CONSTRUCTIVE MATHEMATICAL ANALYSIS
**5** (2022), No. 3, pp. 168-182
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

*Research Article*

# Lower estimates on the condition number of a Toeplitz sinc matrix and related questions

LUDWIG KOHAUPT* AND YAN WU

ABSTRACT. As one new result, for a symmetric Toeplitz sinc $n \times n$-matrix $A(t)$ depending on a parameter $t$, lower estimates (tending to infinity as t vanishes) on the pertinent condition number are derived. A further important finding is that prior to improving the obtained lower estimates it seems to be more important to determine the lower bound on the parameter $t$ such that the smallest eigenvalue $\mu_n(t)$ of $A(t)$ can be reliably computed since this is a precondition for determining a reliable value for the condition number of the Toeplitz sinc matrix. The style of the paper is expository in order to address a large readership.

**Keywords:** Condition number, eigenvalues and eigenvectors, inverse power method, power method, Toeplitz sinc matrix.

**2020 Mathematics Subject Classification:** 15B05, 65F15, 65F22.

## 1. INTRODUCTION

This paper is organized as follows. In Section 2, a symmetric Toeplitz sinc $n \times n$-matrix $A(t) = A_n(t)$ is defined and the problem with its pertinent condition number $\kappa_2(t)$ is described. The entries of this $n \times n$-matrix are made up of $s(0) := 1$ and $s(jt) := \sin(j\pi t)/(j\pi t),\ j = 1, \ldots, n-1$ and are investigated for $0 < t < 1$. Such a matrix appears frequently in the study of minimum phase filter designs [10] and numerical integration/differentiation of bandlimited systems [11]. As properties of the matrices $A(t)$, we found that the limit $\lim_{t\to 0} A(t) = A$ exists and also that, for the eigenvalues $\mu_j(t),\ j = 1, \ldots, n$ of $A(t)$, the limits $\lim_{t\to 0} \mu_j(t) = \mu_j = \mu_j(A),\ j = 1, \ldots, n$ exist and, further, that the values of the entries of $A$ and $\mu_j,\ j = 1, \ldots, n$ can be given explicitly. In Section 3, two-sided estimates on $\mu_j(t),\ j = 1, \ldots, n$ are derived. The eigenvalues are arranged according to $\mu_1(t) \geq \cdots \geq \mu_n(t)$ and $\mu_1 \geq \cdots \geq \mu_n$. In Section 4, two upper bounds on the smallest eigenvalue $\mu_n(t)$ are obtained. Thereby, in Section 5, three lower estimates on the condition number $\kappa_2(t) = \mu_1(t)/\mu_n(t)$ can be derived. These lower bounds are new and tend to infinity as $t$ tends to zero. For comparison reasons, in Section 6, a lower bound on $\mu_1(t)$ and an upper bound on $\mu_n(t)$ are stated from a paper of D. Hertz delivering an upper bound on the condition number. Section 7 contains numerical verifications of the obtained estimates on $\kappa_2(t)$ for some examples. In Section 8, linearly independent eigenvectors of the matrix $A = \lim_{t\to 0} A(t)$ are derived that form a basis of $\mathbb{R}^n$. Then, in Section 9, appropriate computational methods for the determination of $\mu_n(t)$ and $\mu_1(t)$ are presented, and in Section 10, these computational methods are applied to a series of matrices $A(t) = A_n(t)$. Finally, Section 11 contains the conclusions followed by the References.

Received: 10.07.2022; Accepted: 08.08.2022; Published Online: 15.08.2022
*Corresponding author: Ludwig Kohaupt; lkohaupt4@web.de
DOI: 10.33205/cma.1142905

## 2. Some Properties of a Toeplitz Sinc Matrix $A(t)$

A Toeplitz sinc matrix is defined as

$$(2.1)\quad A(t) = A_n(t) = \begin{bmatrix} s(0) & s(t) & s(2\,t) & \cdots & s((n-2)\,t) & s((n-1)\,t) \\ s(t) & s(0) & s(t) & \cdots & s((n-3)\,t) & s((n-2)\,t) \\ s(2\,t) & s(t) & s(0) & \cdots & & s((n-3)\,t) \\ & & & \vdots & & \\ s((n-1)\,t) & s((n-2)\,t) & & \cdots & s(t) & s(0) \end{bmatrix},$$

where $0 < t < 1$ and $s(0) = 1$ as well as $s(t) = \mathrm{sinc}(t) = \sin(\pi t)/(\pi t)$. From [9, Theorem 2.2], it follows that this matrix is positive definite by setting there $t_1 = 0$, $t_i = (i-1)\,t$, $i = 2, \ldots, n$ and taking into account that s(-t)=s(t) for $0 < t < 1$. As $t$ gets smaller, the condition number of this matrix deteriorates quickly. The question that one might ask therefore is: Can one find a way to estimate the largest and smallest eigenvalues of this matrix? This would help us to monitor the condition number of the above Toeplitz sinc matrix and is the starting point of our investigation.

The following theorem presents some properties of the matrices $A(t)$.

**Theorem 2.1.** *Let the matrix $A(t) = A_n(t)$ in (2.1) be given. Further, let the eigenvalues $\mu_j(t) = \mu_j(A(t)) = \mu_j(A_n(t))$ be arranged according to*

$$(2.2)\qquad \mu_1(t) \geq \mu_2(t) \geq \cdots \geq \mu_n(t).$$

*Then, the limits*

$$(2.3)\qquad A := \lim_{t\to 0} A(t)$$

*as well as*

$$(2.4)\qquad \mu_j = \mu_j(A) := \lim_{t\to 0} \mu_j(t) = \lim_{t\to 0} \mu_j(A(t)), \quad j = 1, \ldots, n$$

*exist and*

$$(2.5)\qquad A = A_n = \lim_{t\to 0} A(t) = \lim_{t\to 0} A_n(t) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ & & & \vdots & & \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

*Further, the limits in (2.4) are eigenvalues of $A$, and with appropriate enumeration of the eigenvalues $\mu_j := \mu_j(A)$, $j = 1, \ldots, n$, one has*

$$(2.6)\qquad \lim_{t\to 0} \mu_1(A(t)) = \lim_{t\to 0} \mu_1(A_n(t)) = \mu_1(A) = \mu_1 = n,$$

$$(2.7)\qquad \lim_{t\to 0} \mu_j(A(t)) = \lim_{t\to 0} \mu_j(A_n(t)) = \mu_j(A) = \mu_j = 0,\ j = 2, \ldots, n.$$

