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Research Article

Inequalities for the A-joint numerical radius of two operators and their applications

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Abstract

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and A be a positive (semidefinite) bounded linear operator on $\mathcal{H}$. The semi-inner product induced by A is given by $\langle x, y\rangle_A := \langle Ax, y\rangle$, $x, y \in \mathcal{H}$ and defines a seminorm $\|\cdot\|_A$ on $\mathcal{H}$. This makes $\mathcal{H}$ into a semi-Hilbert space. The A-joint numerical radius of two A-bounded operators T and S is given by

$$\omega_{A,\mathrm{e}}(T,S) = \sup_{\|x\|_A=1} \sqrt{\left|\langle Tx, x\rangle_A\right|^2 + \left|\langle Sx, x\rangle_A\right|^2}.$$

In this paper, we aim to prove several bounds involving $\omega_{A,\mathrm{e}}(T,S)$. This allows us to establish some inequalities for the A-numerical radius of A-bounded operators. In particular, we extend the well-known inequalities due to Kittaneh [Numerical radius inequalities for Hilbert space operators, Studia Math. **168** (1), 73-80, 2005]. Moreover, several bounds related to the A-Davis-Wielandt radius of semi-Hilbert space operators are also provided.

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1. Introduction and preliminaries

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Throughout this paper, by an operator we mean a bounded linear operator. Let T^* denote the adjoint of an operator T. Further, the range and the kernel of T are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. In addition, the cone of all positive operators on $\mathcal{H}$ is given by

$$\mathbb{B}(\mathcal{H})^+ := \{A \in \mathbb{B}(\mathcal{H})\,;\, \langle Ax, x\rangle \geq 0,\ \forall\, x \in \mathcal{H}\,\}.$$

Any $A \in \mathbb{B}(\mathcal{H})^+$ induces the following semi-inner product:

$$\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{C},\ (x,y) \longmapsto \langle x,y\rangle_A := \langle Ax, y\rangle.$$

Observe that the seminorm induced by $\langle\cdot,\cdot\rangle_A$ is given by $\|x\|_A = \langle x,x\rangle_A^{1/2}$, for every $x \in \mathcal{H}$. This makes $\mathcal{H}$ into a semi-Hilbert space. It is not difficult to verify that $\|\cdot\|_A$ is a norm

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on $\mathcal{H}$ if and only if A is injective, and that $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is a closed subspace of $\mathcal{H}$. For very recent contributions concerning operators acting on semi-Hilbert spaces, we refer the reader to [2, 6, 9, 11] and the references therein. From now on, we suppose that $A \in \mathbb{B}(\mathcal{H})$ is always a positive (nonzero) operator and we denote the A-unit sphere of $\mathcal{H}$ by $\mathbb{S}^A(0,1)$, that is,

$$\mathbb{S}^A(0,1) := \{x \in \mathcal{H}\,;\ \|x\|_A = 1\}.$$

For $T \in \mathbb{B}(\mathcal{H})$, the A-numerical radius and the A-Crawford number of T are given by

$$\omega_A(T) = \sup\left\{\left|\langle Tx, x\rangle_A\right|;\ \ x \in \mathbb{S}^A(0,1)\right\}$$

and

$$c_A(T) = \inf\left\{\left|\langle Tx, x\rangle_A\right|;\ \ x \in \mathbb{S}^A(0,1)\right\},$$

respectively (see [5, 24] and the references therein). It should be emphasized here that it may happen that $\omega_A(T) = +\infty$ for some $T \in \mathbb{B}(\mathcal{H})$ (see [16]).

Let $T \in \mathbb{B}(\mathcal{H})$. An operator $S \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint of T if for every $x, y \in \mathcal{H}$, the identity $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ holds (see [3]). So, S is an A-adjoint of T if and only if S is solution in $\mathbb{B}(\mathcal{H})$ of the equation $AX = T^*A$. This kind of equations can be studied by using Douglas theorem [12] which says that the operator equation $TX = S$ has a solution $X \in \mathbb{B}(\mathcal{H})$ if and only if $\mathcal{R}(S) \subseteq \mathcal{R}(T)$ which in turn equivalent to the existence of a positive number λ such that $\|S^*x\| \leq \lambda\|T^*x\|$ for all $x \in \mathcal{H}$. In addition, in the same theorem by Douglas [12], it is shown that: if $TX = S$ has solutions, then there exists only one, denoted by Q, which satisfies $\mathcal{R}(Q) \subseteq \overline{\mathcal{R}(T^*)}$. Such Q is said to be reduced solution of the equation $TX = S$. Obviously, the existence of an A-adjoint operator is not guaranteed. The subspace of all operators admitting A-adjoints is denoted by $\mathbb{B}_A(\mathcal{H})$. By Douglas theorem, it holds that

$$\mathbb{B}_A(\mathcal{H}) = \left\{T \in \mathbb{B}(\mathcal{H})\,;\ \mathcal{R}(T^*A) \subset \mathcal{R}(A)\right\}.$$

Let $T \in \mathbb{B}_A(\mathcal{H})$. The reduced solution of the operator equation $AX = T^*A$ is denoted by $T^{\sharp_A}$. Moreover we have, $T^{\sharp_A} = A^{\dagger}T^*A$ (see [3]). Here $A^{\dagger}$ denotes the Moore-Penrose inverse of A (for more details, see [3, 4] and the references therein). From now on, for simplicity we will write $X^{\sharp}$ instead of $X^{\sharp_A}$ for every $X \in \mathbb{B}_A(\mathcal{H})$. Notice that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp} \in \mathbb{B}_A(\mathcal{H})$, $(T^{\sharp})^{\sharp} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$ and $((T^{\sharp})^{\sharp})^{\sharp} = T$. Here $P_{\overline{\mathcal{R}(A)}}$ denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Further, if $S \in \mathbb{B}_A(\mathcal{H})$ then $TS \in \mathbb{B}_A(\mathcal{H})$ and $(TS)^{\sharp} = S^{\sharp}T^{\sharp}$. For an account of results concerning $T^{\sharp}$, we refer the reader to [3, 4]. Again, an application of Douglas theorem gives

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \left\{T \in \mathbb{B}(\mathcal{H})\,;\ \exists\,\lambda > 0\,;\ \|Tx\|_A \leq \lambda\|x\|_A,\ \forall\, x \in \mathcal{H}\right\}.$$

It $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then T is called A-bounded. Notice that $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$ (see [15]). The seminorm of an operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ is given by

$$\|T\|_A := \sup_{\substack{x\in\overline{\mathcal{R}(A)},\\ x\neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup\left\{\|Tx\|_A\,;\ x \in \mathbb{S}^A(0,1)\right\} < \infty, \tag{1.1}$$

(see [15] and the references therein). We mention here that $\|\cdot\|_A$ and $\omega_A(\cdot)$ are equivalent seminorms on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. More precisely, for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A, \tag{1.2}$$

(see [5]). Further, it was shown in [5] that

$$\omega_A(T^n) \leq \omega_A^n(T), \tag{1.3}$$

for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ and all positive integer n. Before we move on, it is crucial to recall that for every $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ we have

$$\|TS\|_A \leq \|T\|_A \|S\|_A, \tag{1.4}$$

(see [5]). Recall that an operator $T \in \mathbb{B}(\mathcal{H})$ is said to be A-selfadjoint if AT is selfadjoint. Observe that if T is A-selfadjoint, then $T \in \mathbb{B}_A(\mathcal{H})$. It was shown in [15] that for every A-selfadjoint operator T we have

$$\|T\|_A = \omega_A(T). \tag{1.5}$$

Further, an operator T is called A-positive if $AT \geq 0$ and we write $T \geq_A 0$. Obviously, an A-positive operator is A-selfadjoint since $\mathcal{H}$ is a complex Hilbert space. It can be checked that $T^\sharp T \geq_A 0$ and $TT^\sharp \geq_A 0$. Moreover, for every $T \in \mathbb{B}_A(\mathcal{H})$ we have

$$\|T^\sharp T\|_A = \|TT^\sharp\|_A = \|T\|_A^2 = \|T^\sharp\|_A^2, \tag{1.6}$$

(see [4, Proposition 2.3.]). Now, an operator $T \in \mathbb{B}_A(\mathcal{H})$ is called A-normal if $TT^\sharp = T^\sharp T$ (see [24]). It is obvious that every selfadjoint operator is normal. However, an A-selfadjoint operator is not necessarily A-normal (see [15, Example 4]).

