiv P(V \leq v | U = u) = \frac{\partial C(u, v, \emptyset)}{\partial u}$$

and the conditional distribution function for U given $V = v$ is denoted by $C_v(u)$:

$$C_v(u) \equiv P(U \leq u | V = v) = \frac{\partial C(u, v, \emptyset)}{\partial v},$$

where $\emptyset = (\theta_1, \theta_2, \alpha, \beta)$ is the unknown parameter set at the beginning.

Definitions of the copula regression functions (in the directions of U to V $(U \to V)$ and V to U $(V \to U)$) and the coefficients of directional dependence according to these conditional distributions are given below. The copula regression functions for $U \to V$ and $V \to U$ are defined as

$$r_{V|U}(u) = E[V|U = u] = 1 - \int_0^1 C_u(v)\, dv,$$

$$r_{U|V}(v) = E[U|V = v] = 1 - \int_0^1 C_v(u) du.$$

Thereafter, the coefficients of directional dependence in joint behavior can be calculated by plug-in the above quantities:

$$\rho^{(2)}_{U \to V} = \frac{Var(r_{V|U}(U))}{Var(V)} = 12E\left[\left(r_{V|U}(u)\right)^2\right] - 3,$$

$$\rho^{(2)}_{V\to U} = \frac{Var(r_{U|V}(V))}{Var(U)} = 12E\left[\left(r_{U|V}(v)\right)^2\right] - 3,$$

from $U \to V$ and $V \to U$, respectively. Here, $\rho^{(2)}_{U\to V}$ can be interpreted as the proportion of total variation of V that can be explained by the copula regression of V on U. In the opposite direction, $\rho^{(2)}_{V\to U}$ explains the proportion of total variation of U that can be explained by the copula regression of U on V.

Above mentioned measurements, which we defined as directional dependence measures, were equivalently called copula correlation ratios. Although there is a closed form solution for the FGM family, directional dependency measurements cannot be obtained theoretically [36]. The reason for this is related to the non-existent closed form representations on conditional distribution functions for parametric copulas. For this reason, the formulas given below can be used to provide close estimates of the directional dependence coefficients in various cases.

$$\widetilde{\rho}^{(2)}_{U\to V} = \frac{12}{S}\sum_{s=1}^{S}\left(\tilde{r}_{V|U}(u_s)\right)^2 - 3,$$

$$\widetilde{\rho}^{(2)}_{V\to U} = \frac{12}{S}\sum_{s=1}^{S}\left(\tilde{r}_{U|V}(v_s)\right)^2 - 3,$$

where, the vector (u_s, v_s) is pseudo observations transformed from data to $U(0,1)$; S is the size of pseudo observations; $\tilde{r}_{V|U}(u) = 1 - \frac{1}{S}\sum_{s=1}^{S} C_u(v_s)$ and $\tilde{r}_{U|V}(v) = 1 - \frac{1}{S}\sum_{s=1}^{S} C_v(u_s)$ are approximately calculated copula regression functions over the pseudo observations, $(u_s, v_s) \in (0,1)^2$.

2.4. Parameter estimation

In this section, the parameter estimation methods are mentioned simply for the asymmetric copulas presented in Table 1. For the asymmetric Khoudraji copula family, the maximum likelihood (ML) method, the inference functions for margins (IFM) method, and the maximum pseudo-likelihood (MPL) method are commonly used parameter estimation methods in the literature [10]. While ML and IFM are affected by the choice of the marginal distributions, Kim et al. [24] has shown that MPL is not affected by the marginals. Besides, Kim and Kim [23] used the MPL method in their study for the inference of directional dependence in joint behaviour based on asymmetric copula regression. For these reasons, to obtain the parameter estimations, the MPL method is preferred having the detailed steps presented below.

Here, R_i and S_i are the ranks of observations x_i and y_i, $\phi = (\theta_1, \theta_2, \alpha, \beta)$ is the parameter vector where θ_1, θ_2 are the dependence parameters of each copula family with α, β are the shape parameters. Corresponding to the pseudo observations, $c(u, v, \phi)$ is a copula density given by $c(u, v, \phi) = \frac{\partial^2 C(u,v,\phi)}{\partial u \partial v}$. Within this setting, the following steps are followed to obtain the directional dependence measures.

Step 1. Obtaining pseudo observations u_i and v_i: $u_i = \frac{R_i}{n+1}$ and $v_i = \frac{S_i}{n+1}$, $\{(u_i,\ v_i)$, where $i = 1, \ldots, n\} \in (0,1)^2$.

Step 2. Replacing pseudo observations in the pseudo-maximum likelihood function, which is written as follows:

$$\ell(\phi) = \log \prod_{i=1}^{n} c(u_i, v_i, \phi) = \sum_{i=1}^{n} \log c(u_i, v_i, \phi).$$

Step 3. Estimating the set of parameters by Nelder Mead optimization technique to derive:

$$(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\alpha}, \widehat{\beta}).$$

Step 4. Use the estimated parameters for the calculation of directional dependence measures $\widetilde{\rho}^{(2)}_{U\to V}$ and $\widetilde{\rho}^{(2)}_{V\to U}$.

The primary goal is to derive the estimates for $\widetilde{\rho}^{(2)}_{U\to V}$ and $\widetilde{\rho}^{(2)}_{V\to U}$ eventually. To accomplish this goal, the approximate formulas given in Section 2.3 are considered for the directional dependence measures.

2.5. Model selection

In this subsection, we are interested in determining the best fitting asymmetric copula model among different candidate models for the given dataset. For this purpose, we adopt the criteria mentioned in [23] according to the directional dependence measures. According to these criteria, firstly the copula models with the largest/smallest pair of $\rho^{(2)}_{U\to V}$ and $\rho^{(2)}_{V\to U}$ directional dependence measures are selected, and then the goodness of fit (GOF) test is performed to see the compatibility of these selected copulas with the data. The reason for that is about deciding the best copula regression function, which best explains the change in u with v. Secondly, since there may be more than one copula matching the data, the GOF test criterion is applied for the overall decision in the empirical findings.

There are various GOF methods available in the literature based on empirical copula. In this study, Cramer-von Mises statistics (S_n) are used, described in [9, 13, 23] given below:

$$S_n = \sum_{i=1}^{n} \left\{ C_n\left(u_i, v_i\right) - C_{\phi_n}\left(u_i, v_i; \widehat{\phi}\right) \right\}^2,$$

where $u_i = \frac{R_i}{n+1}$ and $v_i = \frac{S_i}{n+1}$ are the normalized ranks, $C_n\left(u, v\right)$ is the empirical copula and $\widehat{\phi}$ is the MPL for ϕ.

After selecting the appropriate copula model for the data, the graph $\tilde{r}_{U|V}\left(w\right)$ vs $\tilde{r}_{V|U}\left(w\right)$ with a 45° reference line called the CR graph to empirically detect the directional dependence in the joint behavior of the data, is drawn (see [23] for more details related to CR plots). Any departure from the reference line in the CR plot is a sign of directional dependence. More deviation along the reference line is an indicator of higher asymmetric behavior. Additionally, by checking whether the curve is located above or below the line v = u, one may suspect the form of directional dependence in the data. For this purpose, the mentioned CR plots are presented for each application in the upcoming section.

