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HYBRINOMIALS RELATED TO HYPER-LEONARDO NUMBERS

Efruz Özlem MERSİN[1] and Mustafa BAHŞİ[2]

[1]Department of Mathematics, Aksaray University, Aksaray, TÜRKİYE
[2]Department of Mathematics and Science Education, Aksaray University, Aksaray, TÜRKİYE

Abstract. In this paper, we define hybrinomials related to hyper-Leonardo numbers. We study some of their properties such as the recurrence relation and summation formulas. In addition, we introduce hybrid hyper-Leonardo numbers.

1. Introduction

Integer sequences are the subject of many studies which are shown in recent literature [1–8]. The most famous integer sequence is called Fibonacci sequence and is defined by the following recurrence relation ($n \geq 1$) [1]:

$$F_{n+1} = F_n + F_{n-1} \quad \text{with} \quad F_0 = 0, \quad F_1 = 1.$$

Leonardo sequence, which has similar properties to the Fibonacci sequence, is defined by Catarino and Borges [5], as follows:

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \qquad (n \geq 2),$$

with the initial conditions $Le_0 = Le_1 = 1$. Although commonly called "Leonardo numbers" in the literature, Kürüz et al. [9] preferred to call them "Leonardo Pisano numbers" and introduced Leonardo Pisano polynomials as

$$Le_n(x) = \begin{cases} 1, & n = 0, 1 \\ x + 2, & n = 2 \\ 2xLe_{n-1}(x) - Le_{n-3}(x), & n \geq 3. \end{cases}$$

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[1] efruzmersin@aksaray.edu.tr-Corresponding author; 0000-0001-6260-9063
[2] mhvbahsi@yahoo.com; 0000-0002-6356-6592

Hyper Leonardo numbers $Le_n^{(r)}$ are defined as a generalization of the Leonardo numbers by the formula

$$Le_n^{(r)} = \sum_{s=0}^{n} Le_s^{(r-1)} \quad \text{with} \quad Le_n^{(0)} = Le_n, \quad Le_0^{(r)} = Le_0 \quad \text{and} \quad Le_1^{(r)} = r+1,$$

where r is a positive integer [10]. The hyper-Leonardo numbers have the following recurrence relation for $n \geq 1$ and $r \geq 1$ [10]:

$$Le_n^{(r)} = Le_{n-1}^{(r)} + Le_n^{(r-1)}.$$

Hyper-Leonardo polynomials are defined as:

$$Le_n^{(r)}(x) = \sum_{s=0}^{n} Le_s^{(r-1)}(x)$$

with the initial conditions $Le_n^{(0)}(x) = Le_n(x)$, $Le_0^{(r)}(x) = 1$ and $Le_1^{(r)}(x) = r+1$ [11]. Note that, for $x = 1$, hyper-Leonardo polynomials $Le_n^{(r)}(x)$ give the hyper-Leonardo numbers $Le_n^{(r)}$. Hyper-Leonardo polynomials have the following recurrence relation for $n \geq 1$ and $r \geq 1$ [11]:

$$Le_n^{(r)}(x) = Le_{n-1}^{(r)}(x) + Le_n^{(r-1)}(x). \tag{1}$$

For $n \geq 3$ and $r \geq 1$, there is also the recurrence relation for hyper-Leonardo polynomials [11]:

$$\begin{aligned} Le_n^{(r)}(x) =& 2xLe_{n-1}^{(r)}(x) - Le_{n-3}^{(r)}(x) + \binom{n+r-1}{r-1} \\ &- \binom{n+r-2}{r-1}(2x-1) - \binom{n+r-3}{r-1}(x-2). \end{aligned} \tag{2}$$

If $n \geq 2$ and $r \geq 1$, then there is the summation formula for hyper-Leonardo polynomials [11]:

$$\sum_{s=0}^{r} Le_n^{(s)}(x) = Le_{n+1}^{(r)}(x) + (1-2x)\, Le_n(x) + Le_{n-2}(x). \tag{3}$$

In recent years, hybrid numbers have been the subject of research [12–19]. Özdemir [19] introduced hybrid numbers, as a generalization of complex, hyperbolic and dual numbers, sets by:

$$\mathbb{K} = \{a + bi + c\epsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = hi = \epsilon + i\}.$$