Let $\mathbb{B}(\mathcal{H})^d := \mathbb{B}(\mathcal{H}) \times \cdots \times \mathbb{B}(\mathcal{H})$. The A-joint numerical radius of a d-tuple of operators $(T_1, \ldots, T_d) \in \mathbb{B}(\mathcal{H})^d$ was defined in [5] by

$$\omega_{A,\mathrm{e}}(T_1, \ldots, T_d) = \sup\left\{ \left(\sum_{k=1}^d |\langle T_k x, x\rangle_A|^2 \right)^{\frac{1}{2}} ;\ x \in \mathbb{S}^A(0,1) \right\}.$$

Notice that the particular case $d = 1$ is the A-numerical radius of an operator T which recently attracted the attention of several mathematicians (see, e.g., [2,7,8,15,16,18–20,23] and the references therein). Some interesting properties of A-joint numerical radius of A-bounded operators were given in [5,17]. In particular, it is established that for an operator tuple $(T_1, \ldots, T_d) \in \mathbb{B}_A(\mathcal{H})^d$ we have

$$\frac{1}{2\sqrt{d}} \left\| \sum_{k=1}^d T_k^\sharp T_k \right\|^{\frac{1}{2}} \leq \omega_{A,\mathrm{e}}(T_1, \ldots, T_d) \leq \left\| \sum_{k=1}^d T_k^\sharp T_k \right\|^{\frac{1}{2}}. \tag{1.7}$$

By using (1.7), the present author proved recently in [16] that for every $T \in \mathbb{B}_A(\mathcal{H})$ we have

$$\frac{1}{16}\|T^\sharp T + TT^\sharp\|_A \leq \omega_A^2(T) \leq \frac{1}{2}\|T^\sharp T + TT^\sharp\|_A. \tag{1.8}$$

Recently, the A-Davis-Wielandt radius of an operator $T \in \mathbb{B}(\mathcal{H})$ is defined by K. Feki et al in [21] by

$$d\omega_A(T) := \sup\left\{ \sqrt{|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4} ;\ x \in \mathbb{S}^A(0,1) \right\}.$$

Notice that it was shown in [21], that $d\omega_A(T)$ may be equal to $+\infty$ for some $T \in \mathbb{B}(\mathcal{H})$. However, if $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, then we have

$$\max\left\{\omega_A(T), \|T\|_A^2\right\} \leq d\omega_A(T) \leq \sqrt{\omega_A(T)^2 + \|T\|_A^4} < \infty.$$

Clearly, if $T \in \mathbb{B}_A(\mathcal{H})$, then the A-Davis-Wielandt radius can be seen as the A-joint numerical radius of the operator tuple $(T, T^\sharp T)$. That is, for $T \in \mathbb{B}(\mathcal{H})$, it holds

$$d\omega_A(T) = \omega_{A,\mathrm{e}}(T, T^\sharp T). \tag{1.9}$$

In this paper we establish several inequalities concerning the A-joint numerical radius of two semi-Hilbert space operators. In particular, some related results connecting the A-joint numerical radius and the classical A-numerical radius are also presented. Some of the obtained results cover and extend the work of Drogomir [13]. Moreover, we prove several inequalities involving the A-Davis-Wielandt radius and the A-numerical radii of A-bounded operators. In particular, we generalize and refine some earlier results established in [26].

2. Results

In this section, we present our result. In order to establish our first upper bound for the A-joint numerical radius of two semi-Hilbert space operators we need the following lemmas.

Lemma 2.1 ([3], Section 2)**.** *Let $T \in \mathbb{B}(\mathcal{H})$ be an A-selfadjoint operator. Then, $T = T^\sharp$ if and only if T is A-selfadjoint and $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$.*

Lemma 2.2. *For every $a, b, c \in \mathcal{H}$*

$$|\langle a, b\rangle_A|^2 + |\langle a, c\rangle_A|^2 \leq \|a\|_A^2 \sqrt{|\langle b, b\rangle_A|^2 + 2|\langle b, c\rangle_A|^2 + |\langle c, c\rangle_A|^2}. \tag{2.1}$$

Proof. Notice first that, by [14, p. 148], we have

$$|\langle x, y\rangle|^2 + |\langle x, z\rangle|^2 \leq \|x\|^2 \Big(|\langle y, y\rangle|^2 + 2|\langle y, z\rangle|^2 + |\langle z, z\rangle|^2\Big)^{\frac{1}{2}}, \tag{2.2}$$

for any $x, y, z \in \mathcal{H}$. Now, let $a, b, c \in \mathcal{H}$. It follows, from (2.2), that

$$\begin{aligned}|\langle a, b\rangle_A|^2 + |\langle a, c\rangle_A|^2 &= |\langle A^{1/2}a, A^{1/2}b\rangle|^2 + |\langle A^{1/2}a, A^{1/2}c\rangle|^2 \\ &\leq \|A^{1/2}a\|^2 \sqrt{|\langle A^{1/2}b, A^{1/2}b\rangle|^2 + 2|\langle A^{1/2}b, A^{1/2}c\rangle|^2 + |\langle A^{1/2}c, A^{1/2}c\rangle|^2}.\end{aligned}$$

This proves (2.1) as desired. □

Our first result in this paper reads as follows.

Theorem 2.3. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_{A,e}(T, S) \leq \sqrt{\|T\|_A^4 + \|S\|_A^4 + 2\omega_A^2(S^\sharp T)} \leq \|T\|_A^2 + \|S\|_A^2.$$

Proof. Let $x \in \mathbb{S}^A(0, 1)$. By choosing in Lemma 2.2 $a = x, b = Tx$ and $c = Sx$ we see that

$$\begin{aligned}\Big(|\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2\Big)^2 &= \Big(|\langle x, Tx\rangle_A|^2 + |\langle x, Sx\rangle_A|^2\Big)^2 \\ &\leq \|x\|_A^4 \Big(|\langle Tx, Tx\rangle_A|^2 + 2|\langle Tx, Sx\rangle_A|^2 + |\langle Sx, Sx\rangle_A|^2\Big) \\ &= |\langle T^\sharp Tx, x\rangle_A|^2 + |\langle S^\sharp Sx, x\rangle_A|^2 + 2|\langle S^\sharp Tx, x\rangle_A|^2 \\ &\leq \omega_{A,\mathrm{e}}^2(T^\sharp T, S^\sharp S) + 2\omega_A^2(S^\sharp T) \\ &\leq \Big\|(T^\sharp T)^\sharp T^\sharp T + (S^\sharp S)^\sharp S^\sharp S\Big\|_A + 2\omega_A^2(S^\sharp T),\end{aligned} \tag{2.3}$$

where the last inequality follows from the second inequality in (1.7). Now, since $T^\sharp T$ is A-selfadjoint and satisfies $\mathcal{R}(T^\sharp T) \subseteq \overline{\mathcal{R}(A)}$, then by Lemma 2.1 we have $(T^\sharp T)^\sharp = T^\sharp T$. Similarly, $(S^\sharp S)^\sharp = S^\sharp S$. So, by (2.3), we have

$$\Big(|\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2\Big)^2 \leq \Big\|(T^\sharp T)^2 + (S^\sharp S)^2\Big\|_A + 2\omega_A^2(S^\sharp T).$$

By taking the supremum over all $x \in \mathbb{S}^A(0, 1)$ in the above inequality we get

$$\omega_{A,\mathrm{e}}(T, S) \leq \sqrt{\|(T^\sharp T)^2 + (S^\sharp S)^2\|_A + 2\omega_A^2(S^\sharp T)}.$$

Moreover, by using the triangle inequality together with (1.4) we obtain

$$\begin{aligned}
\omega_{A,\mathrm{e}}(T,S) &\leq \sqrt{\|T^\sharp T\|_A^2 + \|S^\sharp S\|_A^2 + 2\omega_A^2(S^\sharp T)} \\
&= \sqrt{\|T\|_A^4 + \|S\|_A^4 + 2\omega_A^2(S^\sharp T)} \quad (\text{by } (1.6)) \\
&\leq \sqrt{\|T\|_A^4 + \|S\|_A^4 + 2\|S^\sharp T\|_A^2} \quad (\text{by } (1.2)) \\
&\leq \sqrt{\|T\|_A^4 + \|S\|_A^4 + 2\|S^\sharp\|_A^2\|T\|_A^2} \quad (\text{by } (1.4)) \\
&= \sqrt{\left(\|T\|_A^2 + \|S\|_A^2\right)^2} = \|T\|_A^2 + \|S\|_A^2 .
\end{aligned}$$

This proves the desired result. □

In what follows, we need the following lemmas.

