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Existence and convergence for stochastic differential variational inequalities

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Abstract

In this paper, we consider a class of stochastic differential variational inequalities (for short, SDVIs) consisting of an ordinary differential equation and a stochastic variational inequality. The existence of solutions to SDVIs is established under the assumption that the leading operator in the stochastic variational inequality is P-function and P_0-function, respectively. Then, by using the sample average approximation and time stepping methods, two approximated problems corresponding to SDVIs are introduced and convergence results are obtained.

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1. Introduction

The differential variational inequality (for short, DVI), initially introduced by Aubin and Cellina [1] in 1984 and systematically examined by Pang and Stewart [32] in 2008, is the source of problem in this paper. Precisely speaking, DVI is a system that consists of a differential (evolution) equation and a variational inequality. In the past years, considerable literature has been devoted to the mathematical theory and applications of variational inequalities; see [19–23, 26, 30, 33–35]. DVI has been pointed out in [7, 17, 24] to be a powerful mathematical tool to represent models involving both dynamics and constraints in the form of inequalities. It arises in many applied problems in our real life such as mechanical impact problems, electrical circuits with ideal diodes, the Coulomb friction problems for contacting bodies, economical dynamics, dynamic traffic networks, and so on. Since then, increasing number of scholars have been attracted to both theoretical and numerical aspects of the differential variational inequalities as well as its applications in economical dynamics system and contact mechanics problems; see [6, 16, 25, 27, 42–44] and

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the references therein. It is worth mentioning that the differential linear complementarity system (for short, DLCS) is a specific instance of differential variational inequality.

However, the above mentioned researches are deterministic models. In reality, there are many uncertain factors to influence a system which is described by a stochastic model; see [4,5,13,28,38] and the references therein. Hence, the study on stochastic differential variational inequalities (for short, SDVIs) consisting of differential equations and stochastic variational inequalities, is significant and meaningful. In this paper, we study the existence of solutions for a class of stochastic differential variational inequalities. The convergence results to SDVIs are also obtained via the sample average approximation and time stepping methods.

Let $\xi : \Omega \to \mathbb{R}^m$ be a stochastic variable defined in the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with support set $\Xi := \xi(\Omega) \subseteq \mathbb{R}^m$ and $Q = [0,T]\times\mathbb{R}^n$. Let $\mathcal{X} = C([0,T];\mathbb{R}^n)$ denote the space of the n-dimensional vector-valued continuous functions on $[0,T]$ and $\mathcal{Y} = \{y : [0,T]\times\Xi \to \mathbb{R}^m \,|\, y(\cdot,\xi) \in L^1([0,T];\mathbb{R}^m), y(t,\cdot) \text{ is } \mathcal{P}\text{-measurable}\}$ denote the space of m-dimensional vector-valued functions defined on $[0,T]\times\Xi$ in which $y(t,\cdot)$ is measurable such that the expected value is well defined and finite valued for all $t \in [0,T]$ and $y(\cdot,\xi)$ is integrable on $[0,T]$ for each $\xi \in \Xi$. And let $f : Q \to \mathbb{R}^n, \boldsymbol{B} : Q\times\Xi \to \mathbb{R}^{n\times m}, G : Q\times\Xi \to \mathbb{R}^m$ and $F : \Xi\times\mathbb{R}^m \to \mathbb{R}^m$ be given functions. The stochastic differential variational inequality considered in this paper is as follows: find $(x, y) \in \mathcal{X}\times\mathcal{Y}$ such that

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + \mathbb{E}(\boldsymbol{B}(t, x(t), \xi)y(t, \xi)), \\ y(t, \xi) \in \mathbf{SOL}(K, G(t, x(t), \xi) + F(\xi, \cdot)), \\ x(0) = x_0, \quad t \in [0,T], \end{cases} \tag{1.1}$$

where $K \subseteq \mathbb{R}^m$ is closed and convex. $\mathbf{SOL}(K, G(t, x(t), \xi) + F(\xi, \cdot)) \subseteq \mathbb{R}^m$ is the solution set of the following SVI: find $y \in \mathcal{Y}$ such that $y(t, \xi) \in K$ and

$$(\nu - y(t, \xi))^T(G(t, x(t), \xi) + F(\xi, y(t, \xi))) \geq 0, \forall \nu \in K, \text{ a.e. } \xi \in \Xi. \tag{1.2}$$

For convenience, we write $\xi = \xi(\omega)$, $\omega \in \Omega$ in (1.1) and (1.2). Usually, the meaning of such notation will be clear from the context and will not cause any confusion. To highlight the generalization of (1.1), we mention below cases.

(i) In [6], the following differential variational inequality

$$\begin{cases} \dot{x}(t) = f(t, x(t), y(t)), \\ y(t) \in \mathbf{SOL}(K, G(t, x(t), \cdot)), \\ x(0) = x_0, \quad t \in [0,T], \end{cases} \tag{1.3}$$

is considered. Existence of solutions and the convergence analysis of regularized time-stepping methods are obtained there. It is clear that the stochastic variable hasn't been considered in the system (1.3). However, in reality there are numerous uncertain factors to influence systems which should be described by stochastic models; see [4,5,28,37,40]. To our knowledge, the literature on SDVIs is scare.

(ii) In case that $f(t, x(t)) = Ax(t) + f(t)$, $\mathbb{E}(\boldsymbol{B}(t, x(t), \xi)y(t, \xi)) = \mathbb{E}(\boldsymbol{B}(\xi)y(t, \xi))$, $G(t, x(t), \xi) = N(\xi)x(t) + q(t, \xi)$ and $F(\xi, y(t, \xi)) = M(\xi)y(t, \xi)$, $K = \mathbb{R}^m_+$, problem (1.1) reduces to the following differential stochastic linear complementarity problem

$$\begin{cases} \dot{x}(t) = Ax(t) + \mathbb{E}(\boldsymbol{B}(\xi)y(t, \xi)) + f(t), \\ 0 \leq y(t, \xi) \perp N(\xi)x(t) + q(t, \xi) + M(\xi)y(t, \xi) \geq 0, \quad \text{a.e. } \xi \in \Xi, \\ x(0) = x_0, \quad t \in [0,T], \end{cases} \tag{1.4}$$

which has been examined by Luo, Wang and Zhao [28] where $A \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B}(\cdot) : \Xi \to \mathbb{R}^{n\times m}$, $N(\cdot) : \Xi \to \mathbb{R}^{m\times n}$, $M(\cdot) : \Xi \to \mathbb{R}^{m\times m}$, $f : [0,T] \to \mathbb{R}^n$ and $q : [0,T]\times\Xi \to \mathbb{R}^m$ are given functions. Therefore, the stochastic linear complementarity problem (1.4) is extended to the stochastic nonlinear variational inequality (1.1) in this paper. The latter has more generality and wider applications. In the following we present an optimization problem whose KKT condition is described by (1.1) rather than (1.4).

Example 1.1. Let $x : [0, T] \to \mathbb{R}^n$ be a absolutely continuous function and $g : \mathbb{R}^{1+n+m} \ni (t, x(t), \xi) \to g(t, x(t), \xi) \in \mathbb{R}^m$ be continuously differentiable on $x(t)$, t and measurable on ξ such that each component function $g_i : \mathbb{R}^{1+n+m} \to \mathbb{R}$, $i = 1, \cdots, m$, is convex on ($x(t)$, t). Consider stochastic programming problems with inequalities constraints (P):

$$\min_{x_j(t)\in\mathbb{R}} -x_j(t), \quad j = 1, \cdots, n,$$
$$s.t. \;\; g_i(t, x(t), \xi) \le 0, \; i = 1, \cdots, m, \;\; a.e. \; \xi \in \Xi,$$

where $x_j : [0, T] \to \mathbb{R}$, $j = 1, 2, \cdots, n$, is component function of $x(t)$.

To solve this problem, we define a Lagrange function:

$$L(t, x, u) = -x(t) + \mathbb{E}(g(t, x(t), \xi)^T y) I_{n\times n},$$

where $y \in \mathbb{R}^m$ is a Lagrange multiplier of $g(t, x(t), \xi)$ and $I_{n\times n}$ is a n-dimensional identity matrix. Then, the Karush-Kuhn-Tucker (KKT) stochastic condition for problem (P) is

$$\begin{cases} \dot{x}(t) = \mathbb{E}(\boldsymbol{J}g(t, x(t), \xi)^T y), & \text{a.a. } t \in [0, T] \\ y \in \mathbf{SOL}(\mathbb{R}^m_+, -g(t, x(t), \xi)), \end{cases}$$

which can be identified as a special case of SDVI (1.1) with $f(t, x(t)) = 0$, $G(t, x(t), \xi) + F(\xi, y(t, \xi)) = -g(t, x(t), \xi)$ and $\boldsymbol{B}(t, x(t), \xi) = \boldsymbol{J}g(t, x(t), \xi)$, where $\boldsymbol{J}g(t, x(t), \xi)$ denotes the Jacobian matrix of g at $x(t)$.

In fact, KKT condition, as one of the most important theoretical achievements in nonlinear programming, is a significant method to solve optimization problems. Details can be found in [11, 12] and the references therein.

Next, we give a more general example which is a stochastic differential variational inequality rather than just a stochastic differential nonlinear complementarity problem.

Example 1.2. Let F be a given (single-valued) mapping from $\mathbb{R}^n$ into itself. Consider the following problem (P):

$$\dot{x}(t) = -F(x(t)) + \mathbb{E}(w(t, \xi)),$$
$$0 = x(t) - \mathbb{E}(u(t, \xi)),$$
$$u(t, \xi) \in \tilde{K}, \quad (\nu - u(t, \xi))^T w(t, \xi) \ge 0, \; \forall \nu \in \tilde{K},$$

where $\tilde{K} \subseteq \mathbb{R}^m$ is closed and convex cone.

In the past couple decades, differential algebraic equations have become a very important generalization of ordinary differential equations and have been studied extensively. More details can be found in [2, 3, 32, 36].