*Proof.* (2.5): The Toeplitz matrix $A(t) = A_n(t) \in \mathbb{R}^{n\times n}$ according to (2.1) reads

$$A(t) = A_n(t) = \begin{bmatrix} s(0) & s(t) & s(2\,t) & \cdots & s((n-2)\,t) & s((n-1)\,t) \\ s(t) & s(0) & s(t) & \cdots & s((n-3)\,t) & s((n-2)\,t) \\ s(2\,t) & s(t) & s(0) & \cdots & & s((n-3)\,t) \\ & & & \vdots & & \\ s((n-1)\,t) & s((n-2)\,t) & & \cdots & s(t) & s(0) \end{bmatrix}$$

for $0 < t < 1$ and $s(0) := \lim_{t\to 0} s(t) := \lim_{t\to 0} \operatorname{sinc}(t) := \lim_{t\to 0} \sin(\pi\, t)/(\pi\, t) = 1$. From this, apparently (2.5) follows.

(2.6) and (2.7): This is seen as follows. Matrix $A = A_n$ in (2.5) is a rank-one symmetric matrix. Hence, there is only one non-zero eigenvalue, namely $\mu_1(A)$, and therefore $\mu_1(A)$ must equal $tr(A) = n$. Since the limits $\lim_{t\to 0} \mu_j(t)$ exist and are equal to $\mu_j$ for $j = 1, \ldots, n$ if one chooses an appropriate enumeration, the assertion follows.

(2.3): This follows immediately from (2.5).

(2.4): This follows immediately from (2.6) and (2.7).

So, on the whole, Theorem 2.1 is proven. □

For later use, we arrange the eigenvalues $\mu_j = \mu_j(A)$ according to

$$\mu_1 \geq \cdots \geq \mu_n. \tag{2.8}$$

**Remark 2.1.** *Another elementary proof of Theorem 2.1 will be given at the end of Section 3.*

**Remark 2.2.** *Theorem 2.1 also follows from* [4, Theorem 17, p. 263] *that is a much more general result.*

## 3. Two-Sided Estimates on the Eigenvalues $\mu_j(t)$, $j = 1, \ldots, n$ of $A(t)$

(i) Upper Estimate on $\mu_1(t)$ According to [5, Section 5.4, Formula (7), p. 89], we have

$$\mu_1(t) = |\mu_1(t)| \leq \|A(t)\|_\infty = \max_{j=1,\cdots,n} \sum_{k=1}^{n} |a_{jk}(t)|\,.$$

Now,

$$s((n-1)\,t) < \cdots < s(2\,t) < s(t) < s(0) = 1$$

yielding the upper estimate

$$0 < \mu_1(t) \leq \sum_{k=0}^{n-1} 1 = n. \tag{3.9}$$

(ii) Lower Estimate on $\mu_1(t)$

We use [5, Section 5.4, Formula (28), p. 94]. Thereby, employing (2.2) and (2.8),

$$|\mu_1 - \mu_1(t)| \leq \|A - A(t)\|_\infty = \max_{j=1,\cdots,n} \sum_{k=1}^{n} |a_{jk} - a_{jk}(t)|$$

with

$$A - A(t) \tag{3.10}$$

$$= \begin{bmatrix} 0 & 1-s(t) & 1-s(2\,t) & \cdots & 1-s((n-2)\,t) & 1-s((n-1)\,t) \\ 1-s(t) & 0 & 1-s(t) & \cdots & 1-s((n-3)\,t) & 1-s((n-2)\,t) \\ 1-s(2\,t) & 1-s(t) & 0 & \cdots & & 1-s((n-3)\,t) \\ & & & \vdots & & \\ 1-s((n-1)\,t) & 1-s((n-2)\,t) & & \cdots & 1-s(t) & 0 \end{bmatrix}.$$

Because of

$$1 > s(t) > s(2\,t) > \cdots > s((n-1)\,t),$$

we obtain

$$-s(t) < -s(2\,t) < \cdots < -s((n-1)\,t)$$

and thus

$$1 - s(t) < 1 - s(2\,t) < \cdots < 1 - s((n-1)\,t). \tag{3.11}$$

Therefore,

$$\begin{aligned} \mu_1(t) &= \mu_1(t) - \mu_1 + \mu_1 \geq \mu_1 - |\mu_1 - \mu_1(t)| \\ &\geq \mu_1 - \|A - A(t)\|_\infty \geq \mu_1 - n\,[1 - s((n-1)\,t)] \\ &= n - n\,[1 - s((n-1)\,t)] = n\,s((n-1)\,t) \end{aligned}$$

so that we obtain the lower estimate

$$\mu_1(t) \geq n\,s((n-1)\,t)\,. \tag{3.12}$$

(iii) Two-Sided Estimate on $\mu_1(t)$

On the whole, we have the two-sided estimate

$$n\,s((n-1)\,t) \leq \mu_1(t) \leq n\,. \tag{3.13}$$

(iv) Two-Sided Estimates on $\mu_j(t),\ j = 2, \ldots, n$

Since $\mu_j = 0,\ j = 2, \ldots, n$, one has

$$0 < \mu_j(t) = |-\mu_j(t)| = |\mu_j - \mu_j(t)| \leq \|A - A(t)\|_\infty \leq \max_{j=1,\ldots,n} \sum_{k=1}^{n} |a_{jk} - a_{jk}(t)|.$$