Lemma 2.4 ([26], Lemma 2.9)**.** *For any $z_1, z_2 \in \mathbb{C}$, we have*

$$\sup\left\{\left|\alpha z_1 + \beta z_2\right|^2;\ (\alpha,\beta)\in\mathbb{C}^2,\ |\alpha|^2+|\beta|^2\leq 1\right\} = |z_1|^2 + |z_2|^2.$$

Lemma 2.5. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then, for every $\alpha, \beta \in \mathbb{C}$, we have*

$$\|\alpha T + \beta S\|_A^2 \leq (|\alpha|^2 + |\beta|^2)\|T^\sharp T + S^\sharp S\|_A.$$

Proof. Let $x \in \mathbb{S}^A(0,1)$. Then, by applying the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}
\|\alpha Tx + \beta Sx\|_A^2 &= \|\alpha A^{1/2}Tx + \beta A^{1/2}Sx\|^2 \\
&\leq (|\alpha|^2 + |\beta|^2)(\|A^{1/2}Tx\|^2 + \|A^{1/2}Sx\|^2) \\
&= (|\alpha|^2 + |\beta|^2)(\|Tx\|_A^2 + \|Sx\|_A^2) \\
&= (|\alpha|^2 + |\beta|^2)\langle (T^\sharp T + S^\sharp S)x, x\rangle_A \\
&\leq (|\alpha|^2 + |\beta|^2)\omega_A(T^\sharp T + S^\sharp S) \\
&= (|\alpha|^2 + |\beta|^2)\|T^\sharp T + S^\sharp S\|_A,
\end{aligned}$$

where the last equality follows from (1.5) since $T^\sharp T + S^\sharp S \geq_A 0$. Hence,

$$\|(\alpha T + \beta S)x\|_A^2 \leq (|\alpha|^2 + |\beta|^2)\|T^\sharp T + S^\sharp S\|_A.$$

So, by taking the supremum over all $x \in \mathbb{S}^A(0,1)$ in the above inequality and then using (1.1) we get the desired result. □

Now, we are in a position to prove the following result.

Theorem 2.6. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_{A,e}(T,S) \leq \left[\omega_A\Big((T^\sharp T)^2 + (S^\sharp S)^2\Big) + 2\omega_A^2(S^\sharp T)\right]^{\frac{1}{4}}. \tag{2.4}$$

Proof. Let $x \in \mathbb{S}^A(0,1)$. As in the proof of Theorem 2.3, by choosing in Lemma 2.2 $a = x, b = Tx$ and $c = Sx$, we get

$$\Big(|\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2\Big)^2 \leq \sup_{x\in\mathbb{S}^A(0,1)}\Big(|\langle T^\sharp Tx, x\rangle_A|^2 + |\langle S^\sharp Sx, x\rangle_A|^2\Big) + 2\omega_A^2(S^\sharp T).$$

Hence, by applying Lemma 2.4 we obtain

$$\begin{aligned}
&\Big(|\langle Tx,x\rangle_A|^2+|\langle Sx,x\rangle_A|^2\Big)^2\\
&\leq \sup_{x\in\mathbb{S}^A(0,1)}\left(\sup_{|\alpha|^2+|\beta|^2\leq 1}\Big|\alpha\langle T^\sharp Tx,x\rangle_A+\beta\langle S^\sharp Sx,x\rangle_A\Big|^2\right)+2\omega_A^2(S^\sharp T)\\
&=\sup_{x\in\mathbb{S}^A(0,1)}\left(\sup_{|\alpha|^2+|\beta|^2\leq 1}\Big|\Big\langle\Big[\alpha T^\sharp T+\beta S^\sharp S\Big]x,x\Big\rangle_A\Big|^2\right)+2\omega_A^2(S^\sharp T)\\
&=\sup_{|\alpha|^2+|\beta|^2\leq 1}\left(\sup_{x\in\mathbb{S}^A(0,1)}\Big|\Big\langle\Big[\alpha T^\sharp T+\beta S^\sharp S\Big]x,x\Big\rangle_A\Big|^2\right)+2\omega_A^2(S^\sharp T).
\end{aligned}$$

On the other hand, it can be see that the operator $\alpha T^\sharp T+\beta S^\sharp S$ is an A-selfadjoint operator and then by (1.5), we have

$$\sup_{x\in\mathbb{S}^A(0,1)}\Big|\Big\langle\Big[\alpha T^\sharp T+\beta S^\sharp S\Big]x,x\Big\rangle_A\Big|=\|\alpha T^\sharp T+\beta S^\sharp S\|_A.$$

So, by using Lemma 2.5, we get

$$\begin{aligned}
&\Big(|\langle Tx,x\rangle_A|^2+|\langle Sx,x\rangle_A|^2\Big)^2\\
&\leq \sup_{|\alpha|^2+|\beta|^2\leq 1}\|\alpha T^\sharp T+\beta S^\sharp S\|_A^2+2\omega_A^2(S^\sharp T)\\
&\leq \sup_{|\alpha|^2+|\beta|^2\leq 1}(|\alpha|^2+|\beta|^2)\Big\|\,(T^\sharp T)^\sharp T^\sharp T+\Big[S^\sharp S\Big]^\sharp S^\sharp S\,\Big\|_A+2\omega_A^2(S^\sharp T)\\
&=\sup_{|\alpha|^2+|\beta|^2\leq 1}(|\alpha|^2+|\beta|^2)\Big\|\,(T^\sharp T)^2+(S^\sharp S)^2\,\Big\|_A+2\omega_A^2(S^\sharp T)\\
&=\Big\|\,(T^\sharp T)^2+(S^\sharp S)^2\,\Big\|_A+2\omega_A^2(S^\sharp T)\\
&=\omega_A\Big[(T^\sharp T)^2+(S^\sharp S)^2\Big]+2\omega_A^2(S^\sharp T),
\end{aligned}$$

where the last equality follows from (1.5) since $(T^\sharp T)^2+(S^\sharp S)^2\geq_A 0$. Thus, we get

$$|\langle Tx,x\rangle_A|^2+|\langle Sx,x\rangle_A|^2\leq\sqrt{\omega_A\left[(T^\sharp T)^2+(S^\sharp S)^2\right]+2\omega_A^2(S^\sharp T)},$$

for all $x\in\mathbb{S}^A(0,1)$. Finally, by taking the supremum over all $x\in\mathbb{S}^A(0,1)$ in the above inequality we get (2.4) as required. □

The following corollary is an immediate consequence of Theorem 2.6 and extends a result by Zamani et al. (see [26, Theorem 2.11]).

Corollary 2.7. *Let $T\in\mathbb{B}_A(\mathcal{H})$. Then,*

$$d\omega_A(T)\leq\left[\omega_A\Big((T^\sharp T)^2+(T^\sharp T)^4\Big)+2\omega_A^2(T^\sharp T^2)\right]^{\frac{1}{4}}.$$

Proof. By Lemma 2.1, we have $(T^\sharp T)^\sharp=T^\sharp T$. So, by replacing S by $T^\sharp T$ in (2.4) and then using (1.9) we get the required result. □

The following lemma is useful in the sequel.

Lemma 2.8. *For any $a,b,c\in\mathcal{H}$, we have*

$$|\langle a,b\rangle_A|^2+|\langle a,c\rangle_A|^2\leq\|a\|_A^2\Big(\max\{\|b\|_A^2,\|c\|_A^2\}+|\langle b,c\rangle_A|\Big).\tag{2.5}$$

Proof. Let $a, b, c \in \mathcal{H}$. By applying the Cauchy-Schwarz inequality we see that

$$\begin{aligned}\Big(|\langle a, b\rangle_A|^2 + |\langle a, c\rangle_A|^2\Big)^2 &= \Big(\langle a, b\rangle_A\langle b, a\rangle_A + \langle a, c\rangle_A\langle c, a\rangle_A\Big)^2\\ &= \Big(\langle a, (\langle a, b\rangle_A b + \langle a, c\rangle_A c)\rangle_A\Big)^2\\ &\leq \|a\|_A^2\|\langle a, b\rangle_A b + \langle a, c\rangle_A c\|_A^2. \end{aligned}\tag{2.6}$$

On the other hand, if we denote by $\Re z$ the real part of any complex number z, then one observes that

$$\begin{aligned}\|\langle a, b\rangle_A b + \langle a, c\rangle_A c\|_A^2 &= |\langle a, b\rangle_A|^2\|b\|_A^2 + |\langle a, c\rangle_A|^2\|c\|_A^2 + 2\Re(\langle a, b\rangle_A\langle c, a\rangle_A\langle b, c\rangle_A)\\ &\leq |\langle a, b\rangle_A|^2\|b\|_A^2 + |\langle a, c\rangle_A|^2\|c\|_A^2 + 2|\langle a, b\rangle_A|\cdot|\langle c, a\rangle_A|\cdot|\langle b, c\rangle_A|\\ &\leq |\langle a, b\rangle_A|^2\|b\|_A^2 + |\langle a, c\rangle_A|^2\|c\|_A^2 + \Big(|\langle a, b\rangle_A|^2 + |\langle a, c\rangle_A|^2\Big)|\langle b, c\rangle_A|\\ &\leq \Big(|\langle a, b\rangle_A|^2 + |\langle a, c\rangle_A|^2\Big)\Big(\max\{\|b\|_A^2, \|c\|_A^2\} + |\langle b, c\rangle_A|\Big). \end{aligned}\tag{2.7}$$

By combining (2.6) together (2.7), we get (2.5) as desired. $\square$

Now, we are in a position to prove the following theorem.