2.6. Time series model

For the application part, one dataset is already modeled whereas the second one needs to undertake a univariate time series analysis. For this purpose, similar to the applied method in the first one, the core ingredient is the standardized residuals. In order to model the financial time series, one should start with modeling the original series to extract the standardized residuals, the input for copula analysis.

To model both the trend and non-constant volatility inherent in financial time series data we opt to use an ARMA-GARCH model. The return series is described as an Auto Regressive Moving Average model, ARMA(p, q), as follows:

$$r_t = c + \sum_{i=1}^{p} \alpha_i r_{t-1} + \sum_{j=1}^{q} \beta_j \varepsilon_{t-j} + \varepsilon_t, \tag{2.2}$$

where, r_t denotes the return at time t, ε_t is a white noise series, c is the model constant, $\alpha_i \neq 0$ and β_j are coefficients within AR(p) and MA(q) components

As a generalized version of autoregressive conditional heteroscedasticity, GARCH model is useful for modeling volatility. Generally, GARCH(m, n) model can defined as

$$\varepsilon_t = \sigma_t z_t, \tag{2.3}$$

$$\sigma_t^2 = w + \sum_{i=1}^{m} a_i \varepsilon_{t-i}^2 + \sum_{j=1}^{n} b_j \sigma_{t-j}^2, \tag{2.4}$$

where $z_t \sim \mathrm{D}(0,1)$ are iid, w is a constant, $a_i \geq 0$, $b_j \geq 0$, and $\sum a_i + \sum b_j \leq 0$ are GARCH parameters.

In the mentioned formula above, z_t can be taken as any distribution to correctly reflect the features of the data. In order to model any univariate financial time series under this setting, ARMA(p,q)-GARCH(m,n) model with suitable distributions should be tested. By using the guidance of the literature and the built in functions in "rugarch" package, skewed student's-t distribution is determined as the base distribution for z_t. Additionally, various combinations of lags ($p = 0,1$; $q = 0,1$; $m = n = 1$) over different variance models ("sGARCH", "fGARCH", "eGARCH", and "iGARCH") are compared for each financial series in the second application. The reason for that is about the possibilities of different model selections rather than just filtering with ARMA$(1,1)$-GARCH$(1,1)$.

3. Numerical findings

In this section, firstly, a small simulation study was examined for the symmetric and asymmetric copula models in subsection 3.1. Thereafter, asymmetric dependencies are analyzed for two real stock market datasets in 3.2 and 3.3 below. The first application is about the SP500-DAX and NIKKEI-FTSE pairs in the world indices dataset available in the CDVine R package [5]. The second dataset consists of the SP500-NIKKEI pair, which belongs to the beginning months of Covid-19 period (https://tr.investing.com/indices/world-indices).

Firstly, based on various combinations of and as initial values, the model fit summary is examined to get estimates having the highest log-likelihood to discuss the sensitivity over the initials. Instead of starting with a fixed value, equally spaced distinct values of α and β on the interval $[0.1, 0.9]$ satisfying that $\alpha \neq \beta$, $\alpha \neq \frac{1}{2}$ and $\beta \neq \frac{1}{2}$ are examined and totally 36 different model fits are tested. Besides, the parameter estimation part is jointly modeled on the pseudo-loglikelihood function. As an initial parameter of archimedean copulas, the estimated parameters of symmetric copula fits are considered in the joint maximization process. For the GOF test, as it was previously suggested, the multiplier method is preferred for the cost of computational time. The joint numerical maximization was executed with the help of "copula" R package [16]. Then, directional dependence modeling was performed to show the existence of asymmetric patterns and to complete the model choice. In addition, log-likelihood (LL) and Cramer-von Mises (S_n) statistics values are calculated as supplementary tools for model selection. Finally, using the CR-plots and GOF test results, the most suitable asymmetric copula model for each pair is determined.

3.1. Khoudraji's copula

To illustrate the structural differences over symmetric and asymmetric copulas, a small simulation study with certain dependence parameters was discussed. For the models C12, C13, and C14, as special cases including independence copula, the parameter estimation results are given in Table 2. Here, the built functions in copula package in the R software program were used for this computation [16]. Table 2, simply illustrates the parameter estimations with standard errors in the parenthesis for the generated dataset corresponding to Kendall's $\tau = 0.8$ value. For each asymmetric model (C12, C13, C14), the relevant dependence parameter (matching with the $\tau = 0.8$) and the pre-defined shape parameters are summarized under the Model title of Table 2. In this setting, the initials for the dependence parameter are selected around the true value and the maximum pseudo-loglikelihood estimator approach is considered. The performance of the dependence parameter estimation is plausible enough with less accuracy for the C14 model ($\widehat{\theta} = 23.7251$ whereas the true parameter $\theta = 18.19$ with the largest attained standard errors). On the other hand, the estimated shape parameters ($\widehat{\alpha}$, $\widehat{\beta}$) are really close to the true observations with small standard errors.

Table 2. Parameter estimations with (standard errors) for the simulated dataset (N=1000, $\tau = 0.8$).

Model	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\beta}$
C12 ($\theta = 8$, $\alpha = 0.3$, $\beta = 0.7$)	7.5895	0.2973	0.7383
	(1.845)	(0.029)	(0.064)
C13 ($\theta = 5$, $\alpha = 0.3$, $\beta = 0.7$)	4.1947	0.3085	0.6589
	(0.492)	(0.029)	(0.045)
C14 ($\theta = 18.19$, $\alpha = 0.3$, $\beta = 0.7$)	23.7251	0.2848	0.6598
	(4.040)	(0.023)	(0.045)

Besides, the dependence pattern exhibits different structures for the considered asymmetric models compared to the symmetric ones. The difference between symmetric and asymmetric models can be seen more clearly when one looks at the contour lines of both symmetric and asymmetric copulas in Figure 1. Here, the dependence patterns are changing when the asymmetric copula is constructed on the same values with defined shape parameters. Such graphical tools can be useful for identifying the possible asymmetric pattern preserved by the dataset. To illustrate, for the selected value, different tail dependencies can be seen for Clayton and Gumbel families via dense contour lines at a specific corner (left panel of Figure 1), whereas C12 and C13 models reveal more counter lines over the region (right panel of Figure 1). Overall, the asymmetric behaviour of the generated models is easy to guess by looking at the skewed contours for the models mentioned above. Certainly, suitable statistical tests are more reliable for model selection. With this motivation, S_n values are considered in the upcoming applications with the log-likelihood (LL) values for the given model. As such, the above findings in Table 2 are important to highlight the considered parameter estimation tool before going further on the directional dependence measures. For the application part, all of the mentioned asymmetric models presented in Table 1 are considered for two different stock data.

3.2. Dataset 1

In this first application, the available dataset in "CDVine" package is preferred as a motivating example (dataset 1 [5]). Generally, this dataset was used before in different finance applications related to copulas. It contains the transformed standardized residuals of daily log returns of major world stock indices in 2009 and 2010 (396 observations on 6 variables). The considered indices are the leading stock exchanges of the six largest economies in the world, the USA (S&P 500), Japanese (Nikkei 225), Chinese (SSE Composite Index), German (DAX), French (CAC 40) and British (FTSE 100) Index. Each time series was filtered already by ARMA(1,1)-GARCH(1,1) model with Student t innovations. As a starting point, Kendall's correlation coefficient and the exchangeability test results are presented in Table 3.