By introducing an auxiliary variable, the problem (P) is a standard stochastic differential algebraic equations as a special case of the SDVI (1.1) with $y = (w, u)$, $f(t, x(t)) + \mathbb{E}(\boldsymbol{B}(t, x(t), \xi) y(t, \xi)) = -F(x(t)) + \mathbb{E}(w(t, \xi))$, $G(t, x(t), \xi) + F(\xi, y(t, \xi)) = (x - u, w)$ and $K = \mathbb{R}^n \times \tilde{K}$.

Recently, numerous stochastic variational inequalities and stochastic nonlinear problems have been studied; see [8, 13, 18, 39, 40] and the references therein. Motivated by these researches, in this paper, we study a class of stochastic differential variational inequalities given by (1.1).

The remaining of this paper is organized as follows. Section 2 collects some notations and preliminaries materials. In section 3, an existence theorem of solutions to SDVI (1.1) is established in case that F is a P-function via the theory of variational inequality and projection method. The uniqueness of the solution is discussed. Then, when F is a P_0-function, existence of weak solutions to SDVI (1.1) is proved by using the theory of differential inclusions and Filippov's implicit function theorem. Sample average and time stepping approximated problems to SDVI are introduced and convergence results are obtained in section 4.

2. Preliminaries

Let $\mathcal{L} = L^2([0,T];\mathbb{R}^n)$ denote the square integrable function space on $[0,T]$ and $\mathcal{U}$ be the space taken either as $\mathcal{U} = \mathcal{X} \times \mathcal{L}$ or $\mathcal{U} = \mathcal{X} \times \mathcal{X}$. We use $\|\cdot\|_2$ to denote the l_2-norm for vectors. In this section, we recall some definitions and lemmas.

Definition 2.1. [9] A mapping $\widehat{\Psi} : \mathrm{U} \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is said to be weakly univalent on its domain if it is continuous and there exists a sequence of univalent (i.e., continuous and injective) function $\{\widehat{\Psi}^k\}$ from U into $\mathbb{R}^m$ such that $\{\widehat{\Psi}^k\}$ uniformly converges to $\widehat{\Psi}$ on bounded subsets of U.

Lemma 2.2. [9] Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a weakly univalent function. Suppose $f^{-1}(0) \neq \emptyset$ and $f^{-1}(0)$ is compact. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that for weakly univalent function $h : \mathbb{R}^m \to \mathbb{R}^m$ satisfying

$$\sup\left\{\|h(y) - f(y)\|_2 : y \in cl(f^{-1}(0) + B(0,\varepsilon))\right\} \leq \delta,$$

then, we have

$$\emptyset \neq h^{-1}(0) \subseteq f^{-1}(0) + B(0,\varepsilon).$$

Lemma 2.3. [32] Let $\mathbb{F} : Q \to \mathbb{R}^n$ be an upper semi-continuous (usc) set-valued map with nonempty closed and convex value. Suppose that there exists a scalar $\rho_{\mathbb{F}} > 0$ satisfying

$$\sup\{\|u\|_2 : u \in \mathbb{F}(t,x)\} \leq \rho_{\mathbb{F}}(1 + \|x\|_2), \quad \forall\,(t,x) \in Q.$$

Then the Cauchy problem of the differential inclusion

$$\dot{x} \in \mathbb{F}(t,x),\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n$$

has a weak solution in the sense of Carathéodory.

Lemma 2.4. [10] Let $G : Q \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function and $\Phi : Q \to \mathbb{R}^m$ be a closed set-valued map such that for some constant $\eta_\Phi > 0$,

$$\sup\{\|y\|_2 : y \in \Phi(t,x)\} \leq \eta_\Phi(1 + \|x\|_2), \quad \forall\,(t,x) \in Q.$$

Let $\hat{\nu} : [0,T] \to \mathbb{R}^n$ be a measurable function and $x : [0,T] \to \mathbb{R}^n$ be a continuous function satisfying $\hat{\nu}(t) \in G(t, x(t), \Phi(t, x(t)))$ for almost all (a.a.) $t \in [0,T]$. Then, there exists a measurable function $y : [0,T] \to \mathbb{R}^m$ such that $y(t) \in \Phi(t, x(t))$ and $\hat{\nu}(t) = G(t, x(t), y(t))$ for a.a. $t \in [0,T]$.

Lemma 2.5. [6] Let $T > 0, \alpha \geq 0, \gamma \geq 0$, and $\beta > 0$, and let $\Theta : [0,T] \to \mathbb{R}_+$ be (Lebesgue) integrable. If

$$\Theta(t) \leq \alpha + \int_0^t [\beta\Theta(s) + \gamma]ds, \qquad \forall\, t \in [0,T],$$

then

$$\Theta(t) \leq \alpha \exp(\beta t) + \frac{\gamma}{\beta}(\exp(\beta t) - 1),\ \forall\, t \in [0,T].$$

Definition 2.6. [29] The epigraph of a function θ over $\mathcal{U}$ is defined as the set

$$epi\ \theta := \{(x,y,\alpha) \in \mathcal{U} \times \mathbb{R} \mid \theta(x,y) \leq \alpha\}.$$

Let

$$\|u\|_{\mathcal{X}} = \sup_{t\in[0,T]} \|u(t)\|_2,$$

for $u \in \mathcal{X}$ and

$$\|\chi\|_{L^2} = \left(\int_0^T \chi(\tau)^T \chi(\tau) d\tau\right)^{\frac{1}{2}},$$

for $\chi \in \mathcal{L}$. And we define the norms:

$$\|(u,\chi)\|_{\mathcal{X}\times\mathcal{L}} := \|u\|_{\mathcal{X}} + \|\chi\|_{L^2},\ for\ (u,\chi) \in \mathcal{X}\times\mathcal{L},$$

and

$$\|(u,\chi)\|_{\mathcal{X}\times\mathcal{X}} := \|u\|_{\mathcal{X}} + \|\chi\|_{\mathcal{X}},\ for\ (u,\chi) \in \mathcal{X}\times\mathcal{X}.$$

A sequence of functions $\{\theta^k\}_{k=1}^{\infty}$ is said to be epigraphically convergent to a functional θ, denoted by $\theta^k \to^{epi} \theta$, if the following two statements hold:

$$\liminf_{k\to\infty} \theta^k(x^k, y^k) \geq \theta(x,y)\ for\ all\ \{x^k, y^k\}_{k=1}^{\infty} \subset \mathcal{U}\ with\ (x^k, y^k) \to (x,y),$$

$$\limsup_{k\to\infty} \theta^k(x^k, y^k) \leq \theta(x,y)\ for\ some\ \{x^k, y^k\}_{k=1}^{\infty} \subset \mathcal{U}\ with\ (x^k, y^k) \to (x,y).$$

Here the convergence of $(x^k, y^k) \to (x,y)$ is characterized by the norm $\|\cdot\|_{\mathcal{U}}$.

Definition 2.7. [14] A function θ over $\mathcal{U} \times \Xi$ is a random lower semi-continuous (lsc) function if θ is jointly measurable in (x, y, ξ) and $\theta(\cdot,\cdot,\xi)$ is lsc for every $\xi \in \Xi$..

Definition 2.8. [40] A function $\theta(x, y, \xi)$ is said to be a Carathéodory function if $\theta(x, y, \cdot)$ is measurable for every $(x, y) \in \mathcal{U}$ and $\theta(\cdot,\cdot,\xi)$ is continuous for a.e. $\xi \in \Xi$. Obviously, θ is a random lsc if it is a Carathéodory function.

Remark 2.9. [28] A sequence of random lsc function $\{\theta^k\}_{k=1}^{\infty}$ over $\mathcal{U}\times\Xi$ epiconverges to θ on $\mathcal{U}$ a.s. (almost sure), written $\theta^k \to^{epi} \theta$ a.s., if for a.e. $\xi \in \Xi$, $\{\theta(\cdot,\cdot,\xi)\}_{k=1}^{\infty}$ over $\mathcal{U}$ epiconverges to θ over $\mathcal{U}$.

According to Pang-Stewart [32], we give the definition of a weak solution to (1.1).

Definition 2.10. For a fixed $\xi \in \Xi$, a pair of trajectories $(x(t), y(t,\xi))$ is called a solution to (1.1) in the weak sense of Carathéodory if $x(t)$ is absolutely continuous on [0,T] and $y \in \mathcal{Y}$ such that

$$x(t) = x(0) + \int_0^t f(t, x(\tau)) + \mathbb{E}(\boldsymbol{B}(\tau, x(\tau), \xi) y(\tau,\xi)) d\tau, \quad \forall\ t \in [0,T],$$

and $y(t,\xi) \in \mathbf{SOL}(K, G(t, x(t), \xi) + F(\xi,\cdot))$ for a.a. $t \in [0,T]$.

3. Existence of solutions

K is defined as the cartesian product of a finite number of lower-dimensional sets:

$$K = \prod_{\vartheta=1}^{N} K^{\vartheta}, \tag{3.1}$$

where K^{ϑ} is a convex subset of $\mathbb{R}^{m_\vartheta}$ and $\sum_{\vartheta=1}^{N} m_\vartheta = m$. We postulate the following conditions:

(A_0) $F(\xi,\cdot)$ is a continuous and uniform P_0-function [6] on K, i.e.

$$\max_{1\leq\vartheta\leq N} (y_\vartheta - y'_\vartheta)^T (F_\vartheta(\xi, y) - F_\vartheta(\xi, y')) \geq 0, \qquad a.e.\ \xi \in \Xi,$$

for any $y \equiv (y_\vartheta)_{\vartheta=1}^N$ and $y' \equiv (y'_\vartheta)_{\vartheta=1}^N$ in K.