Along with (3.10) and (3.11), we herewith conclude that

$$0 < \mu_j(t) < (n-1)\,[1 - s((n-1)\,t)],\ j = 2, \ldots, n\,. \tag{3.14}$$

(v) Elementary Proof of $\lim_{t\to 0} \mu_1(t) = n$

Taking the limit as $t \to 0$ in the two-sided estimate (3.13), we get

$$\lim_{t\to 0} \mu_1(t) = n \tag{3.15}$$

since

$$\lim_{t\to 0} s((n-1)\,t) = 1\,.$$

(vi) Elementary Proof of $\lim_{t\to 0} \mu_j(t) = 0,\ j = 2, \ldots, n$

Taking the limit as $t \to 0$ in the two-sided estimate (3.14), we get

$$\lim_{t\to 0} \mu_j(t) = 0,\ j = 2, \ldots, n \tag{3.16}$$

since

$$\lim_{t\to 0} s((n-1)\,t) = 1\,.$$

(vii) Elementary Proof of $\lim_{t\to 0} \mu_1(t) = \mu_1 = n$

One has the chain of implications

$$det(A(t) - \mu_1(t)\,I) = 0$$

$\Rightarrow$

$$\lim_{t\to 0} det(A(t) - \mu_1(t)\,I) = 0$$

$\Rightarrow$

$$det(\lim_{t\to 0} A(t) - \lim_{t\to 0} \mu_1(t)\,I) = 0$$

$\Rightarrow$

$$det(A - \lim_{t\to 0} \mu_1(t)\,I) = 0.$$

Thus, $\lim\limits_{t\to 0} \mu_1(t)$ is an eigenvalue of $A$ that is denoted by $\mu_1$. Therefore,

$$det(A - \mu_1\,I) = 0.$$

Together with

$$\mu_1 = n,$$

one obtains

(3.17) $$\lim_{t\to 0} \mu_1(t) = n = \mu_1\,.$$

(viii) Elementary Proof of $\lim_{t\to 0} \mu_j(t) = \mu_j = 0,\ j = 2,\ldots,n$

The Proof is similar to that in (vii).

**Remark 3.3.** *The points (v) - (viii) deliver an elementary proof of Theorem 2.1. This is because they show that the limits* $\lim_{t\to 0} \mu_j(t),\ j = 1,\ldots,n$ *exist and are eigenvalues of* $A$. *In particular, the elementary proof is independent of [4] the application of which is in a way like using a sledge-hammer to crack a nut.*

## 4. Two Upper Estimates on Smallest Eigenvalue $\mu_n(t)$

(i) First Upper Estimate

As is known,

$$0 < |A(t)| := det(A(t)) = \mu_1(t)\,\mu_2(t)\,\cdots\,\mu_n(t) < 1$$

at least for sufficiently small t in $0 < t < 1$ since from Section 3 we know that $\mu_j = \mu_j(A) = 0,\ j = 2,\ldots,n$ so that in particular $\mu_n(t) \to 0$ as $t \to 0$. This entails

$$0 < [\mu_n(t)]^n \le |A(t)| < 1$$

for $0 < t \le t_1$ with sufficiently small $t_1$ or the first upper estimate

(4.18) $$0 < \mu_n(t) \le |A(t)|^{\frac{1}{n}} < 1$$

for $0 < t \le t_1$ with sufficiently small $t_1$.

(ii) Second Upper Estimate

The derivation of the second upper estimate is based on [3, Corollary 8.1,4, p.411] that, in turn, is proven in [8, pp. 103-104] using the Courant-Fischer Minimax Theorem. The cited corollary is called Theorem 4.2 here and, in our notation, reads as follows:

**Theorem 4.2.** *If $A_r$ denotes the leading r-by-r principal submatrix of an n-by-n symmetric matrix $A$, then for $r = 1 : n-1$ the following* interlacing property *holds:*

$$\mu_{r+1}(A_{r+1}) \le \mu_r(A_r) \le \mu_r(A_{r+1}) \le \cdots \le \mu_2(A_{r+1}) \le \mu_1(A_r) \le \mu_1(A_{r+1})\,.$$

For $r = n-1$, Theorem 4.2 delivers

$$\mu_n(A_n) \le \mu_{n-1}(A_{n-1}),$$

where $A_n = A$. Now, we apply the last estimate to the Toeplitz sinc matrix (for short: Tsinc matrix) $A(t) = A_n(t) \in \mathbb{R}^{n\times n}$ and remark that the leading $(n-1)\times(n-1)$ submatrix of this matrix is the Tsinc matrix $A_{n-1}(t)$. This entails the chain of inequalities

$$\mu_n(A_n(t)) \le \mu_{n-1}(A_{n-1}(t)),$$
$$\mu_{n-1}(A_{n-1}(t)) \le \mu_{n-2}(A_{n-2}(t)),$$
$$\cdots$$
$$\mu_3(A_3(t)) \le \mu_2(A_2(t)),$$

where

$$\mu_2(A_2(t)) = 1 - s(t)$$

and

$$\mu_1(A_2(t)) = 1 + s(t)$$

which follows from

$$|A_2(t) - \mu(t)I| = \left| \begin{array}{c|c} 1-\mu(t) & s(t) \\ \hline s(t) & 1-\mu(t) \end{array} \right| = 0\,.$$

This yields the second upper estimate

$$0 < \mu_n(t) := \mu_n(A_n(t)) \le 1 - s(t), \quad 0 < t < 1\,. \tag{4.19}$$

## 5. Three Lower Estimates on Condition Number $\kappa_2(t) := \mu_1(t)/\mu_n(t)$

(i) First Lower Estimate on $\kappa_2(t)$

From the first upper estimate on $\mu_n(t)$, we obtain

$$\frac{1}{\mu_n(t)} \ge \frac{1}{|A(t)|^{\frac{1}{n}}} > 1$$

for $0 < t \le t_1$ with sufficiently small $t_1$. This yields the first lower estimate

$$\kappa_2(t) = \mu_1(t)/\mu_n(t) \ge \frac{n\, s((n-1)\,t)}{|A(t)|^{\frac{1}{n}}} := e_1(t) \tag{5.20}$$

for $0 < t \le t_1$ with sufficiently small $t_1$.

(ii) Second Lower Estimate on $\kappa_2(t)$

From the second upper estimate on $\mu_n(t)$, we obtain

$$\frac{1}{\mu_n(t)} \ge \frac{1}{1-s(t)}$$

for $0 < t \le t_1$ with sufficiently small $t_1$. This yields the second lower estimate

$$\kappa_2(t) = \mu_1(t)/\mu_n(t) \ge \frac{n\, s((n-1)\,t)}{1-s(t)} := e_2(t) \tag{5.21}$$

for all $t$ in $0 < t < 1$.

(iii) Third Lower Estimate on $\kappa_2(t)$

Combining the preceding results, one gets the third lower estimate

$$\kappa_2(t) = \mu_1(t)/\mu_n(t) \ge \max\{e_1(t), e_2(t)\} := e_3(t) \tag{5.22}$$

for $0 < t \le t_1$ with sufficiently small $t_1$.