In Table 3, one can see the degree of the association among the stock indices in terms of Kendall's (upper diagonal) and the corresponding p-values of the pairwise ex-changeability test (lower diagonal). Firstly, the dependence among the stock market in USA and the other stocks in Europe reveals a larger correlation. This finding is similar among the stock indices in the European market, for instance, the largest value belongs to the pair of (DAX, CAC40). In terms of the symmetric or asymmetric pattern of the dataset, only the pairs (SP500, DAX) and (NIKKEI225, FTSE100) fail for the ex-changeability of the variables, ie. asymmetric copula modeling seems more suitable for them (p values < 0.05 means the symmetric behavior fails at the %95 significance level). For that reason, since the main aim of the study is fitting suitable asymmetric copula models, the (SP500, DAX) and (NIKKEI225, FTSE100) pairs are considered for exploring the large ($\tau = 0.50$) and small ($\tau = 0.16$) correlation cases, respectively.

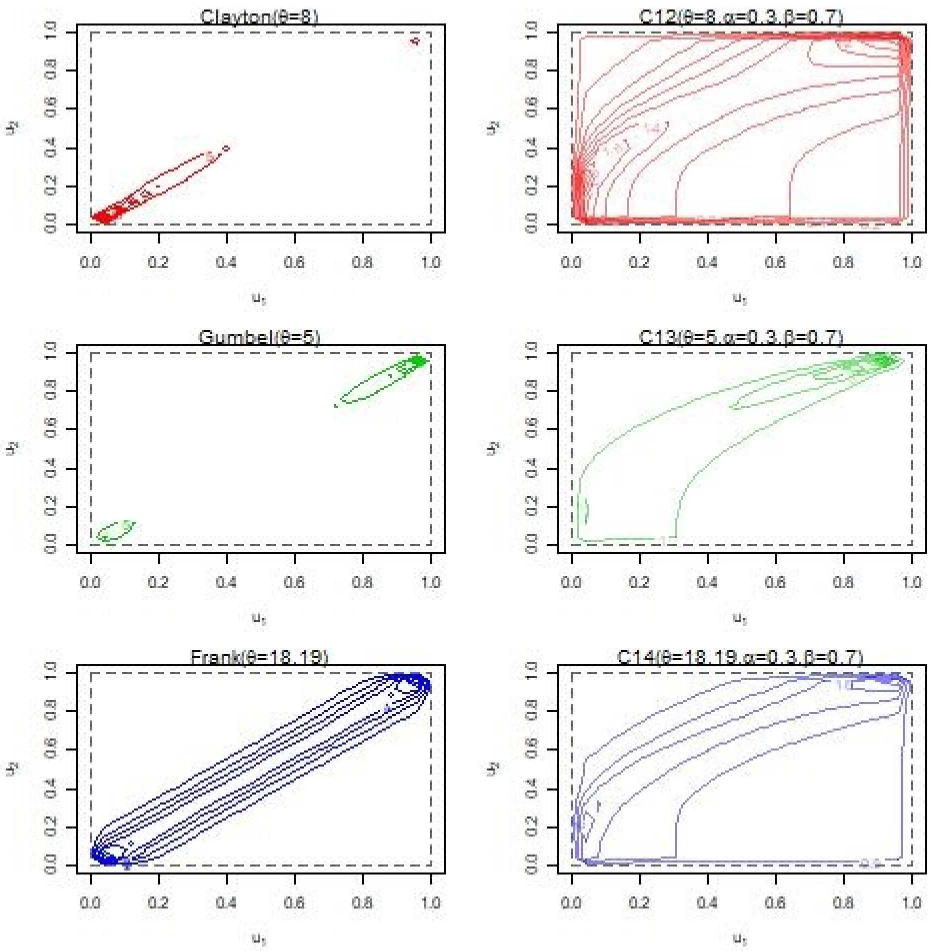


Figure 1. The contour lines for the symmetric (left) and asymmetric (right) copula models ($\tau = 0.8$ and $N = 1000$).

Table 3. Kendall's correlation coefficients (above the diagonal) and symmetric - asymmetric test results (p-value) (below the diagonal).

	SP 500	NIKKEI225	SSEC	DAX	CAC 40	FTSE 100
SP 500		0.10	0.10	0.50	0.50	0.49
NIKKEI225	(0.911)		0.20	0.16	0.18	0.16
SSEC	(0.110)	(0.157)		0.12	0.14	0.15
DAX	(0.025*)	(0.138)	(0.873)		0.82	0.73
CAC 40	(0.071)	(0.470)	(0.617)	(0.161)		0.77
FTSE 100	(0.088)	(0.020*)	(0.282)	(0.371)	(0.762)	

In Table 4, the parameters of main symmetric families are derived in terms of, Cramer-von Mises (S_n) with the faster multiplier approach for bootstrapping. Since the inversion of Kendall's τ is obliged to apply to the Clayton family for NIKKEI-FTSE pair data, the log-likelihood part is not presented for this case. Generally, all the p-values are smaller than the threshold value 0.05 so none of the symmetric copula models is suitable for the considered stock pairs. The motivation behind the use of symmetric modeling is to describe the initial parameters for the asymmetric model. Since the considered non-gradient optimization method is quite sensitive to the initial values, this strategy is preferred, rather than starting with arbitrary initials. The more extensive search on the impact of initial values is useful but out of the scope of this study.

Table 4. The parameter estimation summary of symmetric copula families for SP500-DAX and NIKKEI-FTSE pairs (S_n: Cramer-von Mises statistics, LL: Log-Likelihood).

SP500-DAX				
Copula	**Parameter**	LL	S_n	p**-value**
Clayton	1.690	141.4	0.14279	0.0004995
Gumbel	1.984	140.2	0.07477	0.0004995
Frank	5.828	127.7	0.11090	0.0004995
Joe	2.224	106.7	0.24028	0.0004995
NIKKEI-FTSE				
Copula	**Parameter**	LL	S_n	p**-value**
Clayton	0.385	-	0.03664	0.03147
Gumbel	1.192	15.5	0.04030	0.03147
Frank	1.490	11.6	0.03392	0.03846
Joe	1.234	11.9	0.07091	0.00550

By starting with the initial parameters coming from Table 4, different asymmetric copula models are applied to the residuals of the SP500-DAX stock data pair. In terms of different shape parameter initiations, the models attaining the highest log-likelihood values are presented in Table 5. Meanwhile, the shape parameters of the models C12, C13, and C35 reveal a certain doubt about being as a plausible model since they satisfy $\alpha = \beta = 1$ values for C12 and C13, while $\beta \approx 0$ for the model C35. In terms of LL values, C23 might be the best model and this result is supported by the lowest S_n value. On the other hand, the main focus of the next step is the calculation of directional dependence and CR-plotting to identify the best one empirically. In that respect, 11 models from Table 5 are used for this comparison with the help of directional dependence measures and CR plots.

Similar modeling steps are followed for the second pair, for the NIKKEI-FTSE. In this case, since the optimization results in a non-finite value for the Clayton family, the only difference is required to use the inversion of Kendall's tau method for that copula. Overall, in Table 6, the parameter estimations and S_n values are presented for each of them. Most of the time, p-values of GOF test results support that the considered asymmetric model is useful except for C15, C33, C34, C35, C45, and C55. To make a fair comparison in terms of directional dependence, totally 11 models are compared except the models C12, C13, and C35 since they have shape parameters close to the boundary of (0,1). For that reason, the directional dependence part is evaluated for the models C14, C22, C23, C24, C25, C33, C34, C44, C45, and C55.