(A_1) $F(\xi,\cdot)$ is a continuous and uniform P-function [9] on K, i.e.

$$\max_{1\leq\vartheta\leq N} (y_\vartheta - y'_\vartheta)^T (F_\vartheta(\xi, y) - F_\vartheta(\xi, y')) \geq \eta_F(\xi)\|y - y'\|_2^2, \quad a.e.\ \xi \in \Xi,$$

for any $y \equiv (y_\vartheta)_{\vartheta=1}^N$ and $y' \equiv (y'_\vartheta)_{\vartheta=1}^N$ in K, where $\eta : \Xi \to (0,\infty)$ is a $\mathcal{P}$-measurable function and $\mathbb{E}(\|\eta_F(\xi)\|_2) < \infty$.

(A_2) $G(\cdot,\cdot,\xi)$ is Lipschitz continuous, i.e., exists a constant $L_G > 0$ such that

$$\|G(t,x,\xi) - G(t',x',\xi)\|_2 \leq_{a.s.} L_G[|t-t^{'}| + \|x-x'\|_2],$$

for all (t,x) and (t',x') in Q.

(A_3) $f(\cdot,\cdot)$ is Lipschitz continuous with constant L_f on Q. And $\boldsymbol{B}(\cdot,\cdot,\xi)$ also is Lipschitz continuous with constant $L_{\boldsymbol{B}}$, i.e.

$$\|\boldsymbol{B}(t,x,\xi) - \boldsymbol{B}(t',x',\xi)\|_2 \leq_{a.s.} L_{\boldsymbol{B}}[|t-t'| + \|x-x'\|_2],$$

where $\mathbb{E}(\|\boldsymbol{B}(t,x,\xi)\|_2) < \infty$, i.e., there exists a constant $\boldsymbol{B}_s > 0$ such that $\|\boldsymbol{B}(t,x,\xi)\|_2 \leq_{a.s.} \boldsymbol{B}_s$.

Remark 3.1. $\|\boldsymbol{B}(t,x,\xi)\|_2 \leq_{a.s.} \boldsymbol{B}_s$ implies that $\mathcal{P}(\|\boldsymbol{B}(t,x,\xi)\|_2 \leq \boldsymbol{B}_s) = 1$.

Remark 3.2. Notice that (A_2) and (A_3) imply that G and f have linear growth on Q in x, i.e., for some positive constants ρ_f and ρ_G and all $(t,x) \in Q$, we have

$$\|G(t,x,\xi)\|_2 \leq_{a.s.} \rho_G(1+\|x\|_2),\ \|f(t,x)\|_2 \leq \rho_f(1+\|x\|_2)\ . \tag{3.2}$$

Lemma 3.3. Suppose (A_1), (A_2) and (A_3) hold. Then $\bar{y}(t,x,\xi)$ is a solution of (1.2) if and only if

$$\bar{y}(t,x,\xi) = \boldsymbol{P}_K[\bar{y}(t,x,\xi) - \gamma(\xi)(G(t,x,\xi) + F(\xi,\bar{y}(t,x,\xi)))],$$

where $\gamma : \Xi \to (0,\infty)$ is a $\mathcal{P}$-measurable function and $\mathbb{E}(\|\gamma(\xi)\|_2) < \infty$ and P_K is projection operator on K; i.e. for $l(t,\xi) \in \mathbb{R}^m$, we have

$$P_K(l(t,\xi)) := \{u \in K : \|l(t,\xi) - u\|_2 = d(l(t,\xi),K)\},\ a.e.\ \xi \in \Xi.$$

Moreover, if

$$\sqrt{1 + h_\gamma^2 h_\sigma^2 - 2\|\gamma(\xi)\|_2 h_F} < 1$$

and

$$\|F(\xi,y_1(t,x,\xi)) - F(\xi,y_2(t,x,\xi))\|_2 \leq \sigma(\xi)\|y_1(t,x,\xi) - y_2(t,x,\xi)\|_2,$$

for all $y_1(t,x,\xi)$, $y_2(t,x,\xi) \in K$, where $\mathbb{E}(\sigma(\xi)) < \infty$. In view of $\mathbb{E}(\|\gamma(\xi)\|_2) < \infty$, $\mathbb{E}(\sigma(\xi)) < \infty$ and $\mathbb{E}(\eta_F(\xi)) < \infty$, we can conclude that there exist h_γ, h_σ and h_F such that $\|\gamma(\xi)\|_2 \leq_{a.s.} h_\gamma$, $\sigma(\xi) \leq_{a.s.} h_\sigma$ and $\eta_F(\xi) \leq_{a.s.} h_F$. Then (1.2) has a unique solution $\bar{y}(t,x,\xi) \in \mathcal{Y}$ which is Lipschitz continuous w.r.t. (t,x) for a.e. $\xi \in \Xi$.

Proof. The first part of this lemma follows from [28, Lemma 1] and [31]. We turn to the proof of the second part. To this end, let Ψ_d be defined as

$$\Psi_d(t,x,y,\xi) := \boldsymbol{P}_K[y(t,x,\xi) - \gamma(\xi)(G(t,x,\xi) + F(\xi,y(t,x,\xi)))].$$

We aim to prove that Ψ_d is a stochastic contractive mapping. For every $y_1(t,x,\xi)$, $y_2(t,x,\xi) \in K$, one has

$$\begin{aligned}
&\|\Psi_d(t,x,y_1,\xi) - \Psi_d(t,x,y_2,\xi)\|_2 \\
&= \|\boldsymbol{P_K}[y_1(t,x,\xi) - \gamma(\xi)(G(t,x,\xi) + F(\xi,y_1(t,x,\xi)))] \\
&- \boldsymbol{P_K}[y_2(t,x,\xi) - \gamma(\xi)(G(t,x,\xi) + F(\xi,y_2(t,x,\xi)))]\|_2 \\
&\leq \|y_1(t,x,\xi) - y_2(t,x,\xi) - \gamma(\xi)[F(\xi,y_1(t,x,\xi)) - F(\xi,y_2(t,x,\xi))]\|_2.
\end{aligned}$$

From (A_1), it follows

$$\begin{aligned}
&\|y_1(t,x,\xi) - y_2(t,x,\xi) - \gamma(\xi)[F(\xi,y_1(t,x,\xi)) - F(\xi,y_2(t,x,\xi))]\|_2^2 \\
&= \|y_1(t,x,\xi) - y_2(t,x,\xi)\|_2^2 + \|\gamma(\xi)\|_2^2\|F(\xi,y_1(t,x,\xi)) - F(\xi,y_2(t,x,\xi))\|_2^2 \\
&- 2\|\gamma(\xi)\|_2\|\langle y_1(t,x,\xi) - y_2(t,x,\xi), F(\xi,y_1(t,x,\xi)) - F(\xi,y_2(t,x,\xi))\rangle\|_2 \\
&\leq_{a.s.} (1 + h_\gamma^2 h_\sigma^2 - 2\|\gamma(\xi)\|_2 h_F)\|y_1(t,x,\xi) - y_2(t,x,\xi)\|_2^2.
\end{aligned}$$

Consequently,

$$\|\Psi_d(t,x,y_1,\xi) - \Psi_d(t,x,y_2,\xi)\|_2$$
$$\leq_{a.s.} \sqrt{1 + h_\gamma^2 h_\sigma^2 - 2\|\gamma(\xi)\|_2 h_F}\|y_1(t,x,\xi) - y_2(t,x,\xi)\|_2.$$

This shows that Ψ_d is contractive as $\sqrt{1 + h_\gamma^2 h_\sigma^2 - 2\|\gamma(\xi)\|_2 h_F} < 1$. Therefore, the solution of (1.2) is unique. Moreover, from [8] and [32, Theorem 5.1], we can draw the conclusion that

$$\|\bar{y}(t_1,x_1,\xi) - \bar{y}(t_2,x_2,\xi)\|_2 \leq \kappa_F(\xi)[L_G(\|x_1 - x_2\|_2 + |t_1 - t_2|)], \quad \forall\, x_1, x_2 \in \bar{X},$$

where $\bar{X}$ is certain domain such that a solution of (1.2) exists, $\mathbb{E}(\kappa_F(\xi)) < \infty$ and L_G is Lipschitz constant of $G(\cdot,\cdot,\xi)$. □

According to Lemma 3.3, the value $\bar{y}(t,x,\xi)$ is uniquely defined for all $t \in [0,T]$, $x \in \bar{X}$ and a.e. $\xi \in \Xi$. Then the right hand expression of the ordinary differential equation in (1.1) can be written as:

$$\mathcal{F}(t,x) := f(t,x) + \mathbb{E}(\boldsymbol{B}(t,x,\xi)\bar{y}(t,x,\xi)). \tag{3.3}$$

Hence, in terms of the implicitly defined function $\bar{y}(t,x,\xi)$, to study (1.1), it suffice solve the ordinary differential equation:

$$\begin{cases} \dot{x}(t) = \mathcal{F}(t,x), \\ x(0) = x_0, \quad t \in [0,T]. \end{cases} \tag{3.4}$$

The Lipschitz continuity of $\boldsymbol{B}(t,x,\xi)\bar{y}(t,x,\xi)$ w.r.t. t and x implies that $\mathbb{E}(\boldsymbol{B}(t,x,\xi)$ $\bar{y}(t,x,\xi))$ is Lipschitz continuous w.r.t. $t \in [0,T]$ and $x \in \bar{X}$. It follows that the function $\mathcal{F}(t,x)$ in (3.3) is also Lipschitz continuous.

Therefore, using the classical existence and uniqueness theorem of ordinary differential equation (cf. [32, Sect. 5.1]), we arrive at the following conclusion.

Theorem 3.4. Suppose the conditions in Lemma 3.3 hold. Then the ordinary differential equation (3.4) exists a unique solution $\bar{x} \in \mathcal{X}$. Hence, the SDVI (1.1) exists a unique solution $(\bar{x},\bar{y}) \in \mathcal{X} \times \mathcal{Y}$.

Note that if $F(\xi,\cdot)$ is a continuous and uniform P_0-function on K, the solution to (1.1) is unnecessarily unique. See the following example.