## 6. Bounds Stated by D. Hertz and Application

In this section, we apply the bounds on the extreme eigenvalues of Toeplitz matrices stated in [1] to our symmetric Toeplitz matrix $A(t)$ defined in (2.1). As application, one obtains upper bounds on $\kappa_2(t) = \mu_1(t)/\mu_n(t)$.

Let

$$a(t) = [a_1(t),\, a_2(t), \ldots,\, a_n(t)], \qquad 0 < t < 1 \tag{6.23}$$

be the first row of $A(t)$, and define

$$\tilde{a}(t) := [a_1(t),\, |a_2(t)|, \ldots,\, |a_n(t)|], \qquad 0 < t < 1. \tag{6.24}$$

From (2.1), we have

$$\tilde{a}(t) = a(t), \qquad 0 < t < 1. \tag{6.25}$$

Further, define

$$\overline{\lambda}_k = -\underline{\lambda}_k = 2 \cos\left(\frac{\pi}{\text{floor}[(n-1)/(k-1)]+2}\right), \quad k = 2, \dots, n. \tag{6.26}$$

As in Section 2, we assume that the eigenvalues $\mu_k(t)$, $k = 1, \dots, n$ are arranged according to (2.2). Then, one has, in our notation, the following theorem.

**Theorem 6.3.** *The maximal eigenvalue $\mu_1(t)$ of the symmetric Toeplitz matrix $A(t)$ in (2.1) is bounded from above by the inner product*

$$\mu_1(t) \le (a(t), \overline{w}), \qquad 0 < t < 1, \tag{6.27}$$

*where $a(t)$ is as in (6.23) and the vector $\overline{w}$ is defined by*

$$\overline{w} = [1, \overline{\lambda}_2, \dots, \overline{\lambda}_n] \tag{6.28}$$

*and $\overline{\lambda}_k$ is as in (6.26).*

*Proof.* The theorem is a direct consequence of [1, Theorem 1]. □

**Remark 6.4.** *Theorem 6.3 can hold only if $(a(t), \overline{w}) > 0$, of course.*

Further, one has the following theorem.

**Theorem 6.4.** *The minimal eigenvalue $\mu_n(t)$ of the symmetric Toeplitz matrix $A(t)$ in (2.1) is bounded from below by the inner product*

$$\mu_n(t) \ge (a(t), \underline{w}), \qquad 0 < t < 1, \tag{6.29}$$

*where $a(t)$ is as in (6.23) and the vector $\underline{w}$ is defined by*

$$\underline{w} = [1, \underline{\lambda}_2, \dots, \underline{\lambda}_n]. \tag{6.30}$$

*Note that using (6.26), we obtain*

$$\underline{w} = [1, -\overline{\lambda}_2, \dots, -\overline{\lambda}_n]. \tag{6.31}$$

*Proof.* The theorem is a direct consequence of [1, Theorem 2]. □

**Remark 6.5.** *Theorem 6.4 can hold only if $(a(t), \underline{w}) > 0$, of course.*

**Remark 6.6.** *From Theorems 6.3 and 6.4, we get the upper estimates*

$$\kappa_2(t) = \mu_1(t)/\mu_n(t) \le (a(t), \overline{w})/(a(t), \underline{w}), \quad 0 < t < 1 \tag{6.32}$$

*provided that $(a(t), \overline{w}) > 0$ and $(a(t), \underline{w}) > 0$ for $0 < t < 1$.*

## 7. Numerical Verification of the Estimates on $\kappa_2(t) := \mu_1(t)/\mu_n(t)$ for Some Examples

In this section, we present estimates on $\kappa_2(t) = \mu_1(t)/\mu_n(t)$ for fixed $t = 0.1$ and $n = 2, \dots, 6$. For this, corresponding Matlab computations were carried out. The expressions $e_1(t)$, $e_2(t)$, $e_3(t)$ are estimates from below (tending to $\infty$ as $n \to \infty$ and $t \to 0$) on $\kappa_2(t)$, expression $e_4(t)$ is defined as condition number $\kappa_2(t) = \mu_1(t)/\mu_n(t)$, whereas expression $e_5(t)$ is an estimate from above on $\kappa_2(t)$ provided that $(a(t), \overline{w}) > 0$ and $(a(t), \underline{w}) > 0$. Its derivation follows from two theorems stated by D. Hertz. The pertinent upper estimate should at least be positive since $\kappa_2(t)$ is so. But, it turns out to be negative since $(a(t), \underline{w}) < 0$ for $n \ge 3$. Consequently, $e_5(t)$ cannot deliver an upper bound on $\kappa_2(t)$.

In the following estimate $e_1(t)$, the determinant $|A(t)| = \det(A(t))$ enters. This is computed in two ways, namely first with Matlab routine *det* and second, for comparison reasons, as a product of the eigenvalues of $A(t)$. From numerical considerations, it is clear that the determination of $|A(t)|$ can be achieved through elementary operations by casting matrix $A(t)$ into triangular form without changing the determinant so that the product of the diagonal elements gives the determinant. We think that this technique is behind the Matlab routine *det*. The second way via the product of the eigenvalues that is computationally much more costly is used only for comparison reasons. This is because if one computes the determinant via the product of eigenvalues $\mu_j(t)$, $j = 1, \dots, n$, then one has immediately $\kappa_2(t) = \mu_1(t)/\mu_n(t)$ and needs no estimates.

Now, the details of the computations follow.

For $n = 2$, we obtain

$$A(t) = \begin{bmatrix} 1.000000000000000 & 0.983631643083466 \\ 0.983631643083466 & 1.000000000000000 \end{bmatrix},$$

$$\mu(t) = \begin{bmatrix} 0.016368356916534 \\ 1.983631643083466 \end{bmatrix},$$

$$d(n,t) := \mu_1(t)\,\mu_2(t) = 0.032468790724921,$$

$$|A(t)| = \det(A(t)) = det(A_n(t)) = 0.032468790724921,$$

and

$$\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{bmatrix} := \begin{bmatrix} n\,s((n-1)t)/|A(t)|^{\frac{1}{n}} \\ n\,s((n-1)t)/(1-s(t)) \\ \max\{e_1(t), e_2(t)\} \\ \mu_1(t)/\mu_n(t) \\ (a(t), \overline{w})/(a(t), \underline{w}) \end{bmatrix} = \begin{bmatrix} 10.917656600748638 & \\ 1.201869739399289 & \times 10^2 \\ 1.201869739399289 & \times 10^2 \\ 1.211869739399293 & \times 10^2 \\ 1.211869739399306 & \times 10^2 \end{bmatrix}.$$