To evaluate the performance of the models by combining the directional dependence information, one needs to look at the behaviour of both $\rho^{(2)}_{U\to V}$ and $\rho^{(2)}_{V\to U}$ and CR-plot

Table 5. The parameter estimation for the asymmetric model results for SP500-DAX.

Model	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\alpha}$	$\widehat{\beta}$	LL	S_n
C12	-	1.6902	1.0000	1.0000	141.3749	-
C13	-	1.9841	1.0000	1.0000	140.1647	-
C14	-	8.1191	0.9616	0.7779	131.9890	0.08943
C15	-	2.5057	0.8872	0.8881	107.4801	0.22296
C22	2.4574	9.2989	0.4247	0.2769	144.8191	0.10249
C23	2.6956	2.5780	0.4206	0.4125	156.3414	0.06315
C24	2.3623	16.9603	0.4770	0.3352	149.4486	0.09113
C25	2.2912	3.0166	0.2725	0.2770	153.4168	0.07822
C33	1.9628	9.5763	0.0743	0.0371	141.2727	0.07294
C34	1.9934	12.3144	0.4824	0.3376	150.0544	0.07021
C35	1.9503	4.9779	0.1319	0.0694	141.1104	-
C44	11.5443	7.5450	0.3235	0.5632	143.6709	0.09340
C45	9.7947	2.3434	0.2457	0.4985	143.9252	0.08163
C55	2.6091	2.4000	0.3575	0.6630	121.5911	0.13225

Table 6. The parameter estimation for the asymmetric model results for NIKKEI-FTSE.

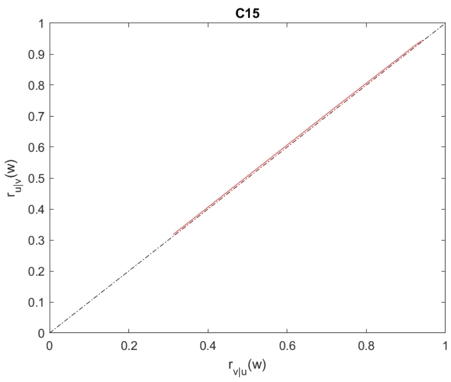
Model	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\alpha}$	$\widehat{\beta}$	LL	S_n
C12	-	2.6816	0.2899	1.0000	16.1767	-
C13	-	1.2583	0.6537	1.0000	15.7781	-
C14	-	8.1498	0.2433	0.7767	15.6513	0.02588
C15	-	1.6016	0.3722	0.5616	12.8794	0.05498
C22	0.3989	37.3298	0.0925	0.3858	20.1520	0.02206
C23	0.3794	1.6705	0.1828	0.3754	18.9704	0.02470
C24	0.4351	37.0056	0.0986	0.3834	23.8956	0.02476
C25	0.3724	1.8399	0.1616	0.3068	18.7919	0.02516
C33	1.6386	1.3133	0.4555	0.9781	16.3314	0.03664
C34	1.2624	2.0349	0.4590	0.2346	16.7950	0.03272
C35	1.3066	1.7090	0.5285	0.0201	16.2840	0.03681
C44	19.8932	1.1389	0.8639	0.4586	16.6855	0.02591
C45	9.0550	1.1706	0.8207	0.3562	17.3002	0.02928
C55	1.1295	2.2342	0.1291	0.2924	13.7030	0.04841

together [23]. Whenever $\rho^{(2)}_{U\to V}$ and $\rho^{(2)}_{V\to U}$ are large or small together, then the directional dependence exists empirically. Besides, the departure from the 45° reference line for the CR plot of $\widetilde{r}_{V|U}(w)$ and $\widetilde{r}_{U|V}(w)$ supports this finding visually. For that reason, when we compare the departure for the models C15 and C44 having smaller and larger values of $\rho^{(2)}_{U\to V}$ and $\rho^{(2)}_{V\to U}$ given in Table 7. The most plausible model can be identified as C44, which can be visualized in Figure 2. Based on the selected model, (SP500, DAX) stock pair is directionally dependent and the total variability of SP500, explained by DAX is 62.56 %, while the variability of DAX is explained by SP500 is 48.07 %. Other CR-plots for the (SP500, DAX) are presented in Appendix part (Figure A1).

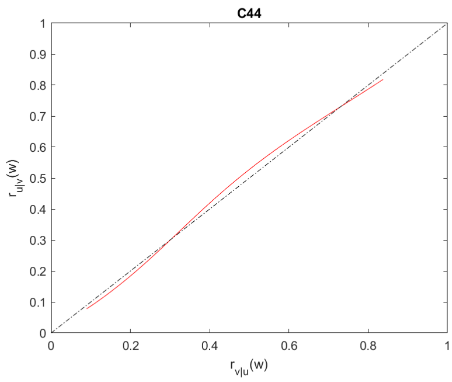
For the stock pair (NIKKEI, FTSE), there are two more likely candidates in the presented results in Table 8, which are C22 and C45. To identify the empirical directional dependence, the departure from the 45° reference line of $\widetilde{r}_{V|U}(w)$ and $\widetilde{r}_{U|V}(w)$ is presented in Figure 3 for C22 and C45. Additionally, LL and S_n values are useful to distinguish the best model when the departure for the models C22 and C45 are high and similar in Figures 3, compared to the other models given in Appendix A (Figure A2). Based on the overall selection, the C22 model is more plausible with smaller S_n and higher LL values. The calculated directional dependence measures can be interpreted as follows: (NIKKEI, FTSE) stock represents a significant directional dependence and the total variability of NIKKEI, explained by FTSE is 5.84 %, while the variability of FTSE is explained by NIKKEI is 15.68 %. Compared to the first pair, the explained variability of the stock pair (NIKKEI, FTSE) is very small. This result matches with the smaller Kendall τ value, presented in Table 3 before.

Table 7. Directional dependence measures for SP500-DAX.

Model	ρ_C	ρ_C^2	τ_C	$\rho^{(2)}_{u\to v}$	$\rho^{(2)}_{v\to u}$
C14	0.5458	0.2979	0.1819	0.4169	0.5557
C15	0.4362	0.1902	0.1454	**0.2998**	**0.4326**
C22	0.5347	0.2859	0.1782	0.4442	0.5788
C23	0.5672	0.3217	0.1891	0.4570	0.5846
C24	0.5560	0.3091	0.1853	0.4500	0.5982
C25	0.5549	0.3079	0.1850	0.4544	0.5766
C33	0.5596	0.3132	0.1865	0.4098	0.5531
C34	0.5949	0.3539	0.1983	0.4636	0.6021
C44	0.5956	0.3547	0.1985	**0.4807**	**0.6256**
C45	0.5905	0.3487	0.1968	0.4642	0.5986
C55	0.5109	0.2611	0.1703	0.3643	0.5036



(a)



(b)