Example 3.5. Let $\xi \sim \mathbb{N}(\mu,\sigma^2)$. Consider an stochastic differential variational inequality:

$$\begin{cases} \dot{x}(t) = Ax(t) + \mathbb{E}(B(\xi)y(t,\xi)) + f(t), \\ y(t,\xi) \in \mathbf{SOL}(\mathbb{R}^m_+, N(\xi)x(t) + q(t,\xi) + M(\xi)y(t,\xi)), \\ x(0) = x_0, \quad t \in [0,T], \end{cases} \tag{3.5}$$

where $A = 1$, $B(\xi) = (1,1)$, $N(\xi) = (\xi,0)^T$, $x(0) = 0$, $f(t) \equiv 0$, $q(t,x) \equiv (-l,0)^T$, $0 < l < 1$ is a constant, and

$$M(\xi) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is a P_0-matrix. It follows from [9, Sect. 3.5] that $F(\xi,y) = M(\xi)y(t,\xi)$ is a P_0-function. It is readily to see that the SDVI (3.5) has infinitely many solutions $(x(t),y(t,\xi))$:

$$x(t) = \begin{cases} lt\ln\mu & if\ 0 \leq t \leq 1, \\ (l+z)e^{-1}e^t - z & if\ t > 1, \end{cases} y(t,\xi) = \begin{cases} (\frac{l-lt}{\xi},0)^T & if\ 0 \leq t \leq 1, \\ (0,z)^T & if\ t > 1, \end{cases}$$

where $z \geq 0$ is an arbitrary constant.

In the following we show the existence of weak solution of SDVI (1.1) in case that $F(\xi,\cdot)$ is a P_0-function. Denote $N(x,r)$ the close ball centered by x with the radius of r in the l_2 norm. Let

$$\Phi(t,x,\xi) = \mathbf{SOL}(K, G(t,x,\xi) + F(\xi,y))$$

and

$$\Psi(t,x,y,\xi) = y - \boldsymbol{P_K}[y - \gamma(\xi)(G(t,x,\xi) + F(\xi,y))].$$

In addition, we also define

$$\mathcal{H}(t,x) := \{f(t,x) + \mathbb{E}(\boldsymbol{B}(t,x,\xi)y) \mid y \in \Phi(t,x,\xi)\} \tag{3.6}$$

and

$$\Omega_\varepsilon := \{y | dist(y, \Phi(0,x_0,\xi)) \leq \varepsilon\}, \tag{3.7}$$

for a positive number $\varepsilon > 0$, where $dist(y,\Phi(0,x_0,\xi)) = \min\limits_{\hat{v}\in\Phi(0,x_0,\xi)} \|y-\hat{v}\|_2$.

Lemma 3.6. Suppose $\Phi(0,x_0,\xi)$ is nonempty and bounded for a.e. $\xi \in \Xi$, and $(A_0), (A_2)$ and (A_3) hold. Then there exist $T_0 > 0$ and $\delta_0 > 0$ such that $\Phi(t,x,\xi)$ and $\mathcal{H}(t,x)$ are nonempty and bounded for every $(t,x) \in [0,T_0] \times N(x_0,\delta_0)$ and a.e. $\xi \in \Xi$.

Proof. We define the mapping $\widehat{\Psi}(y) := \Psi(t,x,y,\xi)$. It is obvious that $\widehat{\Psi}(y)$ is weakly univalent (see Definition 2.1) for any fixed t, x and ξ. Then, according to Lemma 2.2, we know that for every $\varepsilon > 0$ if there exists $\delta > 0$ such that

$$\sup_{y\in\Omega_\varepsilon} \|\Psi(t,x,y,\xi) - \Psi(0,x_0,y,\xi)\|_2 \leq_{a.s.} \delta, \tag{3.8}$$

then $\Phi(t,x,\xi)$ is nonempty and bounded with

$$\emptyset \neq \Phi(t,x,\xi) \subseteq \Omega_\varepsilon.$$

To show (3.8), we choose T_0 and δ_0 such that

$$h_\gamma L_G(T_0 + \delta_0) < \delta.$$

Then we have

$$\begin{aligned}
&\|\Psi(t,x,y,\xi) - \Psi(0,x_0,y,\xi)\|_2 \\
&= \|\boldsymbol{P_K}[y - \gamma(\xi)(G(t,x,\xi) + F(\xi,y))] - \boldsymbol{P_K}[y - \gamma(\xi)(G(0,x_0,\xi) + F(\xi,y))]\|_2 \\
&\leq \|\gamma(\xi)\|_2 \|G(t,x,\xi) - G(0,x_0,\xi)\|_2 \\
&\leq_{a.s.} h_\gamma L_G(t + \|x - x_0\|_2) \\
&\leq h_\gamma L_G(T_0 + \delta_0) < \delta.
\end{aligned}$$

This gives (3.8) and therefore, $\Phi(t,x,\xi)$ is nonempty and bounded for every $(t,x) \in [0,T_0] \times N(x_0,\delta_0)$ and a.e. $\xi \in \Xi$. And then, following [32, Sect. 6.1], we can conclude that $\mathcal{H}(t,x)$ is also nonempty and bounded for every $(t,x) \in [0,T_0] \times N(x_0,\delta_0)$. □

Theorem 3.7. Suppose that the hypotheses of Lemma 3.6 hold. Then $\Phi(\cdot,\cdot,\xi)$ is upper semi-continuous in $[0,T_0]\times N(x_0,\delta_0)$ for a.e. $\xi \in \Xi$ and $\mathcal{H}(\cdot,\cdot)$ is also upper semi-continuous in $[0,T_0] \times N(x_0,\delta_0)$. Moreover, (1.1) has a weak solution on $[0,T_0]$.

Proof. Since $\Phi(t,x,\xi)$ is nonempty and bounded for every $(t,x) \in [0,T_0] \times N(x_0,\delta_0)$ and a.e. $\xi \in \Xi$, we deduce that there exists $\rho > 0$ such that

$$\sup\{\|y\|_2 : y \in \Phi(t,x,\xi)\} \leq \rho(1 + \|x\|_2), \tag{3.9}$$

for every $(t,x) \in [0,T_0] \times N(x_0,\delta_0)$ with fixed $\xi \in \Xi$. Moreover, $G(t,x,\xi) + F(\xi,y)$ is monotone and continuous w.r.t. y by assumptions (A_0). Hence, following [32, Sect. 6.1], we can deduce that $\Phi(t,x,\xi)$ is convex and closed for any fixed (t,x) and $\xi \in \Xi$. This, combination with (3.9), means that $\Phi(\cdot,\cdot,\xi)$ is upper semi-continuous for fixed $\xi \in \Xi$.

On the other hand, by assumption (A_3), there exists a scalar $\lambda_{\mathcal{H}}$ satisfying

$$\sup\{\|\tilde{\nu}\|_2 : \tilde{\nu} \in \mathcal{H}(t,x)\} \leq_{a.s.} \lambda_{\mathcal{H}}(1 + \|x\|_2), \ \forall\ (t,x) \in [0,T_0] \times N(x_0,\delta_0).$$

In the following we aim to prove the upper semi-continuity of $\mathcal{H}(t,x)$ on $(t,x) \in [0,T_0] \times N(x_0,\delta_0)$. It suffices to show that the set-valued mapping $\mathcal{H}$ is closed on $(t,x) \in [0,T_0] \times N(x_0,\delta_0)$. To this end, let the sequence $\{(t_k,x_k)\} \subseteq [0,T_0] \times N(x_0,\delta_0)$ be a sequence converging to some vector $(t_\infty,x_\infty) \in [0,T_0] \times N(x_0,\delta_0)$ and $\{f(t_k,x_k)+\mathbb{E}(\boldsymbol{B}(t_k,x_k,\xi)y_k)\}$ converges to some vector $z_\infty \in \mathbb{R}^n$ as $k \to \infty$, where $y_k \in \Phi(t_k,x_k,\xi)$ for every $k \geq 1$ and a.e. $\xi \in \Xi$. It follows that the sequence $\{y_k\}$ is bounded, and thus has a convergent subsequence with a limit y_∞, which implies that $z_\infty = f(t_\infty,x_\infty) + \mathbb{E}(\boldsymbol{B}(t_\infty,x_\infty,\xi)y_\infty) \in \mathcal{H}(t_\infty,x_\infty)$. This implies that $\mathcal{H}(t,x)$ is closed and also upper semi-continuous. Hence, following [32, Proposition 6.1], it can be derived that

$$\begin{cases} \dot{x} \in \mathcal{H}(t,x), \\ x(0) = x_0, \end{cases}$$

has a weak solution $\tilde{x}$. Therefore, we deduce

$$\|\tilde{x}(t)\|_2 \leq_{a.s.} \|x_0\|_2 + \int_0^t \rho_{\mathcal{H}}(1 + \|x(\tau)\|_2)d\tau.$$

Then, by Gronwall's lemma, we obtain

$$\|\tilde{x}(t)\|_2 \leq_{a.s.} (\|x_0\|_2 + \rho_{\mathcal{H}}T_0)exp(\rho_{\mathcal{H}}T_0).$$

In view of (3.9) and Lemma 2.4, there exists a measurable $\tilde{y}(t,\xi)$ w.r.t. t such that $\tilde{y}(t,\xi) \in \Phi(t,\tilde{x},\xi)$ and $\dot{\tilde{x}} = f(t,\tilde{x}) + \mathbb{E}(\boldsymbol{B}(t,\tilde{x},\xi)\ \tilde{y}(t,\xi))$ for a.a. $t \in [0,T_0]$. This shows that $(\tilde{x}(t),\tilde{y}(t,\xi))$ is a weak solution of (1.1). □

4. Discrete approximation

In this section we study the discrete approximations and convergence analysis to SDVI (1.1) in case that the leading operator F is a P-function. If F is a P_0-function, the convergence analysis of (1.1) can be discussed by a standard regularization method [6,28] to ensure the uniqueness of solution and the convergence results of the former case.