For $n = 3$, we obtain

$$A(t) = \begin{bmatrix} 1.000000000000000 & 0.983631643083466 & 0.935489283788639 \\ 0.983631643083466 & 1.000000000000000 & 0.983631643083466 \\ 0.935489283788639 & 0.983631643083466 & 1.000000000000000 \end{bmatrix},$$

$$\mu(t) = \begin{bmatrix} 0.000145422566712 \\ 0.064510716211361 \\ 2.935343861221926 \end{bmatrix},$$

$$d(n,t) := \prod_{j=1}^{n} \mu_1(t) \dots \mu_j(t) = 0.032468790724921,$$

$$|A(t)| = \det(A(t)) = det(A_n(t)) = 0.032468790724921,$$

and

$$\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{bmatrix} := \begin{bmatrix} n\,s((n-1)t)/|A(t)|^{\frac{1}{n}} \\ n\,s((n-1)t)/(1-s(t)) \\ \max\{e_1(t), e_2(t)\} \\ n\,s((n-1)t)/(1-s(t)) \\ \mu_1(t)/\mu_n(t) \\ (a(t), \overline{w})/(a(t), \underline{w}) \end{bmatrix} = \begin{bmatrix} 92.936401783180301 & \\ 1.714569071090478 & \times 10^2 \\ 1.714569071090478 & \times 10^2 \\ 2.018492677986877 & \times 10^2 \\ -2.507665165149629 & \end{bmatrix}.$$

For $n = 4$, we obtain

$$A(t) = \begin{bmatrix} 1.000000000000000 & 0.983631643083466 & 0.935489283788639 & 0.858393691334140 \\ 0.983631643083466 & 1.000000000000000 & 0.983631643083466 & 0.935489283788639 \\ 0.935489283788639 & 0.983631643083466 & 1.000000000000000 & 0.983631643083466 \\ 0.858393691334140 & 0.935489283788639 & 0.983631643083466 & 1.000000000000000 \end{bmatrix},$$

$$\mu(t) = \begin{bmatrix} 0.000001113119258 \\ 0.000870415161304 \\ 0.157973552463136 \\ 3.841154919256300 \end{bmatrix},$$

$$d(n,t) := \prod_{j=1}^{n} \mu_1(t) \ldots \mu_j(t) = 5.879147433554857 \times 10^{-1},$$

$$|A(t)| = \det(A(t)) = det(A_n(t)) = 5.879147434765446 \times 10^{-1},$$

and

$$\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{bmatrix} := \begin{bmatrix} n\, s((n-1)t)/|A(t)|^{\frac{1}{n}} \\ n\, s((n-1)t)/(1-s(t)) \\ \max\{e_1(t), e_2(t)\} \\ \mu_1(t)/\mu_n(t) \\ (a(t), \overline{w})/(a(t), \underline{w}) \end{bmatrix} = \begin{bmatrix} 6.972971989701032 & \times 10^2 \\ 2.097690551864878 & \times 10^2 \\ 6.972971989701032 & \times 10^2 \\ 3.450802681898942 & \times 10^6 \\ -1.838422415548006 & \end{bmatrix}.$$

For $n = 5$, we obtain

$$A(t) = \begin{bmatrix} 1.000000000000000 & 0.983631643083466 & 0.935489283788639 & 0.858393691334140 & 0.756826728640657 \\ 0.983631643083466 & 1.000000000000000 & 0.983631643083466 & 0.935489283788639 & 0.858393691334140 \\ 0.935489283788639 & 0.983631643083466 & 1.000000000000000 & 0.983631643083466 & 0.935489283788639 \\ 0.858393691334140 & 0.935489283788639 & 0.983631643083466 & 1.000000000000000 & 0.983631643083466 \\ 0.756826728640657 & 0.858393691334140 & 0.935489283788639 & 0.983631643083466 & 1.000000000000000 \end{bmatrix},$$

$$\mu(t) = \begin{bmatrix} 0.000000008008103 \\ 0.000008896854399 \\ 0.003035674827786 \\ 0.307675090716305 \\ 4.689280329593409 \end{bmatrix},$$

$$d(n,t) := \prod_{j=1}^{n} \mu_1(t) \ldots \mu_j(t) = 3.120469248845038 \times 10^{-16},$$

$$|A(t)| = \det(A(t)) = det(A_n(t)) = 3.120469141684447 \times 10^{-16},$$

and

$$\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{bmatrix} := \begin{bmatrix} n\, s((n-1)t)/|A(t)|^{\frac{1}{n}} \\ n\, s((n-1)t)/(1-s(t)) \\ \max\{e_1(t), e_2(t)\} \\ \mu_1(t)/\mu_n(t) \\ (a(t), \overline{w})/(a(t), \underline{w}) \end{bmatrix} = \begin{bmatrix} 4.776639739123309 & \times 10^3 \\ 2.311859194236440 & \times 10^2 \\ 4.776639739123309 & \times 10^3 \\ 5.855669475721616 & \times 10^8 \\ -1.549163591209945 & \end{bmatrix}.$$

For $n = 6$, we obtain

$$A(t) = \begin{bmatrix} 1.000000000000000 & 0.983631643083466 & 0.935489283788639 & 0.858393691334140 & 0.756826728640657 & 0.636619772367581 \\ 0.983631643083466 & 1.000000000000000 & 0.983631643083466 & 0.935489283788639 & 0.858393691334140 & 0.756826728640657 \\ 0.935489283788639 & 0.983631643083466 & 1.000000000000000 & 0.983631643083466 & 0.935489283788639 & 0.858393691334140 \\ 0.858393691334140 & 0.935489283788639 & 0.983631643083466 & 1.000000000000000 & 0.983631643083466 & 0.935489283788639 \\ 0.756826728640657 & 0.858393691334140 & 0.935489283788639 & 0.983631643083466 & 1.000000000000000 & 0.983631643083466 \\ 0.636619772367581 & 0.756826728640657 & 0.858393691334140 & 0.935489283788639 & 0.983631643083466 & 1.000000000000000 \end{bmatrix},$$

