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Pseudo-almost periodic C^0-solution for evolution inclusion with mixed nonlocal plus local initial conditions

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Abstract

This paper is devoted to the study of a class of evolution inclusion in Banach spaces with nonlocal plus local mixed initial conditions. Under some mild assumptions, a unique solvability result to the multivalued evolution problem is obtained via the arguments of fixed point principle and the theory of C^0-semigroup.

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1. Introduction

Let X be a real Banach space, an m-dissipative operator $A : D(A) \subseteq X \rightsquigarrow X$ generate a nonlinear semigroup of contractions $\{S(t) : t > 0\}$ by the exponential formula $S(t) = \lim_{n\to\infty}(I + \frac{t}{n}A)^{-n}$ (see [10]). Given a constant $\tau \geq 0$, and a continuous function $f : \mathbb{R}_+ \times C([-\tau, 0] : \overline{D(A)}) \to X$, in this paper we consider the following evolution inclusion with mixed nonlocal plus local initial conditions

$$\begin{cases} u'(t) \in Au(t) + f(t, u_t), & t \in \mathbb{R}_+, \\ u(t) = p(u)(t) + \psi(t), & t \in [-\tau, 0]. \end{cases} \tag{1.1}$$

Here, the function $p : C([-\tau, +\infty); \overline{D(A)}) \to C([-\tau, 0]; \overline{D(A)})$ shown in the nonlocal condition is non-expansive, and the local initial condition $\psi : [-\tau, 0] \to X$ is continuous, such that $p(u) + \psi \in C([-\tau, 0]; \overline{D(A)})$ for each $u \in C_b([-\tau, +\infty); \overline{D(A)})$. From the control point of view, the nonlocal initial condition p is can be seemed a feedback operator.

It should be pointed out that our problem contains several interesting particular cases. The existence and boundedness of the C^0-solution to problem (1.1) with some suitable

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assumptions have been demonstrated by Vrabie [24] which has proved it by Schaefer Fixed Point Theorem. When ψ vanishes, that is, $\psi(t) \equiv 0$, problem (1.1) reduces to

$$\begin{cases} u'(t) \in Au(t) + f(t, u_t), & t \in \mathbb{R}_+, \\ u(t) = p(u)(t), & t \in [-\tau, 0]. \end{cases} \tag{1.2}$$

Such problem (1.2) has been studied widely. Burlică and Roşu [6] established to problem (1.2) a general existence result for bounded C^0-solutions and proved the uniform asymptotic stability of the C^0-solutions. Further, a sufficient condition for the unique C^0-solution of problem (1.2) to be almost periodic has been given by Vrabie [23], moreover, it is also shown that the trajectory of the C^0-solution for problem (1.2) is relatively compact in $t \in [-\tau, +\infty)$. To make the problem more special, replace $f(t, u_t)$ with $f(t)$ in (1.2), then problem (1.2) turns out to be of the form

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}_+, \\ u(t) = p(u)(t), & t \in [-\tau, 0], \end{cases} \tag{1.3}$$

and this case has already been discussed by Meknani, Zhang and Abdelhamid [18], which states that problem (1.3) has a unique pseudo-almost periodic C^0-solution. These above documents are the starting point for the analysis of our paper.

Some other aspects of the argument on problem (1.1) have been studied by several authors over the past few years. For example, periodic problems, such as Papageorgiou, Rădulescu and Repovš [20], Akagi and Stefanelli [2], and Xue and Cheng [25], anti-periodic problems like Chen Nieto, and O'Regan [8], and Cheng, Cong and Hua [9], mean-value initial condition problems such nondelayed problems. In addition, as their applicability to mathematical modelling in a completely generic condition, it can be referred to Deng [12] and McKibben [16]. Besides, when the operator A is represented as a single-valued operator in problem (1.1), it also has already been studied like Sattayatham, Tangmanee and Wei [21]. Some recent contributions to the C^0-solution of problem (1.1) have been built up in Lizama and Alvarez [3], Lizama and Alvarez-Pardo [15], Meknani [17], Burlică, Roşu and Vrabie [7], Paicu and Vrabie [19], Bilal, Cârjă, Donchev, Javaid and Lazu [5], and Aizicovici and Lee [1]. While, in our paper we prove the uniqueness and the pseudo-almost periodicity of the C^0-solution for problem (1.1), in addition, we got to find out that the trajectory of the unique C^0-solution is relatively compact.