4.1. Convergence analysis of sample average approximation

Let $\xi_1, \xi_2, \cdots, \xi_v$ be the independent identically distributed (i.i.d.) samples. Then on the basis of these i.i.d. samples, we can get the following sample average approximate (SAA) [40, Sect. 5] problem of (1.1):

$$\begin{cases} \dot{x}(t) = f(t,x(t)) + \frac{1}{v}\sum\limits_{l=1}^{v} \boldsymbol{B}(t,x(t),\xi_l)y(t,\xi_l), \\ (w - y(t,\xi_l))^T(G(t,x(t),\xi_l) + F(\xi_l,y(t,\xi_l))) \geq 0, \ l = 1,\cdots,v, \ \forall w \in K, \\ x(0) = x_0, \quad t \in [0,T]. \end{cases} \tag{4.1}$$

If F is a P_0-function, (1.2) might have multiple solutions. In this case, (1.1) can be transformed into the following differential inclusion system:

$$\begin{cases} \dot{x}(t) \in \mathcal{H}(t,x), \\ x(0) = x_0, \quad t \in [0,T], \end{cases} \tag{4.2}$$

where $\mathcal{H}(t,x)$ is defined in (3.6). In what follows, we always assume that F is a P-function. Then, (1.2) has a unique solution $\breve{y}(t,x(t),\xi)$ and system (4.2) can be written as:

$$\begin{cases} \dot{x}(t) = f(t,x(t)) + \Gamma(t,x(t)), \\ x(0) = x_0, \quad t \in [0,T], \end{cases} \tag{4.3}$$

where

$$\Gamma(t,x(t)) := \mathbb{E}(\boldsymbol{B}(t,x(t),\xi)\breve{y}(t,x(t),\xi)).$$

If $\breve{x}(\cdot)$ is a solution of (4.3), then $(\breve{x}(\cdot), \breve{y}(\cdot, \tilde{x}(\cdot), \xi))$ is a solution of (1.1). Conversely, if $(\breve{x}(\cdot), \breve{y}(\cdot, \cdot))$ is a solution of (1.1), then $\breve{x}(\cdot)$ is a solution of (4.3). Similarly, the SAA problem (4.1) can be written as:

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + \Gamma^v(t, x(t)), \\ x(0) = x_0, \quad t \in [0, T], \end{cases} \tag{4.4}$$

where

$$\Gamma^v(t, x(t)) := \frac{1}{v} \sum_{l=1}^{v} \boldsymbol{B}(t, x(t), \xi_l) \breve{y}(t, x(t), \xi_l).$$

System (4.4) can be viewed as the SAA problem of (4.3). If $(\breve{x}^v(t), \breve{y}(t, \breve{x}^v(t), \xi_l))$ is a unique solution of (4.1) with $l = 1, \cdots, v$, then $\breve{x}^v(t)$ is unique solution of (4.4). Analogously, if $\breve{x}^v(t)$ is a unique solution of (4.4), then $(\breve{x}^v(t), \breve{y}(t, \breve{x}^v(t), \xi_l))$ is a unique solution of (4.1) with $l = 1, \cdots, v$.

Define

$$I(x, \phi)(t) = \begin{pmatrix} R(t, x(t)) \\ \phi(t) \end{pmatrix} \quad and \quad I^v(x^v, \phi^v)(t) = \begin{pmatrix} R^v(t, x^v(t)) \\ \phi^v(t) \end{pmatrix}, \tag{4.5}$$

where

$$R(t, x(t)) := x(t) - x_0 - \int_0^t f(\tau, x(\tau)) + \phi(\tau) d\tau, \;\; \phi(\tau) := \Gamma(t, x(t)),$$

$$R^v(t, x^v(t)) := x^v(t) - x_0 - \int_0^t f(\tau, x^v(\tau)) + \phi^v(\tau) d\tau \, and \, \phi^v(\tau) := \Gamma^v(t, x^v(t)).$$

It is obvious that $I(x, \phi) \in \mathcal{X} \times \mathcal{L}$ for an $(x, \phi) \in \mathcal{X} \times \mathcal{L}$, and if $(x, \phi) \in \mathcal{X} \times \mathcal{X}$ then $I(x, \phi) \in \mathcal{X} \times \mathcal{X}$. So is I^v. Therefore, we know that $x(t)$ is a classic solution of (4.3) when $\|R\|_{\mathcal{X}} = 0$ and $\phi \in \mathcal{X}$. If $\|R\|_{\mathcal{X}} = 0$ and $\phi \in \mathcal{L}$, then $x(t)$ is a weak solution (4.3). Similarity, $\|R^v\|_{\mathcal{X}} = 0$ and $\phi^v \in \mathcal{X}$ means $x^v(t)$ is a classic solution of (4.4). And $x^v(t)$ is a weak solution of (4.4) if $\|R^v\|_{\mathcal{X}} = 0$ and $\phi^v \in \mathcal{L}$.

Lemma 4.1. Let $\{v_k\}_{k=1}^{\infty} \to \infty$ be given. For any $\{(x^{v_k}, \phi^{v_k})\}_{k=1}^{\infty} \subset \mathcal{U}$, if $(x^{v_k}, \phi^{v_k}) \to (x, \phi)$ with probability 1 (w.p.1) by the norm $\|\cdot\|_{\mathcal{U}}$, we have $\|I^{v_k}(x^{v_k}, \phi^{v_k})\|_{\mathcal{U}} \to \|I(x, \phi)\|_{\mathcal{U}}$ w.p.1 and $\|I^{v_k}\|_{\mathcal{U}} \to^{epi} \|I\|_{\mathcal{U}}$ a.s., by taking $\mathcal{U} = \mathcal{X} \times \mathcal{L}$ or $\mathcal{U} = \mathcal{X} \times \mathcal{X}$.

Proof. Taking $\mathcal{U} = \mathcal{X} \times \mathcal{L}$, then we have

$$\begin{aligned}
&\|I^{v_k}(x^{v_k}, \phi^{v_k}) - I(x, \phi)\|_{\mathcal{X}\times\mathcal{L}} \\
&= \sup_{t\in[0,T]} \|R^{v_k}(t, x^{v_k}(t)) - R(t, x(t))\|_2 + \|\phi^{v_k} - \phi\|_{L^2} \\
&= \sup_{t\in[0,T]} \left\| x^{v_k}(t) - x(t) - \int_0^t f(\tau, x^{v_k}(\tau)) - f(\tau, x(\tau)) + \phi^{v_k}(\tau) - \phi(\tau) d\tau \right\|_2 \\
&+ \|\phi^{v_k} - \phi\|_{L^2} \\
&\le (1 + TL_f)\|x^{v_k} - x\|_{\mathcal{X}} + \int_0^T \|\phi^{v_k}(\tau) - \phi(\tau)\|_2 d\tau + \|\phi^{v_k} - \phi\|_{L^2} \\
&\le (1 + TL_f)\|x^{v_k} - x\|_{\mathcal{X}} + \sqrt{T} \left(\int_0^T \|\phi^{v_k}(\tau) - \phi(\tau)\|_2^2 d\tau \right)^{\frac{1}{2}} + \|\phi^{v_k} - \phi\|_{L^2} \\
&\le (1 + TL_f)\|x^{v_k} - x\|_{\mathcal{X}} + (1 + \sqrt{T})\|\phi^{v_k} - \phi\|_{L^2}.
\end{aligned}$$

Since $(x^{v_k}, \phi^{v_k}) \to (x, \phi)$ w.p.1 by the norm $\|\cdot\|_{\mathcal{X}\times\mathcal{L}}$, we can conclude that $\|I^{v_k}(x^{v_k}, \phi^{v_k})\|_{\mathcal{X}\times\mathcal{L}} \to \|I(x, \phi)\|_{\mathcal{X}\times\mathcal{L}}$ w.p.1 which implies that $\|I^{v_k}\|_{\mathcal{X}\times\mathcal{L}} \to^{epi} \|I\|_{\mathcal{X}\times\mathcal{L}}$ a.s. since $\|I^{v_k}\|_{\mathcal{X}\times\mathcal{L}}$ is a Carathéodory function and a random lsc function.

Similarly, taking $\mathcal{U} = \mathcal{X} \times \mathcal{X}$, we have

$$\begin{aligned}
&\|I^{v_k}(x^{v_k}, \phi^{v_k}) - I(x,\phi)\|_{\mathcal{X}\times\mathcal{X}} \\
&= \sup_{t\in[0,T]} \|R^{v_k}(t, x^{v_k}(t)) - R(t, x(t))\|_2 + \|\phi^{v_k} - \phi\|_{\mathcal{X}} \\
&= \sup_{t\in[0,T]} \left\| x^{v_k}(t) - x(t) - \int_0^t f(\tau, x^{v_k}(\tau)) - f(\tau, x(\tau)) + \phi^{v_k}(\tau) - \phi(\tau) d\tau \right\|_2 \\
&+ \|\phi^{v_k} - \phi\|_{\mathcal{X}} \\
&\le (1 + TL_f)\|x^{v_k} - x\|_{\mathcal{X}} + \int_0^T \|\phi^{v_k}(\tau) - \phi(\tau)\|_2 d\tau + \|\phi^{v_k} - \phi\|_{\mathcal{X}} \\
&\le (1 + TL_f)\|x^{v_k} - x\|_{\mathcal{X}} + (1+T)\|\phi^{v_k} - \phi\|_{\mathcal{X}}.
\end{aligned}$$

Therefore, we also derive that $\|I^{v_k}(x^{v_k}, \phi^{v_k})\|_{\mathcal{X}\times\mathcal{X}} \to \|I(x,\phi)\|_{\mathcal{X}\times\mathcal{X}}$ w.p.1 and then $\|I^{v_k}\|_{\mathcal{X}\times\mathcal{X}}$ $\to^{epi}$ $\|I\|_{\mathcal{X}\times\mathcal{X}}$ a.s. by $\|I^{v_k}\|_{\mathcal{X}\times\mathcal{X}}$ is is a Carathéodory function and a random lsc function. □

Under the assumptions of Lemma 3.3, the (1.2) exists a unique Lipchitz continuous solution. Hence, (4.4) admits a unique solution for each fixed v.