$$\mu(t) = \begin{bmatrix} 0.000000000055683 \\ 0.000000080040530 \\ 0.000039987658742 \\ 0.008055094169584 \\ 0.521314905500388 \\ 5.470589932575071 \end{bmatrix},$$

$$d(n,t) := \prod_{j=1}^{n} \mu_1(t) \dots \mu_j(t) = 4.094114597934044 \times 10^{-24},$$

$$|A(t)| = \det(A(t)) = det(A_n(t)) = 4.094097171079637 \times 10^{-24},$$

and

$$\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{bmatrix} := \begin{bmatrix} n\,s((n-1)t)/|A(t)|^{\frac{1}{n}} \\ n\,s((n-1)t)/(1-s(t)) \\ \max\{e_1(t), e_2(t)\} \\ \mu_1(t)/\mu_n(t) \\ (a(t),\overline{w})/(a(t),\underline{w}) \end{bmatrix} = \begin{bmatrix} 3.019986597952274 & \times & 10^4 \\ 2.333599306077634 & \times & 10^2 \\ 3.019986597952274 & \times & 10^4 \\ 9.824603336342802 & \times & 10^{10} \\ -1.460059391749488 & & \end{bmatrix}.$$

<u>Discussion of the Computational Results on the Estimates on $\kappa_2(t) := \mu_1(t)/\mu_n(t)$ for the Examples</u>

The computational results underpin the theoretical findings. In particular, they show that the lower estimates $e_1(t)$, $e_2(t)$, $e_3(t)$ on $e_4(t) = \kappa_2(t) = \mu_1(t)/\mu_n(t)$ tend to $\infty$ as $t \to 0$, as it must be. Further, apparently expression $e_3(t)$ is the best lower bound out of the lower bounds $e_j(t)$, $j = 1, 2, 3$. But, with growing dimension $n$, it underestimates the condition number $\kappa_2(t) = \mu_1(t)/\mu_n(t)$ significantly. In order to find out more on the reason for this, in the next sections, it will be investigated for what values of $t$ and to how many decimal places the eigenvalues $\mu_1(t)$, $\mu_n(t)$, and the condition number $\kappa_2(t) = \mu_1(t)/\mu_n(t)$ can be determined. We hope that the pertinent results will deliver upper bounds on $n$ and lower bounds on $t$ that form estimates for the applicability of the best estimate $e_3(t)$ on $e_4(t) = \kappa_2(t)$.

The estimates stated by D. Hertz for $n \geq 3$ are not applicable since $(a(t),\overline{w}) < 0$ for $n \geq 3$.

## 8. The Eigenvectors of $A = \lim_{t\to 0} A(t)$

For symmetric matrices $A(t)$ and $A$, when $A(t) \to A$ $(t \to 0)$, one uses, as a rule, the eigenvectors of $A(t)$ associated with an eigenvalue $\mu(t)$ of $A(t)$ as an approximation of an eigenvector of $A$ provided the eigenvectors of $A(t)$ can be determined much easier than those of $A$. Here, it is almost the other way around. The reason for this is that the eigenvalues of matrix $A$ can be determined very simply, and various linearly independent associated eigenvectors can likewise be determined very easily.

This will be shown in the present section.

In the next section, these eigenvectors of $A$ will be used as initial vectors for the power method resp. the inverse power method to compute $\mu_1(t)$ resp. $\mu_n(t)$.

Now, the computational details follow. <u>$n = 3$:</u>

(i) <u>Determination of the eigenvector $w_1$ associated with $\mu_1 = 3$</u>

One has

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

so that

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the eigenvector $w_1$ pertinent to $\mu_1 = 3$ is equal to

$$w_1 = e := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The associated normed eigenvector reads

$$w_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3.$$

The generalization to the case $A \in \mathbb{R}^{n\times n}$ clearly is

$$w_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n.$$

(ii) Determination of the eigenvector $w_2$ and $w_3$ associated with $\mu_2 = 0$ and $\mu_3 = 0$

From

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we obtain

$$v_1 + v_2 + v_3 = 0$$

or

$$v_3 = -v_1 - v_2.$$

$v_1 = 1,\ v_2 = 1$:

With these values,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ -v_1 - v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

$v_1 = 1,\ v_2 = -1$:

With these values,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ -v_1 - v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The normed eigenvectors are thus

$$w_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad w_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad , \quad w_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Apparently,

$$(w_j, w_k) = \delta_{j,k},\ j,k = 1,\ 2,\ 3\,.$$

There are other eigenvectors, for example,

$$w_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, w_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$\underline{n=5}$: Let

$$w_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix},\ w_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\\0\\0\end{bmatrix},\ w_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix},\ w_4 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\0\\1\\-1\end{bmatrix},\ w_5 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}.$$

Then, $w_1 \in \mathbb{R}^n = \mathbb{R}^5$ is a normed eigenvector corresponding to the largest eigenvalue $\mu_1 = n = 5$ of $A \in \mathbb{R}^{n\times n} = \mathbb{R}^{5\times 5}$, whereas $w_j$, $j = 2,\ldots,n = 5$ are linearly independent normed eigenvectors corresponding to the eigenvalues $\mu_j = 0$, $j = 2,\ldots,n = 5$ that are linearly independent, but not pairwise orthogonal. However, one has

$$(w_1, w_j) = 0,\ j = 2,\ldots,5\,.$$

Let

$$e_{\pm 1} = \begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix},\ e_{\pm 2} = \begin{bmatrix}0\\1\\-1\\0\\0\end{bmatrix},\ e_{\pm 3} = \begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix},\ e_{\pm 4} = \begin{bmatrix}0\\0\\0\\1\\-1\end{bmatrix},\ e_{\pm 5} = \begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix} \in \mathbb{R}^n = \mathbb{R}^5$$

and

$$e = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \in \mathbb{R}^n = \mathbb{R}^5.$$

Then, the components of $e_{\pm j}$, $j = 2,3,4,5$ are cyclic permutations of $e_{\pm 1}$:

$$e_{\pm 2} = P(23451)(e_{\pm 1}),\ e_{\pm 3} = P(34512)(e_{\pm 1}),\ e_{\pm 4} = P(45123)(e_{\pm 1}),\ e_{\pm 5} = P(51234)(e_{\pm 1})$$

so that

$$w_1 = \frac{1}{\sqrt{5}}\,e,\ w_2 = \frac{1}{\sqrt{2}}\,e_{\pm 2},\ w_3 = \frac{1}{\sqrt{2}}\,e_{\pm 3},\ w_4 = \frac{1}{\sqrt{2}}\,e_{\pm 4},\ w_5 = \frac{1}{\sqrt{2}}\,e_{\pm 5}.$$

A set of pairwise orthogonal eigenvectors can be obtained when we apply Schmidt's orthogonalization method to these linear independent eigenvectors $w_j$, $j = 1,\ldots,5$.