Theorem 2.9. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}\omega_{A,e}(T, S) &\leq \frac{\sqrt{2}}{2}\sqrt{\|T^\sharp T + S^\sharp S\|_A + \|T^\sharp T - S^\sharp S\|_A + 2\omega_A(S^\sharp T)}\\ &\leq \sqrt{\|T\|_A^2 + \|S\|_A^2 + \omega_A(S^\sharp T)}. \end{aligned}\tag{2.8}$$

Proof. Notice first that for any two real numbers t and s we have

$$\max\{t, s\} = \frac{1}{2}\Big(t + s + |t - s|\Big).\tag{2.9}$$

Now, let $x \in \mathbb{S}^A(0, 1)$. By letting $a = x$, $b = Tx$ and $c = Sx$ in Lemma 2.8 we get

$$\begin{aligned}&|\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2\\ &\leq \max\Big\{\|Tx\|_A^2, \|Sx\|_A^2\Big\} + |\langle Tx, Sx\rangle_A|\\ &= \frac{1}{2}\Big(\|Tx\|_A^2 + \|Sx\|_A^2 + \Big|\|Tx\|_A^2 - \|Sx\|_A^2\Big|\Big) + |\langle Tx, Sx\rangle_A| \quad \text{(by (2.9))}\\ &= \frac{1}{2}\Big(\langle(T^\sharp T + S^\sharp S)x, x\rangle_A + \Big|\langle(T^\sharp T - S^\sharp S)x, x\rangle_A\Big|\Big) + \omega_A(S^\sharp T)\\ &\leq \frac{1}{2}\Big(\omega_A(T^\sharp T + S^\sharp S) + \omega_A(T^\sharp T - S^\sharp S)\Big) + \omega_A(S^\sharp T)\\ &= \frac{1}{2}\Big(\|T^\sharp T + S^\sharp S\|_A + \|T^\sharp T - S^\sharp S\|_A\Big) + \omega_A(S^\sharp T), \end{aligned}$$

where the last inequality follows from (1.5) since the operators $T^\sharp T \pm S^\sharp S$ are A-selfadjoint. So, we get

$$|\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2 \leq \frac{1}{2}\Big(\|T^\sharp T + S^\sharp S\|_A + \|T^\sharp T - S^\sharp S\|_A + 2\omega_A(S^\sharp T)\Big),$$

for every $x \in \mathbb{S}^A(0, 1)$. Thus, by taking the supremum over all $x \in \mathbb{S}^A(0, 1)$ in above inequality, we get the first inequality in Theorem 2.9. Now, the second inequality in Theorem 2.9 follows immediately by applying the triangle inequality and (1.6). $\square$

We can state the following upper bound for the A-Davis-Wielandt radius which generalizes and improves [26, Theorem 2.14.].

Corollary 2.10. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$d\omega_A(T) \leq \sqrt{\frac{1}{2}\Big[\omega_A\Big((T^\sharp T)^2 + T^\sharp T\Big) + \omega_A\Big((T^\sharp T)^2 - T^\sharp T\Big)\Big] + \omega_A(T^\sharp T^2)}.$$

Proof. Follows immediately by proceeding as in the proof of Corollary 2.7. □

For the sequel, for any arbitrary operator $X \in \mathbb{B}_A(\mathcal{H})$, we write

$$\Re_A(X) := \frac{X + X^\sharp}{2} \quad \text{and} \quad \Im_A(X) := \frac{X - X^\sharp}{2i}.$$

Furthermore, it is useful to recall the following two lemmas. Notice that the second one follows by applying Corollary 3 and Proposition 4 in [15].

Lemma 2.11 ([16])**.** *Let $T \in \mathbb{B}(\mathcal{H})$ be an A-selfadjoint operator. Then, $T^\sharp$ is A-selfadjoint and*

$$(T^\sharp)^\sharp = T^\sharp.$$

Lemma 2.12. *Let $T \in \mathbb{B}(\mathcal{H})$ be an A-selfadjoint operator. Then, for any positive integer n we have*

$$\|T^n\|_A = \|T\|_A^n.$$

As an application of Theorem 2.9, we derive the following upper bound of the A-numerical radius of operators in $\mathbb{B}_A(\mathcal{H})$.

Corollary 2.13. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_A(T) \leq \frac{1}{2}\sqrt{\|T^\sharp T + TT^\sharp\|_A + \|T^2 + (T^\sharp)^2\|_A + \omega_A\Big((T^\sharp + T)(T - T^\sharp)\Big)}. \tag{2.10}$$

Moreover, the inequality (2.10) *is sharp.*

Proof. Let $T \in \mathbb{B}_A(\mathcal{H})$. Clearly we have $T = \Re_A(T) + i\Im_A(T)$. This implies that $T^\sharp = [\Re_A(T)]^\sharp - i[\Im_A(T)]^\sharp$. Moreover, we see that

$$\begin{aligned}\omega_A^2(T^\sharp) &= \sup\Big\{|\langle T^\sharp x, x\rangle_A|^2\,;\ x \in \mathbb{S}^A(0,1)\Big\}\\ &= \sup\Big\{|\langle [\Re_A(T)]^\sharp x, x\rangle_A|^2 + |\langle [\Im_A(T)]^\sharp x, x\rangle_A|^2\,;\ x \in \mathbb{S}^A(0,1)\Big\}\\ &= \omega_{A,\mathrm{e}}^2\Big([\Re_A(T)]^\sharp, [\Im_A(T)]^\sharp\Big). \end{aligned}\tag{2.11}$$

Since $\omega_A(T) = \omega_A(T^\sharp)$, then by using (2.11) and applying (2.8) for $T = [\Re_A(T)]^\sharp$ and $S = [\Im_A(T)]^\sharp$, we observe that

$$\begin{aligned}\omega_A^2(T) &= \omega_{A,\mathrm{e}}^2\Big([\Re_A(T)]^\sharp, [\Im_A(T)]^\sharp\Big)\\ &\leq \omega_A\Big(([\Im_A(T)]^\sharp)^\sharp[\Re_A(T)]^\sharp\Big) + \frac{1}{2}\Big\|([\Re_A(T)]^\sharp)^\sharp[\Re_A(T)]^\sharp + ([\Im_A(T)]^\sharp)^\sharp[\Im_A(T)]^\sharp\Big\|_A\\ &+ \frac{1}{2}\Big\|([\Re_A(T)]^\sharp)^\sharp[\Re_A(T)]^\sharp - ([\Im_A(T)]^\sharp)^\sharp[\Im_A(T)]^\sharp\Big\|_A.\end{aligned}$$

Moreover, it is not difficult to see that the operators $\Re_A(T)$ and $\Im_A(T)$ are A-selfadjoint. So, by Lemma 2.11, we have

$$([\Re_A(T)]^\sharp)^\sharp = [\Re_A(T)]^\sharp \quad \text{and} \quad ([\Im_A(T)]^\sharp)^\sharp = [\Im_A(T)]^\sharp.$$

So, we infer that

$$\begin{aligned}\omega_A^2(T) &\leq \frac{1}{2}\left(\left\|([\Re_A(T)]^\sharp)^2 + ([\Im_A(T)]^\sharp)^2\right\|_A + \left\|[\Re_A(T)]^\sharp)^2 - ([\Im_A(T)]^\sharp)^2\right\|_A\right) \\ &\qquad + \omega_A([\Im_A(T)]^\sharp[\Re_A(T)]^\sharp) \\ &= \frac{1}{2}\left(\left\|([\Re_A(T)]^\sharp)^2 + ([\Im_A(T)]^\sharp)^2\right\|_A + \left\|[\Re_A(T)]^\sharp)^2 - ([\Im_A(T)]^\sharp)^2\right\|_A\right) \\ &\qquad + \omega_A([\Re_A(T)][\Im_A(T)]),\end{aligned} \tag{2.12}$$

where the last equality follows since $\omega_A(X^\sharp) = \omega_A(X)$ for every $X \in \mathbb{B}_A(\mathcal{H})$. On the other hand, by making direct calculations, it can be checked that

$$\left([\Re_A(T)]^\sharp\right)^2 - \left([\Im_A(T)]^\sharp\right)^2 = \frac{(T^\sharp)^2 + [(T^\sharp)^\sharp]^2}{2} = \left(\frac{T^2 + (T^\sharp)^2}{2}\right)^\sharp,$$

and

$$\left([\Re_A(T)]^\sharp\right)^2 + \left([\Im_A(T)]^\sharp\right)^2 = \frac{(T^\sharp)^\sharp T^\sharp + T^\sharp(T^\sharp)^\sharp}{2} = \left(\frac{TT^\sharp + T^\sharp T}{2}\right)^\sharp.$$