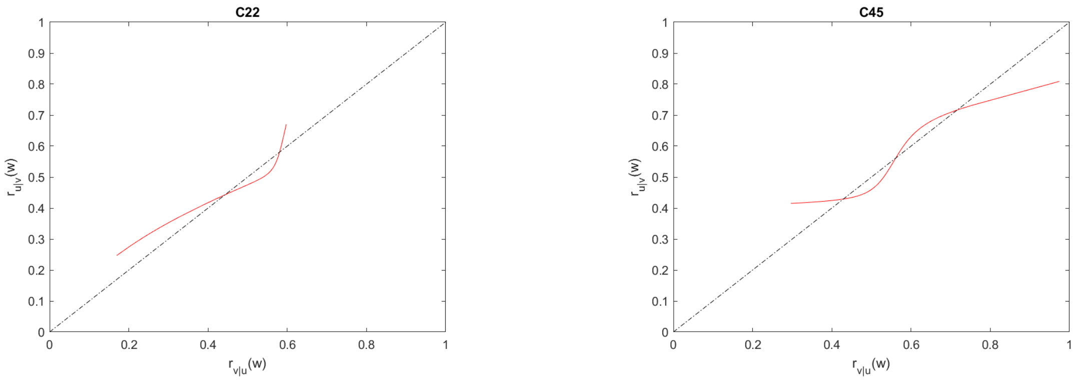
Figure 2. The CR plots of the models (a) C15 and (b) C44 for SP500-DAX.

3.3. Dataset 2

The second dataset used in this study is the stock market index traded on the international stock exchange, taken from the website https://tr.investing.com/indices/world-indices between January 1, 2019 - April 24, 2020. The stock indexes $BIST100$ from Turkey, SSC from China, $Dow30$ and the $SP500$ from America, DAX from Germany, $NIKKEI$ from Japan, $FTSE-MIB$ from Italy were taken, presented in Figure 4. All of the index values exhibit a similar decline pattern during the first months of the pandemic crisis. This is an indication for observing less asymmetric behavior since they are affected by the global health crisis similarly. However, when we look at the pair of stocks, there could be some hidden asymmetry. In this study, daily log returns of stock market index data are calculated to analyze whether directional dependence exists with asymmetric copula models. Here, the log-return formula defined as $r_{i,t} = \log(\frac{x_{i,t}}{x_{i,(t-1)}})$ is used before time series modeling [33].

Table 8. Directional dependence measures for NIKKEI-FTSE.

Model	ρ_C	ρ_C^2	τ_C	$\rho^{(2)}_{u\to v}$	$\rho^{(2)}_{v\to u}$
C14	0.1734	0.0301	0.0578	0.1407	0.0558
C15	0.1262	0.0159	0.0421	0.1216	0.0382
C22	0.2000	0.0400	0.0667	**0.1568**	**0.0584**
C23	0.2022	0.0409	0.0674	0.1510	0.0563
C24	0.2182	0.0476	0.0727	0.1567	0.0351
C25	0.1995	0.0398	0.0665	0.1498	0.0561
C33	0.1874	0.0351	0.0625	0.1308	0.0520
C34	0.1992	0.0397	0.0664	0.1455	0.0538
C44	0.1988	0.0395	0.0663	0.1527	0.0621
C45	0.1990	0.0396	0.0663	**0.1384**	**0.0632**
C55	0.1427	0.0204	0.0476	0.1219	0.0438



(a) **(b)**

Figure 3. The CR plots of the models (a) C22 and (b) C45 for NIKKEI-FTSE

Descriptive statistics regarding the stock market data of the countries are summarized in Table 9. The average and variance values of the return data obtained after transformation are significantly close to each other. On the other hand, the skewness coefficients take negative values for all data, except for NIKKEI. This indicates that these data are skewed left, while the NIKKEI is skewed right. Kurtosis values vary for each but take large numerical values. Since the kurtosis value gives information about the sharpness of the distribution, it indicates heavier tails than a normal distribution. Such insights from Table 9 clearly shows that copulas will be useful for dependence modeling. For the time series modeling, a rich set of ARMA-GARCH models are implemented for filtering the residuals. By referring to the previous studies, the classical ARMA-GARCH setting with different lags and innovations is investigated. In many of the studies, the classical ARMA(1,1)-GARCH(1,1) with student-t innovations are considered for financial time series, similar to the first scenario. However, since the structure might be varying for different stocks, the best candidate among the alternatives is determined for the filtering. In that respect, previously mentioned models are compared with the help of "rugarch" R package to model the features of the financial dataset [14].

Additionally, Kendall's τ correlation coefficients and asymmetric test results are given for each stock exchange pair in Table 10. Accordingly, the country pair with the highest dependency coefficient is DAX (Germany) - $FTSE-MIB$ (Italy) (a high relationship between $DOW30-SP500$ is an expected result). The country pairs $SP500$ (America) - DAX (Germany), $DOW30$ (America) - DAX (Germany), $SP500$ (America) - $FTSEMIB$ (Italy), and $DOW30$ (America) - $FTSE-MIB$ (Italy) follow respectively.