The generalization from $n = 5$ to arbitrary $n \in \mathbb{N}$ of eigenvectors $w_j$, $j = 1,\ldots,n$ as above can be done in a straightforward way.

## 9. Appropriate Computational Methods for the Determination of $\mu_n(t)$ and $\mu_1(t)$

Since $\mu_n(t) \to 0$ $(t \to 0)$ and $\mu_1(t) \to n$ $(t \to 0)$, it is clear that $\kappa_2(t) \to \infty$ $(t \to 0)$ which posed the problem to determine lower estimates on $\kappa_2(t)$. A related important question is how $\mu_n(t)$ and $\mu_1(t)$ can be computed such that the outcome is reliable.

For the determination of the largest eigenvalue $\mu_1(t)$ of $A(t) \in \mathbb{R}^{n\times n}$, the *power method* is appropriate as described, for example, in [5, Section 10.1.1] and for compact symmetric operators in [7, Section 7]. As initial vector $x_0 \in \mathbb{R}^n$, one can use every non-zero real $n$-vector. However, the eigenvector $w_1 = (1/\sqrt{n})\,[1,\ldots,1]^T \in \mathbb{R}^n$ corresponding to $\mu_1 = \mu_1(A)$ seems to be especially advantageous as initial vector $x_0$.

For the determination of the smallest eigenvalue $\mu_n(t)$ of $A(t)$, the *inverse iteration* can be used as described in [5, Section 10.1.3]. This is a modification of the power method where the

| $n$ | $t$ | $\mu_n(t)$ | $\mu_1(t)$ |
|---|---|---|---|
| 5 | 0.1 | $0.800810 \times 10^{-10}$ | 4.68928 |
| 10 | 0.3 | $0.586776 \times 10^{-12}$ | 3.33055 |
| 15 | 0.6 | $0.669565 \times 10^{-8}$ | 1.66666 |

Table 1. Computational Results for $\mu_n(t)$ and $\mu_1(t)$

power method is applied to the inverse of a non-singular matrix. For short, we call this method *inverse power method*.

Based on these methods, pertinent Matlab programs were developed. For comparative reasons, also Matlab routine eig.m is applied that computes not only the largest and smallest eigenvalues of a square matrix, but all eigenvalues which is computationally disadvantages, of course.

## 10. Application of the Computational Methods to a Series of Matrices $A(t)$

First, with the inverse power method mentioned in Section 9, for $n = 5$, 10, 15 and $t = 0.1$, we tried to determine the smallest eigenvalues $\mu_n(t)$. For $n = 5$ and $t = 0.1$, this was possible. For $n = 10$ and $t = 0.1$, the developed Matlab program issued the **error code NaN** meaning **Not a Number**. This error code is typically put out by Matlab, for instance, when a division by zero is tried. For short, the determination of $\mu_n(t)$ by the inverse power method was not possible for $n = 10$ and $t = 0.1$. It was neither possible for $n = 10$ and $t = 0.2$. However, the determination of $\mu_n(t)$ was possible for $n = 10$ and $t = 0.3$. Similarly for $n = 15$ and $t = 0.1, \ldots, 0.5$, $\mu_n(t)$ could not be determined by the inverse power method. However, $\mu_n(t)$ could be successfully determined for $n = 15$ and $t = 0.6$.

Further, for all those pairs $(n, t)$ the smallest eigenvalues $\mu_n(t)$ could be computed successfully for, also the pertinent largest eigenvalues $\mu_1(t)$ could be determined by the power method. In Table 1, the computational results are compiled. For comparison reasons, we applied also the Matlab routine eig.m.

For $n = 5$, $t = 0.1$, we obtained the following vector of not-arranged eigenvalues

$$\mu(t) = [-0.1224,\ -0.6286,\ -0.5261,\ 0.3238,\ 0.4326]^T.$$

Since $\mu_j(t) < 0,\ j = 1, 2, 3$, program eig.m delivers a false result without issuing a warning or error code.

For $n = 10$, $t = 0.3$, we obtained the following vector of not-arranged eigenvalues

$$\mu(t) = \begin{array}{r} [0.0075,\ -0.1683,\ -0.4915,\ -0.2763,\ 0.2189,\ 0.2698, ,,; 0.3038,\ -0.2039, \\ -0.3721,\ 0.0683]^T. \end{array}$$

Since $\mu_j(t) < 0,\ j = 2, 3, 4, 8, 9$, program eig.m delivers a false result without issuing a warning or error code.

For $n = 15$, $t = 0.6$, we obtained the following vector of not-arranged eigenvalues

$$\mu(t) = \begin{array}{r} [0.0037,\ 0.0758,\ -0.3355,\ 0.2519,\ 0.2716,\ 0.1537,\ 0.0616,\ -0.0000, \\ 0.0219,\ -0.0291,\ 0.1980,\ -0.4128,\ 0.2169,\ 0.0254,\ -0.0006]^T. \end{array}$$

Since $\mu_j(t) < 0,\ j = 3, 8, 10, 12, 15$, program eig.m delivers a false result without issuing a warning or error code.

As we see, the computation of $\mu_j(t),\ j = 1, \ldots, n$ by the Matlab routine eig.m is not only costly since it computes all eigenvalues, but it also delivers false results without any error warning.

| $n$ | $t = \underline{t}$ | $\mu_n(t)$ | $\mu_1(t)$ |
|---|---|---|---|
| 5 | 0.004256 | $0.01148 \times 10^{-17}$ | 4.999 |
| 10 | 0.3300 | $0.3514 \times 10^{-11}$ | 3.028 |
| 15 | 0.5350 | $0.1698 \times 10^{-10}$ | 1.868 |

TABLE 2. Determination of minimal $t = \underline{t}$ in $0 < t < 1$ such that $\mu_n(t)$ can be reliably computed

Consequences of the Computational Results

The computational results of this section shows that the critical point in the determination of the condition number $\kappa_2(t) = \mu_1(t)/\mu_n(t)$ is the smallest eigenvalue $\mu_n(t)$ of $A(t) = A_n(t)$. As a consequence, instead of trying to derive better closed-form lower estimates on $\kappa_2(t)$ than those we have already obtained, the efforts should be laid on the reliable computation of the smallest eigenvalue $\mu_n(t)$ as a function of $n$ and $t$.