Hence, by taking into consideration (2.12) we get

$$\omega_A^2(T) \leq \frac{1}{4}\left[\left\|(T^\sharp T + TT^\sharp)^\sharp\right\|_A + \left\|(T^2 + (T^\sharp)^2)^\sharp\right\|_A + \omega_A\Big((T^\sharp + T)(T - T^\sharp)\Big)\right].$$

This proves (2.10) since $\|X^\sharp\|_A = \|X\|_A$ for every $X \in \mathbb{B}_A(\mathcal{H})$. To show the sharpness of the inequality (2.10) we choose $T = S^\sharp$ with S is any A-selfadjoint operator on $\mathcal{H}$. So, by Lemma 2.11, $S^\sharp$ is A-selfadjoint and $(S^\sharp)^\sharp = S^\sharp$. Thus, we deduce that

$$\omega_A\Big(\left[(S^\sharp)^\sharp + S^\sharp\right]\left[S^\sharp - (S^\sharp)^\sharp)\right]\Big) = 0.$$

Further, by taking into account Lemma 2.11, we get

$$\begin{aligned}\frac{1}{2}\sqrt{\|(S^\sharp)^\sharp S^\sharp + S^\sharp(S^\sharp)^\sharp\|_A + \|(S^\sharp)^2 + [(S^\sharp)^\sharp]^2\|_A} &= \frac{1}{2}\sqrt{2\|(S^\sharp)^2\|_A + 2\,\|(S^\sharp)^2\|_A} \\ &= \sqrt{\|(S^\sharp)^2\|_A} \\ &= \|S^\sharp\|_A,\end{aligned}$$

where the last equality follows from Lemma 2.12 since $S^\sharp$ is A-selfadjoint. Thus, by taking into consideration (1.5), we deduce that both sides of (2.10) become $\|S\|_A$. ☐

Corollary 2.14. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_A(T) \leq \frac{1}{2}\sqrt{\|T^\sharp T + TT^\sharp\|_A + \|T^\sharp T - TT^\sharp\|_A + \frac{1}{2}\omega_A(T^2)}. \tag{2.13}$$

Moreover, the inequality (2.13) *is sharp.*

Proof. By replacing T and S by $(T^\sharp)^\sharp$ and $T^\sharp$ respectively and using similar techniques as above we get (2.13). To show the sharpness of the inequality (2.13) we assume that T is any A-normal operator on $\mathcal{H}$. By [15], we have

$$\omega_A(T^2) = \omega_A(T)^2 = \|T\|_A^2. \tag{2.14}$$

So, it be observed that that both sides of (2.13) become $\|T\|_A$. ☐

Another upper bound for $\omega_{A,\mathrm{e}}(T,S)$ is stated as follows.

Theorem 2.15. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_{A,e}(T,S) \leq \sqrt{\max\left(\|T\|_A^2, \|S\|_A^2\right) + \omega_A(S^\sharp T)}. \tag{2.15}$$

Moreover, the inequality (2.15) *is sharp.*

Proof. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. By letting $a = x$, $b = Tx$ and $c = Sx$ in Lemma 2.8 we get

$$\begin{aligned}|\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2 &\leq \max\left(\|Tx\|_A^2, \|Sx\|_A^2\right) + |\langle Tx, Sx\rangle_A| \\ &\leq \max\left(\|T\|_A^2, \|S\|_A^2\right) + |\langle S^\sharp Tx, x\rangle_A| \\ &\leq \max\left(\|T\|_A^2, \|S\|_A^2\right) + \omega_A(S^\sharp T).\end{aligned}$$

Thus, by taking the supremum over all $x \in \mathbb{S}^A(0,1)$ in above inequality, we get the desired result. Now, to prove the sharpness of the inequality (2.15) we choose $T = S$, where T is an A-selfadjoint operator. Then, by using Lemma 2.11, $T^\sharp$ is A-selfadjoint and $(T^\sharp)^\sharp = T^\sharp$. So, we see that

$$\max\left(\|T^\sharp\|_A^2, \|T^\sharp\|_A^2\right) + \omega_A\big((T^\sharp)^\sharp T^\sharp\big) = \|T^\sharp\|_A^2 + \omega_A\big((T^\sharp)^2\big).$$

Since $T^\sharp$ is A-selfadjoint, then $(T^\sharp)^2 \geq_A 0$. So, by (1.5), $\omega_A\big((T^\sharp)^2\big) = \|(T^\sharp)^2\|_A$. This yields, through Lemma 2.12, that $\omega_A\big((T^\sharp)^2\big) = \|T^\sharp\|_A^2$. Thus,

$$\max\left(\|T^\sharp\|_A^2, \|T^\sharp\|_A^2\right) + \omega_A\big((T^\sharp)^\sharp T^\sharp\big) = 2\|T^\sharp\|_A^2.$$

On the other hand,

$$\omega_{A,\mathrm{e}}^2(T^\sharp, T^\sharp) = 2\omega_A^2(T^\sharp) = 2\|T^\sharp\|_A^2.$$

□

Now, we state the following corollary.

Corollary 2.16. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_A(T) \leq \frac{\sqrt{2}}{2}\sqrt{\|T\|_A^2 + \omega_A(T^2)}. \tag{2.16}$$

The constant $\frac{\sqrt{2}}{2}$ is best possible in the sense that it cannot be replaced by a larger constant.

Proof. Let $T \in \mathbb{B}_A(\mathcal{H})$. By replacing T and S in Theorem 2.15 by $T^\sharp$ and T respectively, we get

$$\begin{aligned}2\omega_A^2(T) &\leq \|T\|_A^2 + \omega_A\big((T^\sharp)^2\big) \\ &= \|T\|_A^2 + \omega_A\big((T^2)^\sharp\big) \\ &= \|T\|_A^2 + \omega_A(T^2)\end{aligned}$$

This proves the inequality (2.16). Now, suppose that (2.16) holds with some constant $C > 0$. So, by choosing T any A-normal operator (with $AT \neq 0$) and using (2.14), we easily get $\sqrt{2}C \geq 1$. This finishes the proof of the corollary. □

Remark 2.17. By using (1.2) together with (1.4), we see that

$$\frac{\sqrt{2}}{2}\sqrt{\|T\|_A^2 + \omega_A(T^2)} \leq \|T\|_A.$$

So, the inequality (2.16) refines the second inequality in (1.2).

The following corollary is also an immediate consequence of Theorem 2.15 and its proof is similar to that given in Corollary 2.13 and hence omitted.

Corollary 2.18. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_A(T) \leq \frac{1}{2}\sqrt{\max\left\{\|T + T^\sharp\|_A^2, \|T - T^\sharp\|_A^2\right\} + \omega_A\Big((T^\sharp + T)(T - T^\sharp)\Big)}. \tag{2.17}$$

Moreover, the inequality (2.17) *is sharp.*

The following corollary is an immediate consequence of Theorem 2.15 and provides an upper bound for the A-Davis-Wielandt radius of operators in $\mathbb{B}_A(\mathcal{H})$. The obtained result generalizes and improves [26, Theorem 2.13].

Corollary 2.19. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$d\omega_A(T) \leq \sqrt{\max\{\|T\|_A^2, \|T\|_A^4\} + \omega_A(T^\sharp T^2)}.$$

The following lemma is useful in proving our two next results.

Lemma 2.20. *For every $a, b, c \in \mathcal{H}$, we have*

$$|\langle a, b\rangle_A|^2 + |\langle a, c\rangle_A|^2 \leq \|a\|_A \max\Big\{|\langle a, b\rangle_A|, |\langle a, c\rangle_A|\Big\}\sqrt{\|b\|_A^2 + \|c\|_A^2 + 2|\langle b, c\rangle_A|}.$$

Proof. Let $a, b, c \in \mathcal{H}$. Recall from [14, p. 132] that

$$|\langle x, y\rangle|^2 + |\langle x, z\rangle|^2 \leq \|x\| \max\Big\{|\langle x, y\rangle|, |\langle x, z\rangle|\Big\}\Big(\|y\|^2 + \|z\|^2 + 2|\langle y, z\rangle|\Big)^{\frac{1}{2}},$$

for every $x, y, z \in \mathcal{H}$. So, by choosing $x = A^{1/2}a$, $y = A^{1/2}b$ and $z = A^{1/2}c$ in the above inequality we get the desired result. $\square$

Next, we prove another upper bound for the A-joint numerical radius of a pair of operators.