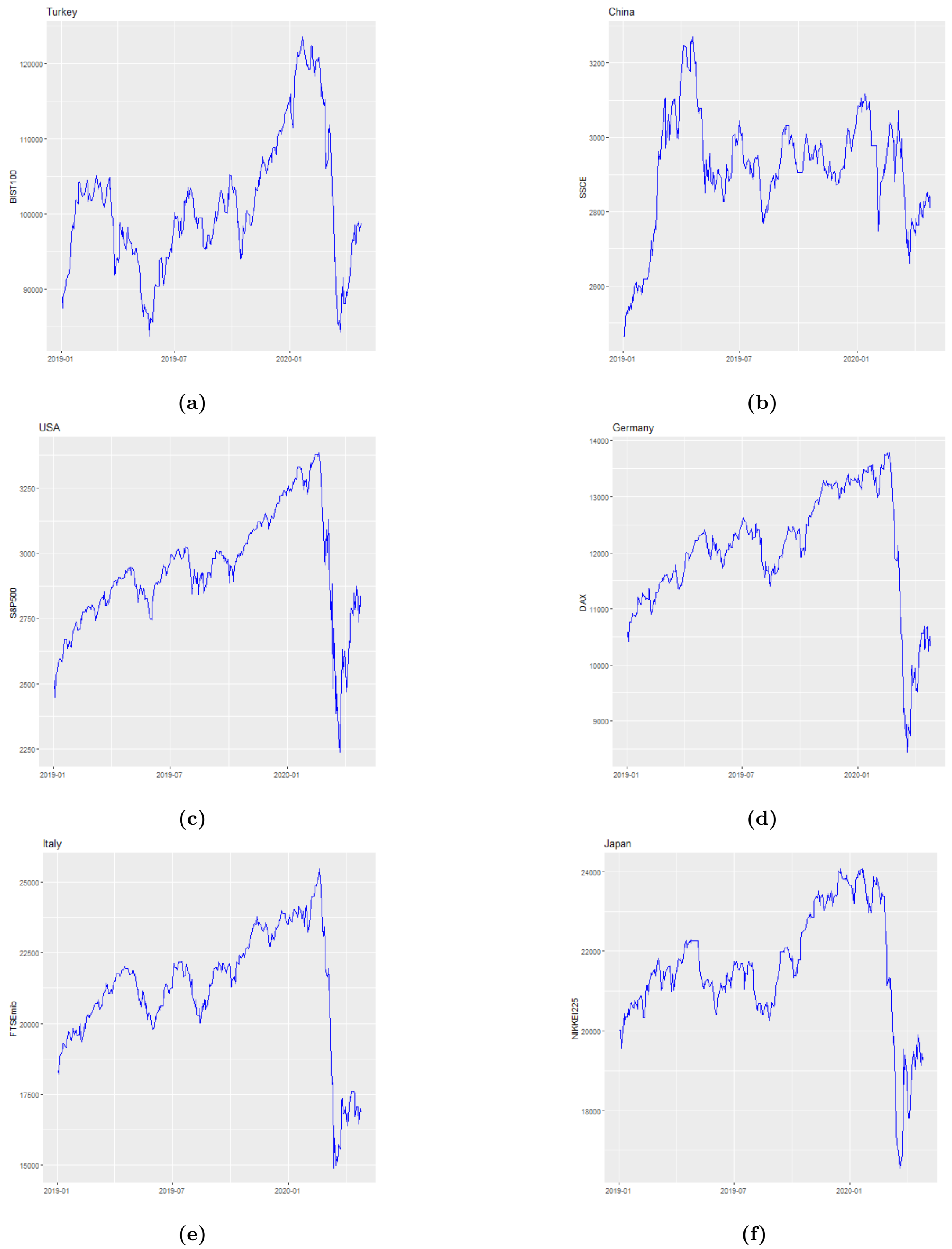


Figure 4. Original stock returns for each country (a) Turkey, (b) China, (c) USA, (d) Germany, (e) Italy and (f) Japan between January 1, 2019 - April 24, 2020.

As for the asymmetry test, there is an asymmetric structure between $SP500$ (America) - $NIKKEI$ (Japan) at a 0.05 significance level, while the dependency between other stock returns does not violate the ex-changeability. For that reason, the rest of the calculations are made for the stock pair ($SP500$, $NIKKEI$) for the selected time period.

Table 9. Descriptive statistics and selected time series models.

	BIST100	**SSCE**	**DOW30**	**SP500**	**DAX**	**NIKKEI**	**FTSE-MIB**
Min	-0.0841	-0.0804	-0.1384	-0.1276	-0.1305	-0.0627	-0.1854
Max	0.0581	0.0545	0.1076	0.0897	0.1041	0.0773	0.0855
Mean	0.0003	0.0004	0.0001	0.0004	-0.0001	-0.0001	-0.0003
Median	0.0003	0.0000	0.0006	0.0008	0.00107	0.0000	0.0009
Variance	0.0002	0.0002	0.0004	0.0003	0.0003	0.0002	0.0003
Skewness	-0.9970	-1.0134	-0.9918	-0.9914	-1.5614	0.2267	-3.8272
Kurtosis	5.4159	7.6759	16.5705	15.0267	20.2778	7.6691	38.1005
Time Series Model	ARMA(0,0) eGARCH(1,1)	ARMA(0,0) iGARCH(1,1)	ARMA(0,0) eGARCH(1,1)	ARMA(0,0) fGARCH(1,1)	ARMA(1,1) eGARCH(1,1)	ARMA(0,0) eGARCH(1,1)	ARMA(0,0) eGARCH(1,1)

Table 10. Kendall's correlation coefficients (above the diagonal) and symmetric - asymmetric test results (p-value) (below the diagonal)
*Asymmetric pair at %95 significance level where p-value <0.05).

	BIST100	SSCE	DOW30	SP500	DAX	NIKKEI	FTSE-MIB
BIST100		0.12	0.19	0.17	0.22	0.11	0.20
SSCE	(0.459)		0.14	0.15	0.21	0.31	0.19
DOW30	(0.358)	(0.122)		0.82	0.46	0.15	0.44
SP500	(0.891)	(0.184)	(0.319)		0.47	0.15	0.45
DAX	(0.933)	(0.972)	(0.989)	(0.711)		0.15	0.62
NIKKEI	(0.830)	(0.540)	(0.058)	(0.046*)	(0.601)		0.13
FTSE-MIB	(0.362)	(0.795)	(0.868)	(0.405)	(0.999)	(0.336)	

The steps of the directional dependence modeling are very similar to the first application. Parameter estimates, log-likelihood (LL) and S_n values for the symmetric and asymmetric models are presented in Table 11 and Table 12, respectively. In Table 11, Clayton copula can show a significant result but not so much powerful for the considered pair (p-value = 0.05544 is a bit higher than the 0.05 level). For the asymmetric models, the models C24, C33, and C35 have certain drawbacks so that they are eliminated for the rest (Table 12). The reason for this elimination is about again touching to the parameter boundaries. For example, model C33 has the dependence parameter $\hat{\theta}_1 = 1$ so that it results in not a constructed asymmetric model since the Gumbel copula exhibits an independence case for that value. For this reason, similar to the first application, totally 11 different asymmetric models are compared in terms of the directional dependence measures and CR plots.

Table 11. The parameter estimation summary of symmetric copula families for SP500-NIKKEI pair (S_n: Cramer-von Mises statistics, LL: Log-Likelihood).

SP500-NIKKEI				
Copula	**Parameter**	LL	S_n	p**-value**
Clayton	0.423	16.24	0.050995	0.05544
Gumbel	1.209	15.34	0.049529	0.03047
Frank	1.476	8.80	0.053948	0.006494
Joe	1.251	11.95	0.062598	0.01748

In Table 13, the calculated $\rho^{(2)}_{U\to V}$ and $\rho^{(2)}_{V\to U}$ measures are presented for the numerical comparison of the candidate models. Clearly, the two candidates are C15 and C23 in terms of the directional dependence values. Additionally, the behavior of both $r_{v|u}(w)$

and $r_{u|v}(w)$ together is investigated within CR-plots, given below in Figure 5 for the best two candidates. In this case, model C15 seems more plausible rather than model C23, supported by the significant S_n values given in Table 12. Based on the final decision, one can interpret that $(SP500, NIKKEI)$ stock pair is directionally dependent and the total variability of $SP500$ explained by $NIKKEI$ is 5.32 %, while the variability of $NIKKEI$ is explained by SP500 is 4.80 %. Certainly, the explained variability for that pair is not large as the previously discussed examples. The remaining CR plots of the other models are presented at the Appendix part (Figure A3).

Table 12. The parameter estimation and asymmetric model results for SP500-NIKKEI.

Model	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\alpha}$	$\widehat{\beta}$	LL	S_n
C12	-	8.1958	0.3676	0.4678	18.6568	0.059716
C13	-	2.4663	0.3101	0.4860	21.2168	0.04293
C14	-	16.0120	0.3225	0.4361	18.3707	0.051249
C15	-	3.1647	0.2622	0.3835	19.1931	0.039166
C22	0.3166	62.2241	0.1668	0.2049	23.2016	0.04673
C23	0.3073	3.2646	0.2032	0.2769	25.0772	0.051691
C24	0.7152	2.0216	0.2702	0.0000	17.3204	-
C25	0.3213	3.6948	0.1778	0.2335	24.3159	0.048257
C33	1.0000	2.4667	0.3099	0.4856	21.2168	0.042938
C34	2.5297	0.2060	0.7034	0.5391	21.2799	0.045677
C35	2.4744	1.0001	0.6902	0.5132	21.2159	-
C44	-0.2310	15.2740	0.3364	0.4565	18.4425	0.047677
C45	0.4448	3.2342	0.2393	0.3457	19.5848	0.042827
C55	1.0190	3.2532	0.2496	0.3628	19.2197	0.039976

Table 13. Directional dependence measures for SP500-NIKKEI.