For $n = 5$, 10, 15, we have determined the minimal $t = \underline{t}$ *up to four significant places* such that $\mu_n(t)$ can be computed by the inverse power method. For these values of $t$, we then determined also $\mu_1(t)$. The results are assembled in Table 2.

## 11. Conclusions

Starting point of this paper was the aim to derive lower estimates on the condition number $\kappa_2(t)$ of the symmetric Toeplitz sinc matrix $A(t) = A_n(t)$. This is of interest since $\kappa_2(t) \to \infty$. The aim was achieved, but numerical calculations showed that the derived lower estimates significantly underestimate the condition number with growing $n$ and vanishing $t$. Thus, this finding shifted the effort to the problem of effectively and reliably determining the smallest eigenvalue $\mu_n(t)$ of the symmetric Toeplitz sinc matrix $A(t) = A_n(t)$. It turned out that the inverse power method is most appropriate to do this. The pertinent computational experiments showed, for instance, that for $n = 5$ and $t = 0.1$, $\mu_n(t)$ can be determined by this method. But, for $n = 10$ and $t = 0.1$ and $t = 0.2$, this was not possible. However, for $n = 10$ and $t = 0.3$, the inverse power method was successful in determining $\mu_n(t)$. For $n = 15$ and $t = 0.1, \ldots, 0.5$, again $\mu_n(t)$ could not be determined, but for $n = 15$ and $t = 0.6$, this was possible. These results were somehow surprising since, for example, $t = 0.6$ is not near zero so that the problems begin (depending on $n$) with much larger values of $t$ than we thought. The reason for the numerical problems are, of course, that the computations are done with a restricted number of digital places of the used machine numbers as opposed to the computation with real numbers that have an unlimited number of places. Comparative computations with the Matlab routine eig.m delivered false results for all the mentioned pairs (n,t) since some of the eigenvalues were negative, which cannot be correct because $A(t)$ is positive definite. The most important implication of all these results is that, for calculations with machine numbers (i.e., on computers), priority should be given to the determination of the lower bound $\underline{t} := \inf t$ of the parameter $t$ such that $\mu_n(t)$ can be reliably computed for $0 < \underline{t} = \inf t \le t < 1$. This was done for $n = 5$, 10, 15 with a precision of four significant places by applying the inverse power method. So, one can also say that, for calculations on computers, the expression $\lim\limits_{\substack{t \to 0 \\ 0<t<1}} \kappa_2(t)$ has to be replaced by $\lim\limits_{\substack{t \to \underline{t} \\ 0<\underline{t}<t<1,\, t \in \mathbb{M}}} \kappa_2(t)$, where $\mathbb{M}$ is the set of machine numbers of the used computer, and further that the problems begin already with around $n = 15$ in the sense that with $n = 15$, the minimal value $t = \underline{t}$ reads $\underline{t} = 0.5350 > 0.5$ indicating that problems must be expected when using machine numbers, i.e., when using a computer in calculations involving the Tsinc matrix for

$n \geq 15$ such as the solution of a system of linear equations. The calculations were carried out in *single precision*. Corresponding computations in double precision might deliver better results, but were not done because we think that this would not give new insight in the problem. For information on the effects of finite precision arithmetic on numerical algorithms, the reader is referred to [2, Chapters 1 and 2] or [6, Sections 13 and 14]. We mention that in the English translation of the First Edition [5], Sections 13 and 14 are not yet contained.

**Acknowledgements**
The authors would like to thank one of the referees for his/her valuable hints to improve the paper.

## References

[1] D. Hertz: *Simple Bounds on the Extreme Eigenvalues of Toeplitz Matrices*, IEEE Transactions on Information Theory, **38** (1) (1992), 175–176.
[2] N. J. Higham: *Accuracy and Stability of Numerical Algorithms*, SIAM, Philadelphia (1996).
[3] G. H. Goloub, Ch. F. van Loan: *Matrix Computations*, The Johns Hopkins University Press, Baltimore and London (1989).
[4] F. Stummel: *Diskrete Approximation linear Operatoren.* II (Discrete Approximation of Linear Operators. II). Math. Z., **120** (1971), 231–264.
[5] F. Stummel, K. Hainer: *Introduction to Numerical Analysis (English Translation by E.R. Dawson of the First Edition of the German Original of 1971)*, Scottish Academic Press, Edinburgh (1980).
[6] F. Stummel, K. Hainer: *Praktische Mathematik (Introduction to Numerical Analysis)*, Second Edition, B.G. Teubner, Stuttgart (1982).
[7] F. Stummel, L. Kohaupt: *Eigenwertaufgaben in Hilbertschen Räumne. Mit Aufgaben und vollständigen Lösungen, (Eigenvalue Problems in Hilbert spaces. With Exercises and Complete Solutions)*, Logos Verlag, Berlin (2021).
[8] J. H. Wilkinson: *The Algebraic Eigenvalue Problem*, Oxford University Press, Oxford (1965).
[9] Y. Wu: *On the positiveness of a functional symmetric matrix used in digital filter design*, Journal of Circuits, Systems, and Computers, **13** (5) (2004), 1105–1110.
[10] Y. Wu, D. H. Mugler: *A robust DSP integrator for acceleration signals*, IEEE Transactions on Biomedical Engineering, Vol. **51** (2) (2004), 385–389.
[11] Y. Wu, N. Sepehri: *Interpolation of bandlimited signals from uniform or non-uniform integral samples*, Electronic Letters, **47** (1) (2011), 6th Jan.

Ludwig Kohaupt
Berlin University of Technology (BHT)
Department of Mathematics
Luxemburger Str. 10, 13353 Berlin, Germany
ORCID: 0000-0003-4364-9144
*E-mail address*: lkohaupt4@web.de

Yan Wu
Georgia Southern University
Department of Mathematical Sciences
65 Georgia Ave, Statesboro, GA, 30460, USA
ORCID: 0000-0002-7202-8980
*E-mail address*: yan@georgiasouthern.edu