Theorem 2.21. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_{A,e}(T, S) \leq \sqrt{\max\{\omega_A(T), \omega_A(S)\}\sqrt{\|T^\sharp T + S^\sharp S\|_A + 2\omega_A(S^\sharp T)}}.$$

Proof. Let $x \in \mathbb{S}^A(0, 1)$. By choosing in Lemma 2.20 $a = x, b = Tx$ and $c = Sx$ one has

$$\begin{aligned}
&|\langle x, Tx\rangle_A|^2 + |\langle x, Sx\rangle_A|^2 \\
&\leq \|x\|_A \max\{|\langle x, Tx\rangle_A|, |\langle x, Sx\rangle_A|\}\sqrt{\|Tx\|_A^2 + \|Sx\|_A^2 + 2|\langle Tx, Sx\rangle_A|} \\
&\leq \max\{\omega_A(T), \omega_A(S)\}\sqrt{\big\langle (T^\sharp T + S^\sharp S)x, x\big\rangle_A + 2|\langle S^\sharp Tx, x\rangle_A|} \\
&\leq \max\{\omega_A(T), \omega_A(S)\}\sqrt{\omega_A(T^\sharp T + S^\sharp S) + 2\omega_A(S^\sharp T)} \\
&= \max\{\omega_A(T), \omega_A(S)\}\sqrt{\|T^\sharp T + S^\sharp S\|_A + 2\omega_A(S^\sharp T)},
\end{aligned}$$

where the last inequality follows from (1.5) since $T^\sharp T + S^\sharp S \geq_A 0$. Thus,

$$|\langle x, Tx\rangle_A|^2 + |\langle x, Sx\rangle_A|^2 \leq \max(\omega_A(T), \omega_A(S)) + \sqrt{\|T^\sharp T + S^\sharp S\|_A + 2\omega_A(S^\sharp T)},$$

for all $x \in \mathbb{S}^A(0, 1)$. Therefore, the desired result follows immediately by taking the supremum over all $x \in \mathbb{S}^A(0, 1)$. $\square$

Corollary 2.22. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\omega_A(T) \leq \frac{\sqrt{2}}{2}\sqrt{\|T\|_A\sqrt{\|T^\sharp T + TT^\sharp\|_A + 2\omega_A(T^2)}} \leq \|T\|_A. \tag{2.18}$$

Proof. By replacing T and S by $(T^\sharp)^\sharp$ and $T^\sharp$ respectively in Theorem 2.21 and then using the facts that $\omega_A(X^\sharp) = \omega_A(X)$ and $\|X^\sharp\|_A = \|X\|_A$ for all $X \in \mathbb{B}_A(\mathcal{H})$, we see that

$$\sqrt{2}\omega_A(T) \leq \sqrt{\omega_A(T)\sqrt{\|T^\sharp T + TT^\sharp\|_A + 2\omega_A(T^2)}}.$$

So, by using the second inequality in (1.2) together with the triangle inequality and (1.3), we infer that

$$\begin{aligned}\omega_A(T) &\le \frac{\sqrt{2}}{2}\sqrt{\|T\|_A\sqrt{\|T^\sharp T\|_A + \|TT^\sharp\|_A + 2\omega_A^2(T)}}\\ &= \frac{\sqrt{2}}{2}\sqrt{\|T\|_A\sqrt{2\|T\|_A^2 + 2\omega_A^2(T)}} \quad (\text{by } (1.6))\\ &\le \frac{\sqrt{2}}{2}\sqrt{\|T\|_A\sqrt{2\|T\|_A^2 + 2\|T\|_A^2}},\end{aligned}$$

where the last inequality follows by applying the second inequality in (1.2). This immediately proves the second inequality in (2.18) as required. □

The following corollary in an immediate consequence of Theorem 2.21 and generalizes [26, Theorem 2.16].

Corollary 2.23. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$d\omega_A(T) \le \sqrt{\max\{\omega_A(T), \omega_A(T^\sharp T)\}\sqrt{\omega_A\left[(T^\sharp T)^2 + T^\sharp T\right] + 2\omega_A(T^\sharp T^2)}}.$$

By using Lemma 2.20, another upper bound for the A-Davis–Wielandt radius of operators in $\mathbb{B}_A(\mathcal{H})$ can be derived as follows.

Theorem 2.24. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$d\omega_A(T) \le \sqrt{\|T\|_A \max\{\omega_A(T), \omega_A(T^\sharp T)\}\sqrt{1 + \|T\|_A^2 + 2\omega_A(T)}}.$$

Proof. Let $x \in \mathbb{S}^A(0,1)$. By choosing in Lemma 2.20 $a = Tx, b = x$ and $c = Tx$ we observe that

$$\begin{aligned}|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 &= |\langle Tx, x\rangle_A|^2 + |\langle Tx, Tx\rangle_A|^2\\ &\le \|Tx\|_A \max\{|\langle Tx, x\rangle_A|, |\langle Tx, Tx\rangle_A|\}\sqrt{1 + \|Tx\|_A^2 + 2|\langle x, Tx\rangle_A|}\\ &= \|Tx\|_A \max\{|\langle Tx, x\rangle_A|, |\langle T^\sharp Tx, x\rangle_A|\}\sqrt{1 + \|Tx\|_A^2 + 2|\langle x, Tx\rangle_A|}\\ &\le \|T\|_A \max\{\omega_A(T), \omega_A(T^\sharp T)\}\sqrt{1 + \|T\|_A^2 + 2\omega_A(T)}.\end{aligned}$$

Thus

$$|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 \le \|T\|_A \max\{\omega_A(T), \omega_A(T^\sharp T)\}\sqrt{1 + \|T\|^2 + 2\omega_A(T)}, \tag{2.19}$$

for all $x \in \mathbb{S}^A(0,1)$. Hence, by taking the supremum over $x \in \mathbb{S}^A(0,1)$ in (2.19) we obtain the required result. □

The next theorem provides an upper and lower bound of the A-joint numerical radius of two operators in $\mathbb{B}_A(\mathcal{H})$.

Theorem 2.25. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\frac{\sqrt{2}}{2}\max\{\omega_A(T+S), \omega_A(T-S)\} \le \omega_{A,e}(T,S) \le \frac{\sqrt{2}}{2}\sqrt{\omega_A^2(T+S) + \omega_A^2(T-S)}.$$

Moreover, the constant $\frac{\sqrt{2}}{2}$ is sharp in both inequalities.

Proof. For every $x \in \mathcal{H}$, we have

$$\begin{aligned}(|\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2)^{\frac{1}{2}} &\geq \frac{\sqrt{2}}{2}(|\langle Tx, x\rangle_A| + |\langle Sx, x\rangle_A|) \\ &\geq \frac{\sqrt{2}}{2}|\langle Tx, x\rangle_A \pm \langle Sx, x\rangle_A| \\ &= \frac{\sqrt{2}}{2}|\langle (T \pm S)x, x\rangle_A|\,.\end{aligned}$$

Taking supremum over all $x \in \mathbb{S}^A(0,1)$ yields that

$$\omega_{A,\mathrm{e}}(T,S) \geq \frac{\sqrt{2}}{2}\omega_A(T \pm S). \tag{2.20}$$

This proves the first inequality in Theorem 2.25. On the other hand, for every $x \in \mathbb{S}^A(0,1)$ we have

$$|\langle Tx, x\rangle_A \pm \langle Sx, x\rangle_A|^2 \leq \omega_A^2(T \pm S). \tag{2.21}$$

So, an application of the parallelogram identity for complex numbers and (2.21) gives

$$\begin{aligned}|\langle Tx, x\rangle_A|^2 + |\langle Sx, x\rangle_A|^2 &= \frac{1}{2}\Big(|\langle Tx, x\rangle_A + \langle Sx, x\rangle_A|^2 + |\langle Tx, x\rangle_A - \langle Sx, x\rangle_A|^2\Big) \\ &\leq \frac{1}{2}\Big(\omega_A^2(T+S) + \omega_A^2(T-S)\Big),\end{aligned}$$

for every $x \in \mathbb{S}^A(0,1)$. Taking supremum over all $x \in \mathbb{S}^A(0,1)$ yields that

$$\omega_{A,\mathrm{e}}^2(T,S) \leq \frac{1}{2}\Big(\omega_A^2(T+S) + \omega_A^2(T-S)\Big).$$

This shows the second inequality in Theorem 2.25. For sharpness one can obtain the same quantity $\sqrt{2}\omega_A(T)$ on both sides of the inequality by putting $T = S$. $\square$

The following corollary in an immediate consequence of Theorem 2.25 and (1.5).

Corollary 2.26. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$ be two A-selfadjoint operators. Then,*

$$\frac{\sqrt{2}}{2}\max\{\|T+S\|_A, \|T-S\|_A\} \leq \omega_{A,e}(T,S) \leq \frac{\sqrt{2}}{2}\sqrt{\|T+S\|_A^2 + \|T-S\|_A^2}.$$

Another bounds of $\omega_{A,\mathrm{e}}(T,S)$ can be stated as follows.