In this paper, we present it in four sections. The second section provides some basic theory on almost periodicity, pseudo-almost periodicity, and the nonlinear evolution equations governed by the m-dissipation operator in Banach spaces. The third section is where the main results of this paper are set out and proved. The final section gives a special example about a pseudo-almost periodic C^0-solution in the form of problem (1.1).

2. Preliminaries

Through the paper, we denote by $C_b([a, +\infty); X)$ the space of all bounded and continuous functions defined on $[a, +\infty)$ with values in X, endowed with the sup-norm $\|\cdot\|_{C_b([a,+\infty);X)}$. Let $C_b([a, +\infty); \overline{D(A)})$ be a closed subset in $C_b([a, +\infty); X)$ i.e., the space of all functions $u \in C_b([a, +\infty); X)$ with $u(t) \in \overline{D(A)}$ for each $t \in [a, +\infty)$. We use the symbol $C([a, b]; X)$ to stand for the space of all continuous functions which are defined on $[a, b]$ with values in X, endowed with the sup-norm $\|\cdot\|_{C([a,b];X)}$. Similarly $C([a, b]; \overline{D(A)})$ is a closed subset in $C([a, b]; X)$ is formed by all functions $u \in C([a, b]; X)$ with $u(t) \in \overline{D(A)}$ for each $t \in [a, b]$. In what follows, let $u \in C([-\tau, +\infty); X)$ and $t \in \mathbb{R}_+$, $u_t \in C([-\tau, 0]; X)$ is defined by $u_t(s) := u(t + s)$ for each $s \in [-\tau, 0]$ (see [14]).

Next, let us recall some necessary materials concerning almost periodic functions, pseudo-almost periodic functions, m-dissipative operators and nonlinear evolution equations in Banach space.

Let us introduce the following notations

$$\begin{cases} S := \{(r_n)_n : \quad r_k \in [0,+\infty), \quad \forall k \in \mathbb{N}\}, \\ S_\infty := \{(r_n)_n : \quad r_k \in [0,+\infty), \ \ \forall k \in \mathbb{N}, \ \ \lim_n r_n = +\infty\}. \end{cases}$$

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Definition 2.1 ([23])**.** For a Banach space $(X, \|\cdot\|_X)$ and $a \in \mathbb{R}$, a function $f : [a,+\infty) \to X$ is said to be almost periodic (a.p.) if the sequence $(t \mapsto f(t + r_n))_n$ has no less than one convergent subsequence in $C_b([a,+\infty); X)$ for each $(r_n)_n \in S$, it means that with a subsequence $(t \mapsto f(t + r_{n_k}))_k$ of $(t \mapsto f(t + r_n))_n$ and there is $\widetilde{f} \in C_b([a,+\infty); X)$ such that

$$\lim_k f(t + r_{n_k}) = \widetilde{f}(t),$$

uniformly for $t \in [a,+\infty)$.

Remark 2.2. Obviously, when an almost periodic function f is continuous, it is uniformly continuous. Furthermore, an uniformly continuous function $f : [a,+\infty) \to X$ is said to be almost periodic if and only if the sequence $(t \mapsto f(t+r_n))_n$ has no less than one convergent subsequence in $C_b([a,+\infty); X)$ for each $(r_n)_n \in S_\infty$.