Model	ρ_C	ρ_C^2	τ_C	$\rho^{(2)}_{u\to v}$	$\rho^{(2)}_{v\to u}$
C12	0.2853	0.0814	0.0951	0.0821	0.0808
C13	0.2608	0.0680	0.0869	0.0674	0.0745
C14	0.2707	0.0733	0.0902	0.0730	0.0738
C15	0.2200	0.0484	0.0733	**0.0480**	**0.0532**
C22	0.2945	0.0867	0.0982	0.0949	0.0918
C23	0.3031	0.0919	0.1010	**0.0980**	**0.0951**
C25	0.2943	0.0866	0.0981	0.0933	0.0909
C34	0.2705	0.0732	0.0902	0.0725	0.0782
C44	0.2603	0.0678	0.0868	0.0673	0.0691
C45	0.2527	0.0638	0.0842	0.0631	0.0662
C55	0.2244	0.0503	0.0748	0.0496	0.0550

According to the above two-step model selection, different asymmetric models are selected for different stock pairs. In dataset 1, C44 and C45 models are the significant ones for the stock pairs $(SP500, DAX)$ and $(NIKKEI, FTSE)$, respectively. Similarly, in dataset 2, C15 asymmetric copula is selected for the pair $(SP500, NIKKEI)$. The magnitude of the established directional dependence measures is reasonable for the existing market all over the world. To illustrate, $SP500$ has more impact on the NIKKEI in dataset 2, including the spreading months of the pandemic. In that respect, since the whole financial system is affected at the same time, the directional dependence measure is not so large. On the other hand, the stock pair $(SP500, DAX)$ for the first data application has larger directional dependence measures comparable to all scenarios.

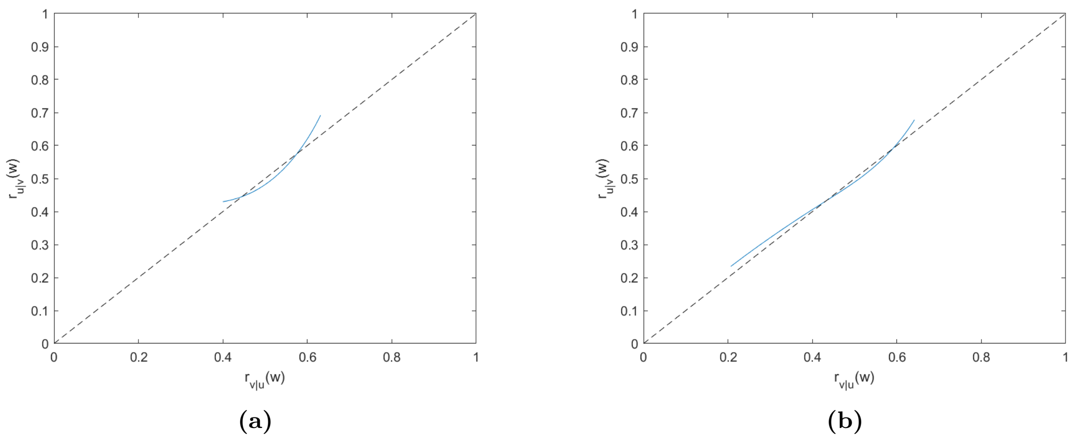


Figure 5. The CR-plots of the models (a) C15 and (b) C23 for SP500-NIKKEI.

4. Conclusion

This study investigates the possible directional dependence measure based on an asymmetric copula setting and demonstrates the application of this procedure for two stock datasets. The first dataset was taken from the CDVine R package while the second up-to-date data is retrieved taken from the world stock market between 01.01.2019-24.04.2020 including early Covid-19 impacts. In terms of the exchangeability test, the identified asymmetric pairs are SP500-DAX and NIKKEI-FTSE in dataset 1, whereas SP500-NIKKEI is in dataset 2. In this study, asymmetric copula modeling and the directional dependence procedure were applied together to select the best asymmetric copula model that explains the nature of the datasets. Here, for all considered stock pairs, the symmetric copulas do not exhibit a significant result, while different asymmetric copula models reveal potential benefits for dependence modeling. Among the considered models, some of the recently proposed models seem to be more successful rather than the classical symmetric and previously studied asymmetric copulas. In that respect, the key findings of the study highlight the importance of ex-changeability and the directional dependence measures interpretation.

For the model selection, the pseudo-likelihood estimation method with the joint maximization over the set of parameters is used with the help of the copula package. For the considered asymmetric models, LL values and Cramer-von Mises (S_n) test statistics were used to compare the models at the first step. Thereafter, the best candidates are discussed according to the empirical directional dependence measures to choose the most powerful one. In this second step, the proposed CR-plots are presented for the top ranked two models having the largest or smallest $\rho^{(2)}_{u\to v}$ and $\rho^{(2)}_{v\to u}$ values together. Especially, such differences on the directional dependence coefficients ($\rho^{(2)}_{u\to v}$ and $\rho^{(2)}_{v\to u}$) might indicate an important occurred event such as a financial crisis within the periods under consideration. Besides, the results of the directional dependence coefficients can help investors in making decisions for their portfolio or risk management in the market. In addition, the key outcomes of the study will contribute to the development of more planned financial strategies before or after any vital change in the market. Last but not least, such directions for the dependence pattern might be useful for understanding other economic activities between countries.

Even if both numerical and graphical diagnostics are used in this study, the main limitation of the work is that the confidence intervals for the considered parameter estimations are not presented. In the next step, the recent findings will be elaborated further on the help of such statistical tools for the model selection process. Besides, it is planning to develop other types of bivariate asymmetric copulas to enlarge the considered pool of models, discussed before in the literature. Furthermore, the proposed approaches can be incorporated for vine copulas to derive more powerful dependence modeling tools including both symmetric and asymmetric patterns. For future studies, directional dependence measures over different time periods can be compared using the change point analysis. Above mentioned research problems are on the list of authors to add further contributions.

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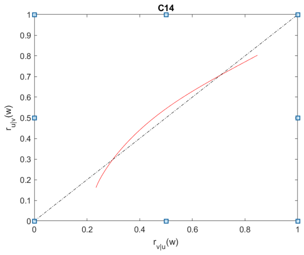
Appendix

Dataset1



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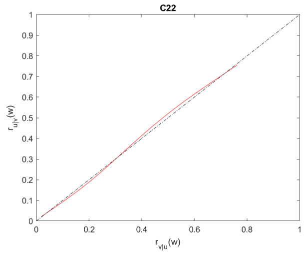
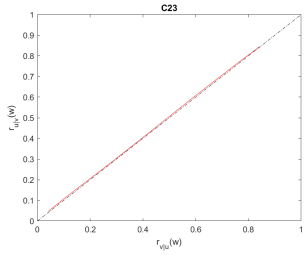


(b) C22



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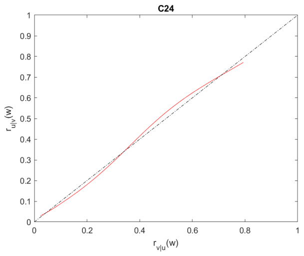
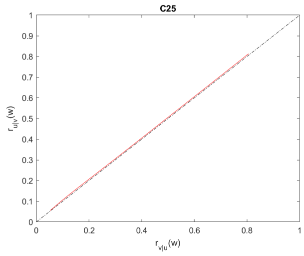
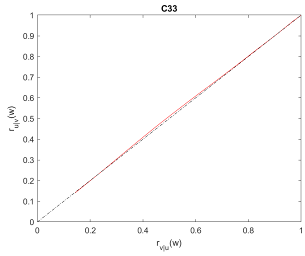