Theorem 2.27. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\frac{\sqrt{2}}{2}\sqrt{\omega_A(T^2+S^2)} \leq \omega_{A,e}(T,S) \leq \sqrt{\|T^\sharp T + S^\sharp S\|_A}. \tag{2.22}$$

Proof. Notice first that the second inequality in (2.22) follows from (1.7). By using (2.20), we observe that

$$\begin{aligned}2\omega_{A,\mathrm{e}}^2(T,S) &\geq \frac{1}{2}\Big(\omega_A^2(T+S) + \omega_A^2(T-S)\Big) \\ &\geq \frac{1}{2}\Big(\omega_A[(T+S)^2] + \omega_A[(T-S)^2]\Big) \quad \text{(by (1.3))} \\ &\geq \frac{1}{2}\Big(\omega_A[(T+S)^2 + (T-S)^2]\Big) \\ &= \omega_A(T^2+S^2).\end{aligned}$$

This proves the first inequality in (2.22). $\square$

The following corollary is also an immediate consequence of Theorem 2.27 and generalizes the well-known inequalities proved by F. Kittaneh in [22, Theorem 1]. Further, the obtained inequalities improve the bounds in (1.8)

Corollary 2.28. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\frac{1}{2}\sqrt{\|T^\sharp T + TT^\sharp\|_A} \leq \omega_A(T) \leq \frac{\sqrt{2}}{2}\sqrt{\|T^\sharp T + TT^\sharp\|_A}. \tag{2.23}$$

Proof. By taking into consideration (2.27) and proceeding as in the proof of Corollary 2.13, we get

$$\frac{\sqrt{2}}{2}\sqrt{\omega_A\Big(([\Re_A(T)]^\sharp)^2 + ([\Im_A(T)]^\sharp)^2\Big)} \leq \omega_A(T) \leq \sqrt{\Big\|([\Re_A(T)]^\sharp)^2 + ([\Im_A(T)]^\sharp)^2\Big\|_A}.$$

Since $\Big([\Re_A(T)]^\sharp\Big)^2 + \Big([\Im_A(T)]^\sharp\Big)^2 \geq_A 0$, then (1.5) gives

$$\frac{\sqrt{2}}{2}\sqrt{\Big\|([\Re_A(T)]^\sharp)^2 + ([\Im_A(T)]^\sharp)^2\Big\|_A} \leq \omega_A(T) \leq \sqrt{\Big\|([\Re_A(T)]^\sharp)^2 + ([\Im_A(T)]^\sharp)^2\Big\|_A}.$$

This proves the desired inequalities by following the proof of Corollary 2.13. □

Remark 2.29.
1. Notice that the inequalities in (2.23) are already proved by the second author in [18] and by Altwaijry et al. in [1]. However, the techniques used here are different from the other proofs.
2. The inequalities in (2.23) are sharp (see [18]).
3. The inequalities in (2.23) improve the bounds in (1.2) (see [18]).
4. A generalization of the inequalities in (2.23) are established in [11].

In the rest of this paper, we prove several inequalities involving the A-Davis-Wielandt radius and the A-numerical radii of operators in $\mathbb{B}_A(\mathcal{H})$.

The following lemma is useful in the proof of our next result.

Lemma 2.30. *Let $S \in \mathbb{B}_A(\mathcal{H})$. Then, for every $a \in \mathbb{S}^A(0,1)$ we have*

$$|\langle Sa, a\rangle_A|^2 \leq \frac{1}{2}|\langle S^2a, a\rangle_A| + \frac{1}{4}\langle (S^\sharp S + SS^\sharp)a, a\rangle_A.$$

Proof. Let $x, y, z \in \mathcal{H}$ with $\|z\|_A = 1$. We first prove that

$$|\langle x, z\rangle_A\langle z, y\rangle_A| \leq \frac{1}{2}\Big(|\langle x, y\rangle_A| + \|x\|_A\,\|y\|_A\Big). \tag{2.24}$$

Since $\|A^{1/2}z\| = 1$, then by using the well-known Buzano's inequality ([10]), we see that

$$\begin{aligned}|\langle x, z\rangle_A\langle z, y\rangle_A| &= |\langle A^{1/2}x, A^{1/2}z\rangle\langle A^{1/2}z, A^{1/2}y\rangle| \\ &\leq \frac{1}{2}\Big(|\langle A^{1/2}x, A^{1/2}y\rangle| + \|A^{1/2}x\|\,\|A^{1/2}y\|\Big).\end{aligned}$$

This proves the desired result.

Now, let $a \in \mathbb{S}^A(0,1)$. By using the arithmetic-geometric mean inequality and applying (2.24) for $x = Sa$, $z = a$ and $y = S^\sharp a$ we infer that

$$\begin{aligned}|\langle Sa, a\rangle_A|^2 &= |\langle Sa, a\rangle_A\langle a, S^\sharp a\rangle_A| \\ &\leq \frac{1}{2}\Big(|\langle Sa, S^\sharp a\rangle_A| + \|Sa\|_A\,\|S^\sharp a\|_A\Big) \\ &\leq \frac{1}{2}|\langle Sa, S^\sharp a\rangle_A| + \frac{1}{4}\Big(\|Sa\|^2 + \|S^\sharp a\|^2\Big) \\ &= \frac{1}{2}|\langle S^2a, a\rangle_A| + \frac{1}{4}\langle (S^\sharp S + SS^\sharp)a, a\rangle_A.\end{aligned}$$

Hence, the proof is complete. □

We present now the following result.

Theorem 2.31. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then, we have*

$$d\omega_A(T) \leq \frac{1}{2}\sqrt{\omega_A\left((T^\sharp T+T)^2\right)+\omega_A\left((T^\sharp T-T)^2\right)+\omega_A\Big(T^\sharp T+2(T^\sharp T)^2+TT^\sharp\Big)}.$$

Proof. Let $x \in \mathbb{S}^A(0,1)$. By applying the well-known parallelogram identity for complex numbers, we see that

$$\begin{aligned}|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 &= \frac{1}{2}\left(\big|\,\|Tx\|_A^2+\langle Tx,x\rangle_A\big|^2+\big|\,\|Tx\|_A^2-\langle Tx,x\rangle_A\big|^2\right)\\ &= \frac{1}{2}\left(\left|\big\langle (T^\sharp T+T)x,x\big\rangle_A\right|^2+\left|\big\langle (T^\sharp T-T)x,x\big\rangle_A\right|^2\right). \qquad (2.25)\end{aligned}$$

On the other hand, by applying Lemma 2.30 we see that

$$\begin{aligned}&\left|\big\langle (T^\sharp T+T)x,x\big\rangle_A\right|^2+\left|\big\langle (T^\sharp T-T)x,x\big\rangle_A\right|^2\\ &\leq \frac{1}{2}\left|\big\langle (T^\sharp T+T)^2x,x\big\rangle_A\right|+\frac{1}{2}\left|\big\langle (T^\sharp T-T)^2x,x\big\rangle_A\right|\\ &\quad+\frac{1}{4}\Big\langle \Big[(T^\sharp T+T)^\sharp(T^\sharp T+T)+(T^\sharp T+T)(T^\sharp T+T)^\sharp\Big]x,x\Big\rangle_A\\ &\quad\;+\frac{1}{4}\Big\langle \Big[(T^\sharp T-T)^\sharp(T^\sharp T-T)+(T^\sharp T-T)(T^\sharp T-T)^\sharp\Big]x,x\Big\rangle_A.\end{aligned}$$

By observing that $(T^\sharp T)^\sharp = T^\sharp T$ and making short calculations, we infer that

$$\begin{aligned}&\left|\big\langle (T^\sharp T+T)x,x\big\rangle_A\right|^2+\left|\big\langle (T^\sharp T-T)x,x\big\rangle_A\right|^2\\ &\leq \frac{1}{2}\left|\big\langle (T^\sharp T+T)^2x,x\big\rangle_A\right|+\frac{1}{2}\left|\big\langle (T^\sharp T-T)^2x,x\big\rangle_A\right|+\frac{1}{2}\Big\langle \Big[T^\sharp T+2(T^\sharp T)^2+TT^\sharp\Big]x,x\Big\rangle_A\\ &\leq \frac{1}{2}\left[\omega_A\left(\left(T^\sharp T+T\right)^2\right)+\omega_A\left(\left(T^\sharp T-T\right)^2\right)+\omega_A\Big(T^\sharp T+2(T^\sharp T)^2+TT^\sharp\Big)\right].\end{aligned}$$

Hence, by taking into account (2.25) we obtain

$$\begin{aligned}&|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4\\ &\leq \frac{1}{4}\left[\omega_A\left(\left(T^\sharp T+T\right)^2\right)+\omega_A\left(\left(T^\sharp T-T\right)^2\right)+\omega_A\Big(T^\sharp T+2(T^\sharp T)^2+TT^\sharp\Big)\right],\end{aligned}$$

for all $x \in \mathbb{S}^A(0,1)$. Finally, by taking the supremum over all $x \in \mathbb{S}^A(0,1)$ in the above inequality we get the desired result. ☐

In order to prove our next upper bound for $d\omega_A(\cdot)$, we need the following lemma.