Szynal-Liana and Wloch [12] defined the n-th Fibonacci hybrid number as

$$HF_n = F_n + iF_{n+1} + \epsilon F_{n+2} + hF_{n+3}.$$

Alp and Koçer [18] defined hybrid-Leonardo numbers by using the Leonardo numbers as:

$$HLe_n = Le_n + Le_{n+1}i + Le_{n+2}\epsilon + Le_{n+3}h.$$

The authors also obtained some identities for the hybrid-Leonardo numbers such as [18]:

$$HLe_n = HLe_{n-1} + HLe_{n-2} + (1 + i + \epsilon + h), \quad (n \geq 2),$$
$$HLe_n = 2HF_{n+1} - (1 + i + \epsilon + h), \quad (n \geq 0),$$
$$HLe_{n+1} = 2HLe_n - HLe_{n-2}, \quad (n \geq 2).$$

Kürüz et al. [9] defined Leonardo Pisano hybrinomials, by using the Leonardo Pisano polynomials, as follows:

$$Le_n^{[H]}(x) = Le_n(x) + iLe_{n+1}(x) + \epsilon Le_{n+2}(x) + hLe_{n+3}(x).$$

The Leonardo Pisano hybrinomials have the following recurrence relation [9]:

$$Le_n^{[H]}(x) = 2xLe_{n-1}^{[H]}(x) - Le_{n-3}^{[H]}(x).$$

Motivated by the above papers, we define hybrinomials related to hyper-Leonardo numbers. We also define hybrid hyper-Leonardo numbers by using the newly defined hybrinomials. Then, we investigate some of their properties such as the recurrence relations and summation formulas.

2. Main Results

Definition 1. *Hybrinomials related to hyper-Leonardo numbers are defined as*

$$LeH_n^{(r)}(x) = Le_n^{(r)}(x) + Le_{n+1}^{(r)}(x)\, i + Le_{n+2}^{(r)}(x)\, \epsilon + Le_{n+3}^{(r)}(x)\, h,$$

where $Le_n^{(r)}(x)$ *are the ordinary hyper-Leonardo polynomials.*

The first few hybrinomials related to the hyper-Leonardo numbers are

$$\begin{aligned}
LeH_0^{(1)}(x) &= 1 + 2i + \epsilon(x+4) + h\left(2x^2 + 5x + 3\right), \\
LeH_1^{(1)}(x) &= 2 + i(x+4) + \epsilon\left(2x^2 + 5x + 3\right) + h\left(4x^3 + 10x^2 + 3x + 2\right), \\
LeH_2^{(1)}(x) &= (x+4) + i\left(2x^2 + 5x + 3\right) + \epsilon\left(4x^3 + 10x^2 + 3x + 2\right) \\
&\quad + h\left(8x^4 + 20x^3 + 6x^2\right)
\end{aligned}$$

and

$$\begin{aligned}
LeH_0^{(2)}(x) &= 1 + 3i + \epsilon(x+7) + h\left(2x^2 + 6x + 10\right), \\
LeH_1^{(2)}(x) &= 3 + i(x+7) + \epsilon\left(2x^2 + 6x + 10\right) + h\left(4x^3 + 12x^2 + 9x + 12\right), \\
LeH_2^{(2)}(x) &= (x+7) + i\left(2x^2 + 6x + 10\right) + \epsilon\left(4x^3 + 12x^2 + 9x + 12\right) \\
&\quad + h\left(8x^4 + 24x^3 + 18x^2 + 9x + 12\right).
\end{aligned}$$

For $x = 1$, the hybrinomials defined in Definition 1 give the hybrid numbers in the following definition:

Definition 2. *The n-th hybrid hyper-Leonardo number* $LeH_n^{(r)}$ *is defined as*

$$LeH_n^{(r)} = Le_n^{(r)} + iLe_{n+1}^{(r)} + \epsilon Le_{n+2}^{(r)} + hLe_{n+3}^{(r)},$$

where $Le_n^{(r)}$ *is the n-th hyper-Leonardo numbers.*

This table contains the values of the hybrid hyper-Leonardo numbers.

	$r=0$	$r=1$	$r=2$	$r=3$
n=0	$1+i+3\epsilon+5h$	$1+2i+5\epsilon+10h$	$1+3i+8\epsilon+18h$	$1+4i+12\epsilon+30h$
n=1	$1+3i+5\epsilon+9h$	$2+5i+10\epsilon+19h$	$3+8i+18\epsilon+37h$	$4+12i+30\epsilon+67h$
n=2	$3+5i+9\epsilon+15h$	$5+10i+19\epsilon+34h$	$8+18i+37\epsilon+71h$	$12+30i+67\epsilon+138h$
n=3	$5+9i+15\epsilon+25h$	$10+19i+34\epsilon+59h$	$18+37i+71\epsilon+130h$	$30+67i+138\epsilon+268h$
n=4	$9+15i+25\epsilon+41h$	$19+34i+59\epsilon+100h$	$37+71i+130\epsilon+230h$	$67+1381i+268\epsilon+498h$

Table 1. The first few hybrid hyper-Leonardo numbers $LeH_n^{(r)}$.