Proposition 4.2. Suppose the conditions in Lemma 3.3 hold. Denote $x^v \in X$ a unique solution of (4.4). Then there exists a constant $M_{\bar{\gamma_k}} > 0$ such that for any $v \in \mathbb{N}^+ := \{v | v \in \{1, 2, 3, \cdots\}\}$

$$\|x^v\|_{\mathcal{X}} \le_{a.s.} (\|x_0\| + 1) exp(\rho_f + \boldsymbol{B}_s M_{\bar{\gamma_k}})T - 1,$$

where $\bar{\gamma_k} > 0$ such that $\|\kappa_F(\xi_l)\|_2 \le_{a.s.} \bar{\gamma_k}$ since $\mathbb{E}(\kappa_F(\xi_l)) < \infty$.

Proof. Under the assumptions in Lemma 3.3, we know that $\breve{y}(t, \bar{x}(t), \xi_l)$ is a unique solution of (1.2) for any $t \in [0.T]$, $\bar{x} \in X$ and $l = 1, \cdots, v$, and it is Lipschitz continuous w.r.r. t and $\bar{x}(t)$. Hence, we can deduce that

$$\|\breve{y}(t, \bar{x}(t), \xi_l)\|_2 \le_{a.s.} M_{\bar{\gamma_k}}(1 + \|\bar{x}(t)\|_2), \tag{4.6}$$

for any $t \in [0, T], \bar{x} \in X$ and $l = 1, \cdots, v$. We know that $\phi^v \in \mathcal{X}$ for a fixed v by the continuity of $\breve{y}(\cdot, \cdot, \xi_l)$ and then

$$\begin{aligned}
\|\phi^v(t)\|_2 &\le \left\| \frac{1}{v} \sum_{l=1}^{v} \boldsymbol{B}(t, \bar{x}(t), \xi_l) \breve{y}(t, \bar{x}(t), \xi_l) \right\|_2 \\
&\le \frac{1}{v} \sum_{l=1}^{v} \|\boldsymbol{B}(t, \bar{x}(t), \xi_l)\|_2 \, \|\breve{y}(t, \bar{x}(t), \xi_l)\|_2 \\
&\le_{a.s.} \boldsymbol{B}_s M_{\bar{\gamma_k}}(1 + \|\bar{x}(t)\|_2).
\end{aligned} \tag{4.7}$$

Since $x^v \in X$, then for any $t \in [0, T]$, we have

$$\begin{aligned}
\|x^v(t)\|_2 &\le \|x_0\|_2 + \left\| \int_0^t f(\tau, x^v(\tau)) + \phi^v(\tau) d\tau \right\|_2 \\
&\le_{a.s.} \|x_0\|_2 + \int_0^t \|f(\tau, x^v(\tau))\|_2 + \boldsymbol{B}_s M_{\bar{\gamma_k}}(1 + \|x^v(\tau)\|_2) d\tau \\
&\le \|x_0\|_2 + \int_0^t \rho_f(1 + \|x^v(\tau)\|_2) + \boldsymbol{B}_s M_{\bar{\gamma_k}}(1 + \|x^v(\tau)\|_2) d\tau \\
&= \|x_0\|_2 + \int_0^t (\rho_f + \boldsymbol{B}_s M_{\bar{\gamma_k}})\|x^v(\tau)\|_2 + (\rho_f + \boldsymbol{B}_s M_{\bar{\gamma_k}}) d\tau.
\end{aligned}$$

Hence, according to Lemma 2.5, we have

$$\|x^v(t)\|_2 \le_{a.s.} \|x_0\|_2 exp(\rho_f + \boldsymbol{B}_s M_{\bar{\gamma_k}})t + exp(\rho_f + \boldsymbol{B}_s M_{\bar{\gamma_k}})t - 1,$$

which means, for $t \in [0,T]$,

$$\begin{aligned}\|x^v\|_{\mathcal{X}} &\leq_{a.s.} \|x_0\|_2 exp(\rho_f + \boldsymbol{B}_s M_{\bar{\gamma}_k})t + exp(\rho_f + \boldsymbol{B}_s M_{\bar{\gamma}_k})t - 1 \\ &\leq (1 + \|x_0\|_2) exp(\rho_f + \boldsymbol{B}_s M_{\bar{\gamma}_k})T - 1.\end{aligned}$$

□

Theorem 4.3. Suppose the conditions in Lemma 3.3 hold. Let x^v be a solution of (4.4). Then there are a sequence $\{v_k\}_{k=1}^{\infty} \to \infty$, $x^* \in \mathcal{X}$ and $\phi^* \in \mathcal{L}$ such that $x^{v_k} \to x^*$ w.p.1 uniformly on $[0,T]$ and $\phi^{v_k} \to \phi^*$ w.p.1 weakly in $\mathcal{L}$. Moreover, if $\phi^{v_k} \to \phi^*$ w.p.1 w.r.t. $\|\cdot\|_{L^2}$, then x^* is a weak solution of system (4.3). If $\phi^{v_k} \to \phi^*$ w.p.1 uniformly on $[0,T]$, then x^* is a classic solution of system (4.3).

Proof. According to Proposition 4.2, we know that $\{x^v\}$ is uniformly bounded on $[0,T]$ and so is $\{\dot{x}^v\}$, which implies $\{x^v\}$ is equicontinuous on $[0,T]$. Then by the Arzelá-Ascoli theorem [15, 41], there exists a sequence $\{v_k\}_{k=1}^{\infty} \to \infty$ such that $\{x^{v_k}\}$ is convergent to an $x^* \in \mathcal{X}$ w.p.1 uniformly on $[0,T]$.

Similarly, from Proposition 4.2 and (4.7), we know that $\{\phi^v\}$ is also uniformly bounded on $[0,T]$ w.p.1 for v sufficiently large. By Alaoglu's theorem [15], there exists a subsequence of $\{\phi^{v_k}\}$, which we may assume without loss of generality to be $\{\phi^{v_k}\}$ itself, has a weak limit, named ϕ^*, as $\mathcal{L}$ is a reflexive Banach space.

Since $x^v(t)$ is a solution of (4.4), we know that $x^v(t)$ is continuous and $\phi^v(t)$ is also continuous under the assumption in Lemma 3.3. Hence, $\|R^v\|_{\mathcal{X}} = 0$ a.s.. In addition, if $\phi^{v_k} \to \phi^*$ w.p.1 w.r.t. $\|\cdot\|_{L^2}$, then we can see that $\|R\|_{\mathcal{X}} = 0$ a.s. and $\phi^* \in \mathcal{L}$ a.s. by

$$\|I^{v_k}(x^{v_k}, \phi^{v_k})\|_{\mathcal{X}\times\mathcal{L}} \to \|I(x^*, \phi^*)\|_{\mathcal{X}\times\mathcal{L}} \; w.p.1 \; as \; k \to \infty,$$

in Lemma 4.1. It means that x^* is a weak solution of system (4.3). Similarly, if $\phi^{v_k} \to \phi^*$ w.p.1 uniformly on $[0,T]$, we know that $\|R\|_{\mathcal{X}} = 0$ a.s. and $\phi^* \in \mathcal{X}$ a.s.. Then x^* is a classic solution of system (4.3). □

Theorem 4.4. Suppose the conditions in Lemma 3.3 hold. Denote x^* and x^v the unique solutions of system (4.3) and (4.4), respectively. Then for any $\varepsilon > 0$ and every compact subset $\check{X} \subseteq \mathbb{R}^n$, there exist $\rho = \rho(\varepsilon, c_1) > 0$, $l = l(\varepsilon) > 0$ (independent of v), a constant c and a constant c_3 such that

$$\mathcal{P}\left\{\sup_{\substack{t\in[0,T]\\ x(t)\in\check{X}}} \|\Gamma^v(t,x(t)) - \Gamma(t,x(t))\|_2 \geq \varepsilon\right\} \leq l(\varepsilon) e^{-v\rho(\varepsilon,c_1)}, \tag{4.8}$$

and

$$\mathcal{P}\{\|x^v - x^*\|_{\mathcal{X}} \geq \varepsilon\} \leq l(\varepsilon/\sigma) e^{-v\rho(\varepsilon/\sigma, c_1^2/\sigma)}, \tag{4.9}$$

where $\sigma = \frac{exp(L_f + M_{\boldsymbol{B}} T - 1)}{L_f + M_{\boldsymbol{B}}}$ and $c_1 = \sqrt{2} c c_3$.