(d) C24



(e) C25



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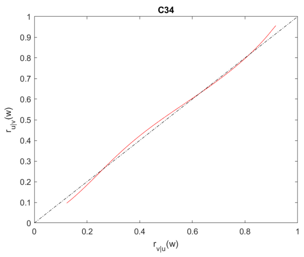

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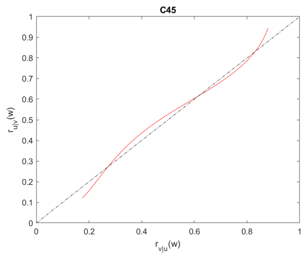

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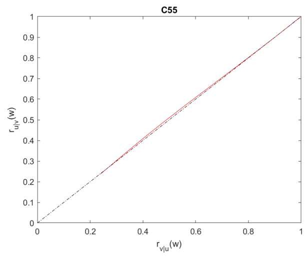
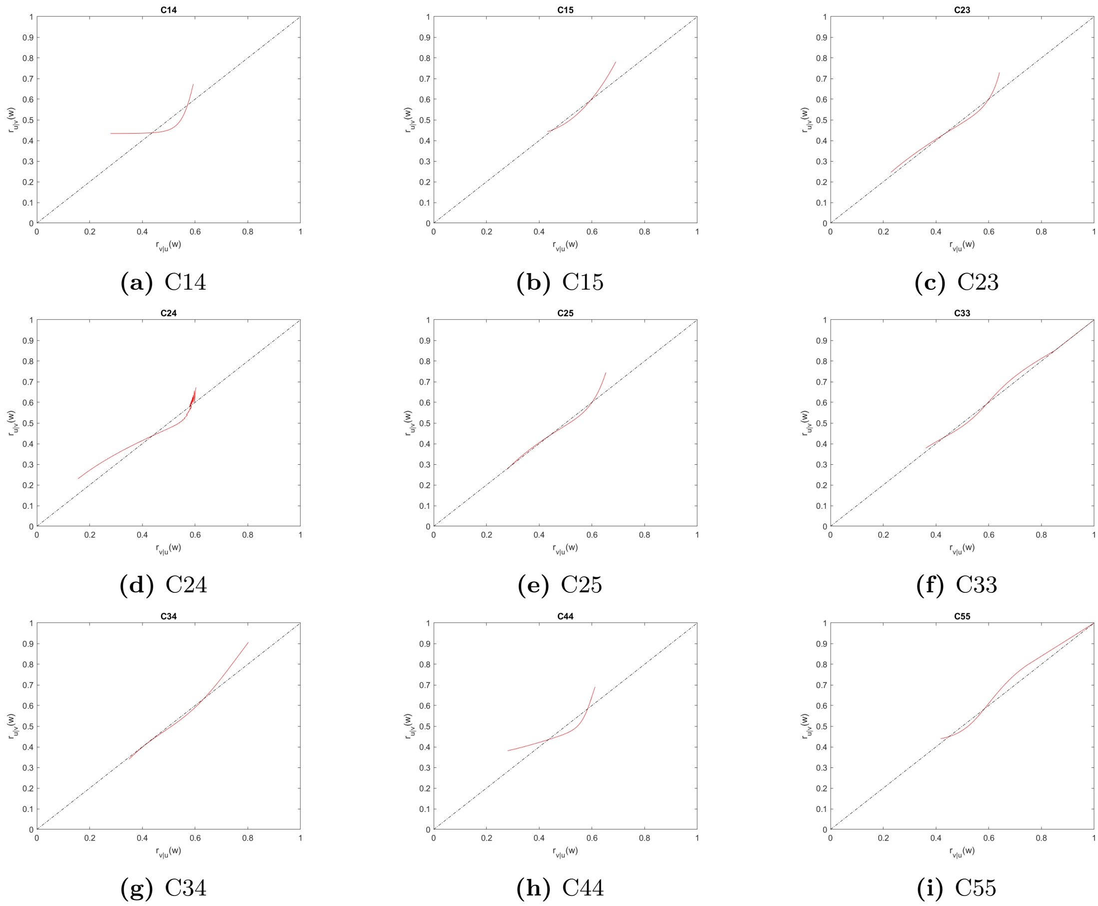


(i) C55

Figure A1. The CR plots of the models for SP500-DAX.



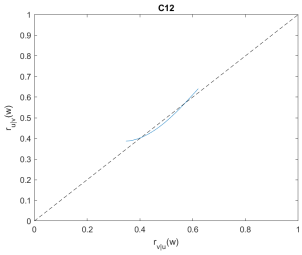
(a) C14 **(b)** C15 **(c)** C23

(d) C24 **(e)** C25 **(f)** C33

(g) C34 **(h)** C44 **(i)** C55

Figure A2. The CR plots of the models for NIKKEI-FTSE.

Dataset2



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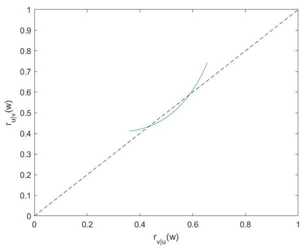
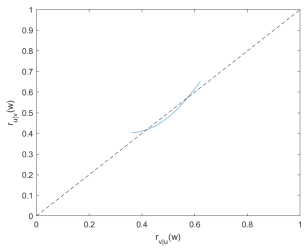
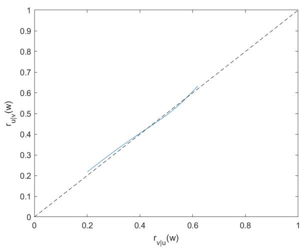


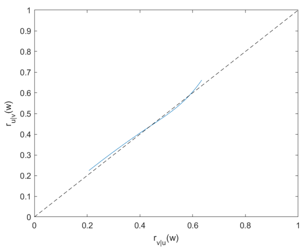
(b) C13



(c) C14



(d) C22



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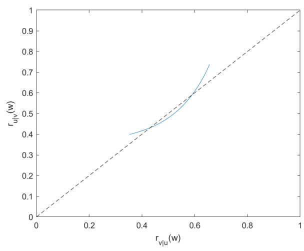

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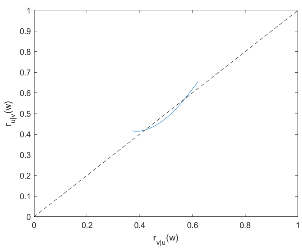

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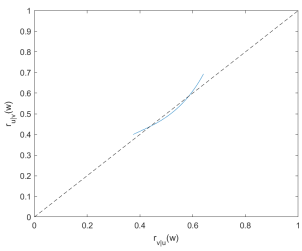
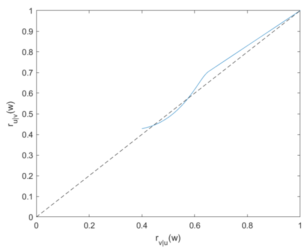


(h) C45



(i) C55

Figure A3. The CR plots of the models for SP500-NIKKEI.