Definition 2.3 ([23])**.** Let Ω be an open subset of Y, and $(Y, \|\cdot\|_Y)$ be a Banach space. For a function $f \in C([a,+\infty) \times Y; X)$, the family $\{t \mapsto f(t,u) : u \in \Omega\}$ is said to be uniformly almost periodic (u.a.p.) if there are a function $\widetilde{f} : [a,+\infty) \times \Omega \to X$ and a subsequence $(r_{n_k})_k \subseteq (r_n)_n$ such that for each $(r_n)_n \in S_\infty$ with

$$\lim_k f(t + r_{n_k}, u) = \widetilde{f}(t,u)$$

uniformly for $(t,u) \in [a,+\infty) \times \Omega$.

Remark 2.4. If there is an almost periodic function $g \in C([a,+\infty); \mathbb{R})$, and a bounded function $h \in C(Y; X)$ on Ω, such that $f(t,u) = g(t)h(u)$ holds, then the family of functions $\{t \mapsto f(t,u); u \in \Omega\}$ is uniformly almost periodic. Likewise, if there are $\widetilde{g} \in C([a,+\infty); X)$ is almost periodic and $h \in C(Y; X)$ such that $f(t,u) = \widetilde{g}(t) + h(u)$ holds, then the family of functions $\{t \mapsto f(t,u); u \in Y\}$ is uniformly almost periodic.

Also, we consider the following function spaces defined by

$$AP(\mathbb{R}_+, X) = \{f \in C(\mathbb{R}_+, X) : f \text{ is } a.p.\}$$

endowed with the norm $\|f\|_{AP(\mathbb{R}_+,X)} = \sup_{s\in\mathbb{R}_+} \{\|f(s)\|_X\}$ and

$$AP(\mathbb{R}_+ \times X, X) = \{f \in C(\mathbb{R}_+ \times X, X) : f \text{ is } u.a.p.\}$$

endowed with the norm $\|f\|_{AP(\mathbb{R}_+\times X,X)} = \sup_{s\in\mathbb{R}_+, u\in X} \{\|f(s,u)\|_X\}$. It is obvious that they are Banach spaces. Besides, we need the following function spaces

$$PAP_0(\mathbb{R}_+, X) = \{f \in C_b(\mathbb{R}_+, X) : \lim_{r\to\infty} \frac{1}{r} \int_0^r \|f(t)\|_X \mathrm{d}t = 0\}$$

and

$$PAP_0(\mathbb{R}_+ \times X, X) = \{f \in C_b(\mathbb{R}_+ \times X, X) : \text{ for each } u \in X, \lim_{r\to\infty} \frac{1}{r} \int_0^r \|f(t,u)\|_X \mathrm{d}t = 0\}.$$

Definition 2.5 ([26])**.** A function $f \in C_b(\mathbb{R}_+, X)$ is called pseudo-almost periodic if there are $g \in AP(\mathbb{R}_+, X)$ and $h \in PAP_0(\mathbb{R}_+, X)$ such that $f = g + h$ holds.

Definition 2.6 ([26])**.** Let Ω be an open subset of Y, and $(Y, \|\cdot\|_Y)$ be a Banach space. For a function $f \in C([a,+\infty) \times Y; X)$, the family $\{t \mapsto f(t,u) : u \in \Omega\}$ is said to be uniformly pseudo-almost periodic if there are $g \in AP(\mathbb{R}_+ \times X, X)$ and $h \in PAP_0(\mathbb{R}_+ \times X, X)$ such that $f = g + h$ uniformly for $(t,u) \in [a,+\infty) \times \Omega$.

Remark 2.7 ([11]). If there is a constant $L > 0$ such that $f(t,x) \in PAP(\mathbb{R}_+ \times X; X)$ satisfies $\|f(t,x) - f(t,y)\|_X \le L\|x-y\|_X$ for each $t \in \mathbb{R}_+$ and $x, y \in X$, then $f(\cdot, u(\cdot)) \in PAP(\mathbb{R}_+; X)$ for each $u(t) \in PAP(\mathbb{R}_+; X)$.