Proof. According to Lemma 3.3, (1.2) has a unique solution $\bar{y}(t, x(t), \xi)$. Naturally, $\bar{y}(t, x(t), \xi) = \breve{y}(t, x(t), \xi)$. Moreover, it is Lipschitz continuous w.r.t t and $x(t)$ for a.e. $\xi \in \Xi$. Hence, by (4.6), we can derive that there exists a constant $Q_1 > 0$ such that, for any $t \in [0,T]$ and $x(t) \in \check{X}$,

$$\|\breve{y}(t, x(t), \xi)\|_2 \leq_{a.s.} Q_1.$$

Let

$$M_\xi(\tau) := \mathbb{E}\left(exp\left(\tau\left(\boldsymbol{B}(t,x(t),\xi)\breve{y}(t,x(t),\xi) - \mathbb{E}\left(\boldsymbol{B}(t,x(t),\xi)\breve{y}(t,x(t),\xi)\right)\right)\right)\right),$$

be the moment-generating function of stochastic variable $\boldsymbol{B}(t, x(t), \xi)\breve{y}(t, x(t), \xi) - \mathbb{E}(\boldsymbol{B}(t, x(t), \xi)\breve{y}(t, x(t), \xi))$. Since $\breve{y}(t, x(t), \xi)$ is uniformly bounded w.p.1 and $\mathbb{E}(\boldsymbol{B}(t, x(t), \xi)) <$

∞, we can derive that

$$\begin{aligned} M_\xi(\tau) &\le \mathbb{E}\left(exp\left(c_3^2\tau^2(\boldsymbol{B}(t,x(t),\xi)\breve{y}(t,x(t),\xi) - \mathbb{E}(\boldsymbol{B}(t,x(t),\xi)\breve{y}(t,x(t),\xi)))^2\right)\right) \\ &\le_{a.s.} \mathbb{E}\left(exp\left(c_3^2\tau^2c^2\right)\right) \\ &= \mathbb{E}\left(exp\left(c_3^2\tau^2\frac{2c^2}{2}\right)\right) \\ &= \mathbb{E}\left(exp\left(\frac{\tau^2}{2}c_1^2\right)\right), \end{aligned} \tag{4.10}$$

where $c_3 > 1$ is easy to be found. By Lemma 3.3, we have

$$\begin{aligned} \|\bar{y}(t,x_1,\xi) - \bar{y}(t,x_2,\xi)\|_2 &\le \kappa_F(\xi)L_G\|x_1 - x_2\|_2 \\ &\le_{a.s.} \bar{\gamma}_k L_G\|x_1 - x_2\|_2. \end{aligned} \tag{4.11}$$

Then, by (4.10), (4.11) and [40, Theorem 7.67], we can conclude that (4.8) holds.

On the other hand, since $x^v(t), x^*(t) \in \check{X}$, we have, for any $t \in [0,T]$,

$$\begin{aligned} \|x^v(t) - x^*(t)\|_2 \le &\int_0^t \|f(\tau,x^v(\tau)) - f(\tau,x^*(\tau))\|_2 \\ &+ \|\Gamma^v(\tau,x^v(\tau)) - \Gamma(\tau,x^v(\tau))\|_2 \\ &+ \|\Gamma(\tau,x^v(\tau)) - \Gamma(\tau,x^*(\tau))\|_2 d\tau \\ \le &\int_{0,}^t \|f(\tau,x^v(\tau)) - f(\tau,x^*(\tau))\|_2 \\ &+ \|\mathbb{E}(\boldsymbol{B}(\tau,x^v(\tau),\xi)\bar{y}(\tau,x^v(\tau),\xi)) \\ &- \mathbb{E}(\boldsymbol{B}(\tau,x^*(\tau),\xi)\bar{y}(\tau,x^*(\tau),\xi))\|_2 \\ &+ \sup_{\substack{t\in[0,T] \\ x^v(t)\in\check{X}}} \|\Gamma^v(t,x^v(t)) - \Gamma(t,x^v(t))\|_2 d\tau. \\ \le_{a.s.} &\int_0^t [L_f + M_{\boldsymbol{B}}]\|x^v(\tau) - x^*(\tau)\|_2 \\ &+ \sup_{\substack{t\in[0,T] \\ x^v(t)\in\check{X}}} \|\Gamma^v(t,x^v(t)) - \Gamma(t,x^v(t))\|_2 d\tau, \end{aligned}$$

where $M_{\boldsymbol{B}}$ is Lipschitz constant of $\mathbb{E}(\boldsymbol{B}(t,x^v(t),\xi)\bar{y}(t,x^v(t),\xi))$. Hence, by lemma 2.5, we derive that

$$\|x^v - x^*\|_{\mathcal{X}} \le_{a.s.} \sigma \sup_{\substack{t\in[0,T] \\ x^v(t)\in\check{X}}} \|\Gamma^v(t,x^v(t)) - \Gamma(t,x^v(t))\|_2,$$

holds for every $t \in [0,T]$ where σ is given in (4.9). Then, we have (4.9) holds. □

4.2. Convergence analysis of time-stepping approximation

Next, the time-stepping method, i.e., a finite-difference formula to approximate the time derivative $\dot{x}$, is used to solve the problem (4.4). It begins with the division of the time interval $[0,T]$ into $\mathcal{K}$ for a fixed $h = T/\mathcal{K} = t_{i+1} - t_i$ where $i = 0,\cdots,\mathcal{K}-1$ subintervals:

$$0 = t_0 < t_1 < \cdots < t_{\mathcal{K}} = T.$$

Starting from $x_0^v = x_0$, we compute two finite sets of vectors

$$\{x_1^v, x_2^v, \cdots, x_{\mathcal{K}}^v\} \subset \mathbb{R}^n \text{ and } \{y_1^{\xi_1}, y_2^{\xi_1}, \cdots, y_{\mathcal{K}}^{\xi_l}, \cdots, y_1^{\xi_l}, \cdots, y_{\mathcal{K}}^{\xi_l}\} \subset \mathbb{R}^m$$

in the following manner: for $i = 0, 1, \cdots, \mathcal{K}-1$,

$$x_{i+1}^v = x_i^v + h\left[f(t_{i+1}, \theta x_i^v + (1-\theta)x_{i+1}^v) + \Gamma^v(t_{i+1}, x_{i+1}^v)\right], \tag{4.12}$$

where $\Gamma^v(t_{i+1}, x_{i+1}^v) = \frac{1}{v}\sum_{l=1}^{v} \boldsymbol{B}(t_i, x_i^v, \xi_l) y_{i+1}^{\xi_l} = \phi^v(t_{i+1})$. Naturally, $y_{i+1}^{\xi_l}$ satisfies

$$y_{i+1}^{\xi_l} \in \mathbf{SOL}\left(K, G(t_{i+1}, x_{i+1}^v, \xi_l) + F(\xi_l, \cdot)\right). \tag{4.13}$$

Let $\hat{x}_h^v(t)$ be the continuous piecewise linear interpolation of the family $\{x_i^v\}$ and $\hat{y}_h^{\xi_l}(t)$ be the continuous piecewise constant function of the family $\{y_i^{\xi_l}\}$, i.e., for $i = 0, 1, \cdots, \mathcal{K}-1$,

$$\begin{cases} \hat{x}_h^v(t) := x_i^v + \frac{t-t_i}{h}(x_{i+1}^v - x_i^v), & \forall t \in [t_i, t_{i+1}], \\ \hat{y}_h^{\xi_l}(t) := y_{i+1}^{\xi_l}, & \forall t \in (t_i, t_{i+1}]. \end{cases} \tag{4.14}$$

Naturally, we define

$$\hat{\phi}_h^v(t) := \frac{1}{v}\sum_{l=1}^{v} \boldsymbol{B}(t_i, x_i^v, \xi_l)\hat{y}_h^{\xi_l}(t) = \frac{1}{v}\sum_{l=1}^{v} \boldsymbol{B}(t_i, x_i^v, \xi_l) y_{i+1}^{\xi_l} = \phi^v(t_{i+1}),$$

for all $t \in (t_i, t_{i+1}]$. Consequently, for every sufficiently small $h > 0$ and sufficiently large $v > 0$, the functions $\hat{x}_h^v : [0,T] \to \mathbb{R}^n, \hat{y}_h^{\xi_l} : [0,T] \to \mathbb{R}^m$ $(l = 1, \cdots, v)$ and $\hat{\phi}_h^v : [0,T] \to \mathbb{R}^n$ are well defined.

Proposition 4.5. Suppose the conditions in Lemma 3.3 hold. Then there exist $h_0 > 0$ and $\gamma > 0$ such that

$$\|x_i^v\|_2 \leq_{a.s.} \alpha_1 := \left(\|x_0\|_2 + \frac{\rho_f + \gamma}{\rho_f}\right) exp\left(\frac{T\rho_f + \gamma}{1 - h\rho_f}\right) - \frac{\rho_f + \gamma}{\rho_f},$$

holds for any $h \in (0, h_0], v \in \mathbb{N}^+$ and $i = 1, 2, \cdots, \mathcal{K}$, where $h_0 < \frac{1}{\rho_f}$.

Proof. Define

$$\gamma = \boldsymbol{B}_s M(\bar{\gamma}_k)(\|x_0\|_2 + 1) exp(\rho_f + \boldsymbol{B}_s M(\bar{\gamma}_k))T.$$

Then we can deduce that

$$\|\phi^v\|_{\mathcal{X}} \leq_{a.s.} \gamma, \tag{4.15}$$

is always ture by (4.7) and Proposition 4.2. According to (4.12), we have

$$\begin{aligned} \|x_{i+1}^v\|_2 &\leq \|x_i^v\|_2 + h\left\|f(t_{i+1}, \theta x_i^v + (1-\theta)x_{i+1}^v) + \phi^v(t_{i+1})\right\|_2 \\ &\leq_{a.s.} \|x_i^v\|_2 + h\rho_f(1 + \|\theta x_i^v + (1-\theta)x_{i+1}^v\|_2) + h\gamma \\ &\leq (1 + h\rho_f\theta)\|x_i^v\|_2 + (1-\theta)h\rho_f\|x_{i+1}^v\|_2 + h(\gamma + \rho_f). \end{aligned}$$

Then, we can get

$$\|x_{i+1}^v\|_2 \leq_{a.s.} c_4\|x_i^v\|_2 + c_5$$

and

$$\|x_i^v\|_2 \leq_{a.s.} c_4^i\|x_0^v\|_2 + \frac{c_4^i - 1}{c_4 - 1}c_5,$$

where $c_4 = \frac{1+\rho_f\theta h}{1-(1-\theta)h\rho_f}$ and $c_5 = \frac{h(\rho_f+\gamma)}{1-(1-\theta)h\rho_f}$. Because $ih \leq \mathcal{K}h = T$ and $c_4 \leq \frac{1}{1-h\rho_f}$, we can obtain that

$$c_4^i = (c_4 - 1 + 1)^i \leq exp(i(c_4 - 1)) \leq exp\left(\frac{ih\rho_f}{1 - h\rho_f}\right) \leq exp\left(\frac{T\rho_f}{1 - h\rho_f}\right).$$

Then by the calculation we can derive that

$$\|x_i^v\|_2 \leq_{a.s.} \alpha_1$$

for $h \in (0, h_0]$. □

Theorem 4.6. Suppose the conditions of Lemma 3.3 are fulfilled. Then there are sequences $\{v_k\} \to \infty$ and $\{h_k\} \downarrow 0$ such that $\{\hat{x}_{h_k}^{v_k}\}$ converges to an x^* w.p.1 uniformly on $[0, T]$ and $\hat{\phi}_{h_k}^{v_k} \to \phi^*$ w.p.1 weakly in $\mathcal{L}$. Moreover, if $\hat{\phi}_{h_k}^{v_k} \to \phi^*$ w.p.1 w.r.t. $\|\cdot\|_{L^2}$, then x^* is a weak solution of system (4.3).