Lemma 2.32. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then, for all $x \in \mathbb{S}^A(0,1)$ we have*

$$|\langle Tx,x\rangle_A|^2 \leq \sqrt{\langle T^\sharp Tx,x\rangle_A}\sqrt{\langle TT^\sharp x,x\rangle_A}.$$

Proof. Let $x \in \mathbb{S}^A(0,1)$. By using the Cauchy-Schwarz inequality we see that

$$\begin{aligned}|\langle Tx,x\rangle_A|^2 &= |\langle Tx,x\rangle_A|\cdot|\langle Tx,x\rangle_A|\\ &= |\langle Tx,x\rangle_A|\cdot|\langle x,T^\sharp x\rangle_A|\\ &= |\langle A^{1/2}Tx,A^{1/2}x\rangle|\cdot|\langle A^{1/2}x,A^{1/2}T^\sharp x\rangle|\\ &\leq \|Tx\|_A\|T^\sharp x\|_A\\ &= \sqrt{\langle T^\sharp Tx,x\rangle_A}\sqrt{\langle TT^\sharp x,x\rangle_A}.\end{aligned}$$

Hence, the proof is complete. ☐

Now, we are in a position to provide the following upper bound for $d\omega_A(\cdot)$.

Theorem 2.33. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$d\omega_A(T) \leq \sqrt{\frac{1}{2}\,\omega_A\Big(T^\sharp T + 2(T^\sharp T)^2 + TT^\sharp\Big) - \frac{1}{2}\inf_{x\in\mathbb{S}^A(0,1)}\Big(\|Tx\|_A - \|T^\sharp x\|_A\Big)^2}.$$

Proof. Notice first that $(T^\sharp T)^\sharp = T^\sharp T$. Now, let $x \in \mathbb{S}^A(0,1)$. By using Lemma 2.32 and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4\\
&= |\langle Tx, x\rangle_A|^2 + |\langle T^\sharp Tx, x\rangle_A|^2\\
&\leq \sqrt{\langle T^\sharp Tx, x\rangle_A}\sqrt{\langle TT^\sharp x, x\rangle_A} + \sqrt{\langle (T^\sharp T)^\sharp (T^\sharp T)x, x\rangle_A}\sqrt{\langle (T^\sharp T)(T^\sharp T)^\sharp x, x\rangle}\\
&= \sqrt{\langle T^\sharp Tx, x\rangle_A}\sqrt{\langle TT^\sharp x, x\rangle_A} + \sqrt{\langle (T^\sharp T)^2 x, x\rangle_A}\sqrt{\langle (T^\sharp T)^2 x, x\rangle_A}\\
&= \frac{1}{2}\left[\langle T^\sharp Tx, x\rangle_A + \langle TT^\sharp x, x\rangle_A - \left(\sqrt{\langle T^\sharp Tx, x\rangle_A} - \sqrt{\langle TT^\sharp x, x\rangle_A}\right)^2\right] + \Big\langle (T^\sharp T)^2 x, x\Big\rangle_A\\
&= \frac{1}{2}\left[\langle T^\sharp Tx, x\rangle_A + \langle TT^\sharp x, x\rangle_A + 2\Big\langle (T^\sharp T)^2 x, x\Big\rangle_A\right] - \frac{1}{2}\left(\sqrt{\langle T^\sharp Tx, x\rangle_A} - \sqrt{\langle TT^\sharp x, x\rangle_A}\right)^2\\
&= \frac{1}{2}\Big\langle \Big[T^\sharp T + 2(T^\sharp T)^2 + TT^\sharp\Big]x, x\Big\rangle_A - \frac{1}{2}\Big(\|Tx\|_A - \|T^\sharp x\|_A\Big)^2\\
&\leq \frac{1}{2}\omega_A\Big[T^\sharp T + 2(T^\sharp T)^2 + TT^\sharp\Big] - \frac{1}{2}\inf_{x\in\mathbb{S}^A(0,1)}\Big(\|Tx\|_A - \|T^\sharp x\|_A\Big)^2.
\end{aligned}$$

This gives

$$|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 \leq \frac{1}{2}\omega_A\Big[T^\sharp T + 2(T^\sharp T)^2 + TT^\sharp\Big] - \frac{1}{2}\inf_{x\in\mathbb{S}^A(0,1)}\Big(\|Tx\|_A - \|T^\sharp x\|_A\Big)^2,$$

for all $x \in \mathbb{S}^A(0,1)$ which in turn shows required inequality by taking the supremum over all $x \in \mathbb{S}^A(0,1)$. ☐

The next theorem provides another bound for $d\omega_A(\cdot)$.

Theorem 2.34. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$d\omega_A(T) \leq \sqrt{\omega_A^2\left(T^\sharp T - T\right) + 2\|T\|_A^2\omega_A(T)}. \tag{2.26}$$

Proof. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Then, by making simple calculations and using the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}
|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 &= \Big|\langle Tx, Tx\rangle_A - \langle Tx, x\rangle_A\Big|^2 + 2\Re\Big(\langle Tx, Tx\rangle_A\langle Tx, x\rangle_A\Big)\\
&= \Big|\Big\langle \Big(T^\sharp T - T\Big)x, x\Big\rangle_A\Big|^2 + 2\|Tx\|_A^2\Re\langle Tx, x\rangle_A\\
&\leq \omega_A^2(T^\sharp T - T) + 2\|T\|_A^2\omega_A(T).
\end{aligned}$$

So, we get

$$|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 \leq \omega_A^2(T^\sharp T - T) + 2\|T\|_A^2\omega_A(T), \tag{2.27}$$

for all $x \in \mathbb{S}^A(0,1)$. Hence, by taking the supremum over all $x \in \mathbb{S}^A(0,1)$ in (2.27), we get (2.26) as required. ☐

To prove our next result, we need the following lemma which is quoted from the proof of [25, Theorem 2.13.].

Lemma 2.35. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\frac{1}{2}\|Tx\|_A \leq \sqrt{\frac{\omega_A^2(T)}{2} + \frac{\omega_A(T)}{2}\sqrt{\omega_A^2(T) - |\langle Tx, x\rangle_A|^2}}$$

for any $x \in \mathbb{S}^A(0,1)$.

Now, we are ready to prove another upper bound for the A-Davis–Wielandt radius of operators in $\mathbb{B}_A(\mathcal{H})$.

Theorem 2.36. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$d\omega_A(T) \leq \frac{\sqrt{2}}{2}\sqrt{\omega_A\left(T^2\right) + \frac{1}{2}\omega_A\left(T^\sharp T + TT^\sharp\right) + 8\mu},$$

where

$$\mu = \omega_A^2(T)\left(2\omega_A^2(T) - c_A^2(T) + 2\omega_A(T)\sqrt{\omega_A^2(T) - c_A^2(T)}\right).$$

Proof. Let $x \in \mathbb{S}^A(0,1)$. It follows, from Lemma 2.30, that

$$\begin{aligned} |\langle Tx, x\rangle_A|^2 &\leq \frac{1}{2}|\langle T^2x, x\rangle_A| + \frac{1}{4}\langle\left(T^\sharp T + TT^\sharp\right)x, x\rangle_A \\ &\leq \frac{1}{2}\omega_A\left(T^2\right) + \frac{1}{4}\omega_A\left(T^\sharp T + TT^\sharp\right). \end{aligned} \tag{2.28}$$

Moreover, by using Lemma 2.35 one has

$$\begin{aligned} \|Tx\|_A^4 &\leq 16\left(\frac{\omega_A^2(T)}{2} + \frac{\omega_A(T)}{2}\sqrt{\omega_A^2(T) - |\langle Tx, x\rangle_A|^2}\right)^2 \\ &\leq 4\left(\omega_A^2(T) + \omega_A(T)\sqrt{\omega_A^2(T) - c_A^2(T)}\right)^2 \\ &\leq 4\omega_A^2(T)\left(2\omega_A^2(T) - c_A^2(T) + 2\omega_A(T)\sqrt{\omega_A^2(T) - c_A^2(T)}\right), \end{aligned} \tag{2.29}$$

By combining (2.28) together with (2.29), we infer that

$$\begin{aligned} |\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 \leq 4\omega_A^2(T)&\left(2\omega_A^2(T) - c_A^2(T) + 2\omega_A(T)\sqrt{\omega_A^2(T) - c_A^2(T)}\right) \\ &+ \frac{1}{2}\omega_A\left(T^2\right) + \frac{1}{4}\omega_A\left(T^\sharp T + TT^\sharp\right), \end{aligned}$$

for all $x \in \mathbb{S}^A(0,1)$. Therefore, we obtain the desired inequality by taking the supremum in the above inequality over all $x \in \mathbb{S}^A(0,1)$. □

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