Theorem 1. *$LeH_n^{(r)}(x)$ has the recurrence relation for $n \geq 1$ and $r \geq 1$:*

$$LeH_n^{(r)}(x) = LeH_{n-1}^{(r)}(x) + LeH_n^{(r-1)}(x). \tag{4}$$

Proof. By using Definition 1 and the recurrence relation in equation (1), we have

$$\begin{aligned}
&LeH_{n-1}^{(r)}(x) + LeH_n^{(r-1)}(x) \\
&= \left(Le_{n-1}^{(r)}(x) + iLe_n^{(r)}(x) + \epsilon Le_{n+1}^{(r)}(x) + hLe_{n+2}^{(r)}(x)\right) \\
&\quad + \left(Le_n^{(r-1)}(x) + iLe_{n+1}^{(r-1)}(x) + \epsilon Le_{n+2}^{(r-1)}(x) + hLe_{n+3}^{(r-1)}(x)\right) \\
&= Le_{n-1}^{(r)}(x) + Le_n^{(r-1)}(x) + i\left(Le_n^{(r)}(x) + Le_{n+1}^{(r-1)}(x)\right) \\
&\quad + \epsilon\left(Le_{n+1}^{(r)}(x) + Le_{n+2}^{(r-1)}(x)\right) + h\left(Le_{n+2}^{(r)}(x) + Le_{n+1}^{(r-1)}(x)\right) \\
&= Le_n^{(r)}(x) + iLe_{n+1}^{(r)}(x) + \epsilon Le_{n+2}^{(r)}(x) + hLe_{n+3}^{(r)}(x) \\
&= LeH_n^{(r)}(x).
\end{aligned}$$

□

Corollary 1. *The hybrid hyper-Leonardo numbers have the recurrence relation for $n \geq 1$ and $r \geq 1$:*

$$LeH_n^{(r)} = LeH_{n-1}^{(r)} + LeH_n^{(r-1)}.$$

Theorem 2. *$LeH_n^{(r)}(x)$ has the summation formula:*

$$\sum_{s=0}^{n} LeH_s^{(r)}(x) = LeH_n^{(r+1)}(x) - \left(iLe_0^{(r+1)}(x) + \epsilon Le_1^{(r+1)}(x) + hLe_2^{(r+1)}(x)\right).$$

Proof. We use the induction method on n. Since,

$$\begin{aligned}
&LeH_0^{(r+1)}(x)-\left(iLe_0^{(r+1)}(x)+\epsilon Le_1^{(r+1)}(x)+hLe_2^{(r+1)}(x)\right)\\
&=Le_0^{(r+1)}(x)+iLe_1^{(r+1)}(x)+\epsilon Le_2^{(r+1)}(x)+hLe_3^{(r+1)}(x)\\
&\quad-\left(iLe_0^{(r+1)}(x)+\epsilon Le_1^{(r+1)}(x)+hLe_2^{(r+1)}(x)\right)\\
&=Le_0^{(r+1)}(x)+i\left(Le_1^{(r+1)}(x)-Le_0^{(r+1)}(x)\right)+\epsilon\left(Le_2^{(r+1)}(x)-Le_1^{(r+1)}(x)\right)\\
&\quad+h\left(Le_3^{(r+1)}(x)-Le_2^{(r+1)}(x)\right)\\
&=Le_0^{(r)}(x)+iLe_1^{(r)}(x)+\epsilon Le_2^{(r)}(x)+hLe_3^{(r)}(x)\\
&=LeH_0^{(r)}(x),
\end{aligned}$$

the result is true for $n=0$. Assume that the result is true for $n=k$. Then,

$$\sum_{s=0}^{k}LeH_s^{(r)}(x)=LeH_k^{(r+1)}(x)-\left(iLe_0^{(r+1)}(x)+\epsilon Le_1^{(r+1)}(x)+hLe_2^{(r+1)}(x)\right).$$

Now, we must show that the result is true for $n=k+1$. Considering the recurrence relation in equation (4), we get

$$\begin{aligned}
\sum_{s=0}^{k+1}LeH_s^{(r)}(x)&=\sum_{s=0}^{k}LeH_s^{(r)}(x)+LeH_{k+1}^{(r)}(x)\\
&=LeH_k^{(r+1)}(x)-\left(iLe_0^{(r+1)}(x)+\epsilon Le_1^{(r+1)}(x)+hLe_2^{(r+1)}(x)\right)\\
&\quad+LeH_{k+1}^{(r)}(x)\\
&=LeH_{k+1}^{(r+1)}(x)-\left(iLe_0^{(r+1)}(x)+\epsilon Le_1^{(r+1)}(x)+hLe_2^{(r+1)}(x)\right).
\end{aligned}$$