Proof. According to Proposition 4.5, we get the family of functions $\{\hat{x}_h^v(t)\}$ is uniformly bounded on $[0, T]$ w.p.1 for $v > 0$ and $h > 0$ small enough. Moreover, for any $v > 0$, we have

$$\begin{aligned}\|x_{i+1}^v - x_i^v\|_2 &\leq_{a.s.} h\left\|f(t_{i+1}, \theta x_i^v + (1-\theta)x_{i+1}^v)\right\|_2 + h\gamma \\ &\leq h\theta\rho_f\|x_i^v\|_2 + h\rho_f(1-\theta)\|x_{i+1}^v\|_2 + h\gamma + h\rho_f \\ &= h(\rho_f\alpha_1 + \rho_f + \gamma).\end{aligned} \tag{4.16}$$

Then for any $t \in [t_k, t_{k+1}], \tau \in [t_{k+p}, k_{k+p+1}]$ $(p \in \mathbb{N}^+)$ and $v \in \mathbb{N}^+$, we have

$$\begin{aligned}&\|\hat{x}_h^v(\tau) - \hat{x}_h^v(t)\|_2 \\ &= \left\|(\hat{x}_h^v(\tau) - x_{k+p}^v) + \sum_{j=1}^{p-1}(x_{k+j+1}^v - x_{k+j}^v) + (x_{k+1}^v - \hat{x}_h^v(t))\right\|_2 \\ &\leq \|\hat{x}_h^v(\tau) - x_{k+p}^v\|_2 + \sum_{j=1}^{p-1}\|x_{k+j+1}^v - x_{k+j}^v\|_2 + \|x_{k+1}^v - \hat{x}_h^v(t)\|_2 \\ &\leq_{a.s} [\tau - t_{k+p} + (p-1)h + t_{k+1} - t][\rho_f\alpha_1 + \rho_1 + \gamma] \\ &\leq |\tau - t|[\rho_f\alpha_1 + \rho_1 + \gamma].\end{aligned} \tag{4.17}$$

It implies that the piecewise interpolations $\hat{x}_h^v(\cdot)$ are Lipschitz continuous almost surely on $[0, T]$ and the Lipschitz constant is independent of h and v. Hence, we obtain that $\{\hat{x}_h^v(t)\}$ is equicontinuous. Then according to the Arzelá-Ascoli theorem [15,41], there are sequence $\{h_k\} \downarrow 0$ and $\{v_k\} \to \infty$ as $k \to \infty$ such that $\{\hat{x}_{h_k}^{v_k}\}$ converges to an x^* w.p.1 uniformly on $[0, T]$. Obviously, $x^*(t)$ is a continuous function on $[0, T]$.

From (4.15), $\{\hat{\phi}_h^v\}$ is also uniformly bounded for $h > 0$ sufficiently small and $v > 0$. By Alaoglu's theorem [15], there exists a subsequence of $\{\hat{\phi}_{h_k}^{v_k}\}$, which we may assume without loss of generality to be $\{\hat{\phi}_{h_k}^{v_k}\}$ itself, has a weak limit ϕ^* w.p.1 in $\mathcal{L}$ as $k \to \infty$.

Next, we show that x^* is a weak solution of system (4.3).

For any $0 \leq t \leq T$ with $\hat{x}_h^v(0) = x_0$ (with loos of generality, we choose $t \in [t_i, t_{i+1}]$),

$$\begin{aligned}&\left\|\hat{x}_h^v(t) - \hat{x}_h^v(0) - \int_0^t f(\tau, \hat{x}_h^v(t)) + \phi^*(\tau)d\tau\right\|_2 \\ &\leq \left\|\hat{x}_h^v(t) - \hat{x}_h^v(0) - \int_0^t f(\tau, \hat{x}_h^v(t)) + \hat{\phi}_h^v(\tau)d\tau\right\|_2 + \left\|\int_0^t \hat{\phi}_h^v(\tau) - \phi^*(\tau)\right\|_2 \\ &\triangleq \|\mathcal{W}_h^v(t)\|_2 + \|\mathfrak{J}_h^v(t)\|_2.\end{aligned}$$

Then,

$$\|\mathfrak{J}_h^v(t)\|_2 \leq \int_0^t \|\hat{\phi}_h^v(\tau) - \phi^*(\tau)\|_2 d\tau \leq \int_0^T \|\hat{\phi}_h^v(\tau) - \phi^*(\tau)\|_2 d\tau,$$

which means that

$$\begin{aligned}\|\mathfrak{J}_h^v\|_{\mathfrak{X}} \leq \int_0^T \|\hat{\phi}_h^v(\tau) - \phi^*(\tau)\|_2 d\tau &\leq \sqrt{T}\left(\int_0^T \|\hat{\phi}_h^v(\tau) - \phi^*(\tau)\|_2^2 d\tau\right)^{\frac{1}{2}} \\ &\leq \sqrt{T}\|\hat{\phi}_h^v - \phi^*\|_{L^2}.\end{aligned}$$

Hence, if $\hat{\phi}_{h_k}^{v_k} \to \phi^*$ w.p.1, as $k \to \infty$ w.r.t. $\|\cdot\|_{L^2}$, then

$$\lim_{k\to\infty} \|\mathfrak{J}_{h_k}^{v_k}\|_{\mathfrak{X}} \leq \sqrt{T}\|\hat{\phi}_{h_k}^{v_k} - \phi^*\|_{L^2} = 0.$$

On the other hand,

$$
\begin{aligned}
&\|\mathcal{W}_h^v(t)\|_2\\
&= \left\| \hat{x}_h^v(t) - \hat{x}_h^v(0) - \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} + \int_{t_i}^{t} \right) \left[f(\tau, \hat{x}_h^v(\tau)) + \hat{\phi}_h^v(\tau) \right] d\tau \right\|_2\\
&\le \Big\| \hat{x}_h^v(t) - \hat{x}_h^v(0) - \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} [f(t_{j+1}, \theta x_j^v + (1-\theta) x_{j+1}^v) + \phi^v(t_{j+1})] d\tau\\
&- \int_{t_i}^{t} \left[f(t_{i+1}, \theta x_i^v + (1-\theta) x_{i+1}^v) + \phi^v(t_{i+1}) \right] d\tau \Big\|_2\\
&+ \Big\| \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left[f(t_{j+1}, \theta x_j^v + (1-\theta) x_{j+1}^v) - f(\tau, \hat{x}_h^v(\tau)) \right] d\tau\\
&+ \int_{t_i}^{t} \left[f(t_{i+1}, \theta x_i^v + (1-\theta) x_{i+1}^v) - f(\tau, \hat{x}_h^v(\tau)) \right] d\tau \Big\|_2\\
&\le \left\| \hat{x}_h^v(t) - \hat{x}_h^v(0) - \sum_{j=0}^{i-1} (x_{j+1}^v - x_j^v) - \frac{t - t_i}{h} (x_{i+1}^v - x_i^v) \right\|_2\\
&+ \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left\| \left[\rho_f \left(1 - \theta - \frac{t - t_j}{h} \right) (x_{j+1}^v - x_j^v) \right] \right\|_2 + \rho_f |t_{j+1} - \tau| d\tau\\
&+ \int_{t_i}^{t} \left\| \left[\rho_f \left(1 - \theta - \frac{t - t_i}{h} \right) (x_{i+1}^v - x_i^v) \right] \right\|_2 + \rho_f |t_{i+1} - \tau| d\tau\\
&\le \sum_{j=0}^{i-1} h \rho_f \|x_{j+1}^v - x_j^v\|_2 + \frac{i}{2} h^2 \rho_f + (t - t_i) \rho_f \|x_{i+1}^v - x_i^v\|_2 + \frac{h^2}{2} \rho_f\\
&\le_{a.s.} \left[\rho_f (\rho_f \alpha_1 + \rho_f + \gamma) + \frac{\rho_f}{2} \right] (T + h) h.
\end{aligned}
$$

Therefore, we can conclude that

$$
\lim_{k\to\infty} \|\mathcal{W}_{h_k}^{v_k}(t)\|_2 = 0,
$$

which implies that $\lim\limits_{k\to\infty} \|\mathcal{W}_{h_k}^{v_k}\|_{\mathcal{X}} = 0$.

Since $\phi^* \in \mathcal{L}$ a.s., we have $f + \phi^* \in \mathcal{L}$ a.s.. By the Lebesgue dominated convergence theorem, we find

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\| x^*(t) - x^*(0) - \int_0^t f(\tau, x^*(\tau)) + \phi^*(\tau) d\tau \right\|_2\\
&= \lim_{k\to\infty} \sup_{t\in[0,T]} \left\| \hat{x}_{h_k}^{v_k}(t) - \hat{x}_{h_k}^{v_k}(0) - \int_0^t f(\tau, \hat{x}_{h_k}^{v_k}(\tau)) + \phi^*(\tau) d\tau \right\|_2\\
&= \lim_{k\to\infty} \|\mathcal{W}_{h_k}^{v_k}\|_{\mathcal{X}} + \lim_{k\to\infty} \|\mathcal{J}_{h_k}^{v_k}\|_{\mathcal{X}} = 0.
\end{aligned}
$$

Hence, we conclude that x^* is a weak solution of system (4.3). □

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