More information about pseudo-almost periodic functions can be found in Diagana [13]. Next, we provide several theory of dissipative operators and nonlinear evolution equations in Banach spaces, for more details readers can refer Barbu [4].

For every $x, y \in X$, let $[x,y]_t := \frac{1}{t}(\|x+ty\|_X - \|x\|_X)$, and $[x,y]_+ = \lim\limits_{t\downarrow 0}[x,y]_t := \inf\{[x,y]_t; t > 0\}$. An operator $A : D(A) \subseteq X \rightsquigarrow X$ is a dissipative operator if $[x_1 - x_2, y_1 - y_2]_+ \le 0$ holds for each $x_i \in D(A)$ and $y_i \in Ax_i, i = 1, 2$. Furthermore, if for $\lambda > 0$, $R(I - \lambda A) = X$ is satisfied, then A is called m-dissipative, where $R(I - \lambda A)$ denotes the range of $I - \lambda A$.

For each $f \in L^1(a, b; X)$, we say that $u \in C([a, b]; \overline{D(A)})$ is a C^0-solution (or an integral solution) on $[a, b]$ of the evolution equation

$$u'(t) \in Au(t) + f(t), \tag{2.1}$$

if the inequality

$$\|u(t) - x\|_X \le \|u(s) - x\|_X + \int_s^t [u(\tau) - x, f(\tau) + y]_+ \mathrm{d}\tau, \tag{2.2}$$

is satisfied for each $x \in D(A)$, $y \in Ax$ and $a \le s \le t \le b$.

Lemma 2.8 ([4]). *If an m-dissipative operator $A : D(A) \subseteq X \rightsquigarrow X$ satisfies $A + \omega I$ is dissipative with the positive constant ω. It follows that for each $f \in L^1(a, b; X)$ and $\xi \in \overline{D(A)}$, on $[a, b]$, the evolution equation (2.1) has a unique C^0-solution u to satisfy the initial condition $u(a) = \xi$. Then*

$$\|u(t) - v(t)\|_X \le e^{-\omega(t-s)}\|u(s) - v(s)\|_X + \int_s^t e^{-\omega(t-\theta)}\|f(\theta) - g(\theta)\|_X \mathrm{d}\theta \tag{2.3}$$

holds for each $a \le s \le t \le b$, with u and v are two C^0-solutions of (2.1) corresponding to f and $g \in L^1(a, b; X)$ respectively. In particular,

$$\|u(t) - x\|_X \le e^{-\omega(t-s)}\|u(s) - x\|_X + \int_s^t e^{-\omega(t-\theta)}\|f(\theta) + y\|_X \mathrm{d}\theta \tag{2.4}$$

holds for each $a \le s \le t \le b$, if $x \in D(A)$ and $y \in Ax$.

Denote by $u(\cdot, \tau, \xi, f)$ the unique C^0-solution $v : [\tau, b) \to \overline{D(A)}$ of the problem (2.1) with the initial condition $v(\tau) = \xi$, for $\xi \in \overline{D(A)}$, $f \in L^1(a, b; X)$ and $a \le \tau < b$. Denoting $\{S_A(t) : \overline{D(A)} \to \overline{D(A)}, t \ge 0\}$ as the contraction semigroup generated by the Crandall-Liggett's exponential formula for A over $\overline{D(A)}$, for each $t \ge 0$ and $\xi \in \overline{D(A)}$, there is $S_A(t)\xi = u(t, 0, \xi, 0)$. In addition, if operator family $(S_A(t))_{t\ge 0}$ is a compact operator family, then A generates a compact semigroup on $\overline{D(A)}$.

Lemma 2.9 ([22]). *If $A : D(A) \subseteq X \rightsquigarrow X$ is an m-dissipative operator, which generates a compact semigroup, then in $C([c, b]; X)$, where $a < c < b$, for $\mathbb{F}$ be uniformly integrable in $L^1(a, b; X)$ and $B \subseteq \overline{D(A)}$ be bounded the C^0-solutions set $\{u(\cdot, a, \xi, f); f \in \mathbb{F}, \xi \in B\}$ is relatively compact. Moreover, the C^0-solutions set is relatively compact in $C([a, b]; X)$ when B is relatively compact in X.*

3. Main results

For problem (1.1), the following assumptions were made about the data.