□

Corollary 2. *The hybrid hyper-Leonardo numbers have the summation formula:*

$$\sum_{s=0}^{n}LeH_s^{(r)}=LeH_n^{(r+1)}-\left(iLe_0^{(r+1)}+\epsilon Le_1^{(r+1)}+hLe_2^{(r+1)}\right).$$

Theorem 3. *For $n\geq 3$ and $r\geq 1$, the recurrence relation*

$$\begin{aligned}
LeH_n^{(r)}(x)&=2xLeH_{n-1}^{(r)}(x)-LeH_{n-3}^{(r)}(x)\\
&\quad+\binom{n+r-1}{r-1}-\binom{n+r-2}{r-1}(2x-1)-\binom{n+r-3}{r-1}(x-2)\\
&\quad+i\left[\binom{n+r}{r-1}-\binom{n+r-1}{r-1}(2x-1)-\binom{n+r-2}{r-1}(x-2)\right]\\
&\quad+\epsilon\left[\binom{n+r+1}{r-1}-\binom{n+r}{r-1}(2x-1)-\binom{n+r-1}{r-1}(x-2)\right]\\
&\quad+h\left[\binom{n+r+2}{r-1}-\binom{n+r+1}{r-1}(2x-1)-\binom{n+r}{r-1}(x-2)\right]
\end{aligned}$$

is true.

Proof. Considering Definition 1 and equation (2), the proof is clear. □

Corollary 3. *For $n \geq 3$ and $r \geq 1$, the hybrid hyper-Leonardo numbers have the recurrence relation:*

$$\begin{aligned} LeH_n^{(r)} &= 2LeH_{n-1}^{(r)} - LeH_{n-3}^{(r)} + \binom{n+r-1}{r-1} - \binom{n+r-2}{r-1} + \binom{n+r-3}{r-1} \\ &+i\left[\binom{n+r}{r-1} - \binom{n+r-1}{r-1} + \binom{n+r-2}{r-1}\right] \\ &+\epsilon\left[\binom{n+r+1}{r-1} - \binom{n+r}{r-1} + \binom{n+r-1}{r-1}\right] \\ &+h\left[\binom{n+r+2}{r-1} - \binom{n+r+1}{r-1} + \binom{n+r}{r-1}\right]. \end{aligned}$$

Theorem 4. *If $n \geq 2$ and $r \geq 1$, then the summation formula*

$$\sum_{s=0}^{r} LeH_n^{(s)}(x) = LeH_{n+1}^{(r)}(x) + (1-2x)\, LeH_n(x) + LeH_{n-2}(x)$$

is true.

Proof. By considering equation (3), we get

$$\begin{aligned} \sum_{s=0}^{r} LeH_n^{(s)}(x) &= \sum_{s=0}^{r} \left(Le_n^{(s)}(x) + iLe_{n+1}^{(s)}(x) + \epsilon Le_{n+2}^{(s)}(x) + hLe_{n+3}^{(s)}(x)\right) \\ &= \sum_{s=0}^{r} Le_n^{(s)}(x) + i\sum_{s=0}^{r} Le_{n+1}^{(s)}(x) + \epsilon\sum_{s=0}^{r} Le_{n+2}^{(s)}(x) \\ &\quad +h\sum_{s=0}^{r} Le_{n+3}^{(s)}(x) \\ &= Le_{n+1}^{(r)}(x) + (1-2x)\, Le_n(x) + Le_{n-2}(x) \\ &\quad +i\left(Le_{n+2}^{(r)}(x) + (1-2x)\, Le_{n+1}(x) + Le_{n-1}(x)\right) \\ &\quad +\epsilon\left(Le_{n+3}^{(r)}(x) + (1-2x)\, Le_{n+2}(x) + Le_n(x)\right) \\ &\quad +h\left(Le_{n+4}^{(r)}(x) + (1-2x)\, Le_{n+3}(x) + Le_{n+1}(x)\right) \\ &= LeH_{n+1}^{(r)}(x) + (1-2x)\, LeH_n(x) + LeH_{n-2}(x). \end{aligned}$$

□

Corollary 4. *If $n \geq 1$ and $r \geq 1$, then there is the relation between the hybrid hyper-Leonardo numbers and Fibonacci hybrid numbers:*

$$\sum_{s=0}^{r} LeH_n^{(s)} = LeH_{n+1}^{(r)} - 2HF_n.$$

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