(H_A) The m-dissipative operator $A : D(A) \subseteq X \rightsquigarrow X$ meets the following requirements:
- (A_1) $A + \omega I$ is dissipative for a positive number ω, and $0 \in D(A)$, $0 \in A0$,
- (A_2) A generates a compact semigroup over $\overline{D(A)}$.

(H_f) The continuous function $f : \mathbb{R}_+ \times C([-\tau, 0]; \overline{D(A)}) \to X$ satisfies:

(f_1) for each $u \in C([-\tau, 0]; \overline{D(A)})$ and $t \in \mathbb{R}_+$,

$$\|f(t,u)\|_X \le m + l\|u\|_{C([-\tau,0];X)},$$

where $m,\, l > 0$,

(f_2) for each $u, v \in C([-\tau, 0]; \overline{D(A)})$ and $t \in \mathbb{R}_+$,

$$\|f(t,u) - f(t,v)\|_X \le l\|u - v\|_{C([-\tau,0];X)},$$

where l is from the last assumption,

(f_3) for $t \in [0, +\infty)$, $\{t \mapsto f(t, v) : v \in \mathcal{C}\}$ is equi-uniformly continuous, where $\mathcal{C}$ is an arbitrary bounded subset of $C([-\tau, 0]; \overline{D(A)})$,

(f_4) for $t \in [0, +\infty)$, $\{t \mapsto f(t, v) : v \in \mathcal{C}\}$ is uniformly pseudo-almost periodic, where $\mathcal{C}$ is an arbitrary bounded subset of $C([-\tau, 0]; \overline{D(A)})$.

(H_c) There are positive constants l, $\tau >$, and ω such that the non-resonance condition is satisfied, that is:

(c_1) $l < \omega$,

(c_2) $le^{\omega\tau} < \omega$.

(H_p) $p : C_b([-\tau, +\infty), \overline{D(A)}) \to C([-\tau, 0]; \overline{D(A)})$ is a nonlocal condition satisfying:

(p_1)

$$\|p(u)\|_{C([-\tau,0];X)} \le \|u\|_{C_b([0,+\infty);X)},$$

for each $u \in C_b([-\tau, +\infty), \overline{D(A)})$,

(p_2) there is $\alpha > 0$ such that

$$\|p(u) - p(v)\|_{C[-\tau,0];X)} \le \|u - v\|_{C_b([\alpha,+\infty);X)},$$

for each $u, v \in C_b([-\tau, +\infty), \overline{D(A)})$,

(p_3) in $C([\tau, 0]; X)$, $P(\widetilde{E})$ is relatively compact, where $\widetilde{E} \subseteq C_b([-\tau, +\infty), \overline{D(A)})$ is a bounded set, $\widetilde{E}$ is relatively compact in $\widetilde{C}_b([\sigma, +\infty); X)$ for each $\sigma > 0$,

(p_4) p is continuous from its domain endowed with the locally convex space topology of $\widetilde{C}_b([-t, +\infty); X)$ to $C([-\tau, 0]; X)$.

$(H_{\psi,p})$ for each $u \in C_b([-\tau, +\infty); \overline{D(A)})$, $p(u)+\psi \in C([-\tau, 0]; \overline{D(A)})$, where $\psi \in C_b([-\tau, 0]; X)$.

Lemma 3.1 ([24]). *Let (H_A), (f_1) and (f_3) in (H_f), (c_1) in (H_c), (H_p) and $(H_{\psi,p})$ be satisfied. Then* (1.1) *has at least one C^0-solution $u \in C_b([-\tau, +\infty); \overline{D(A)})$ which satisfies*

$$\|u\|_{C_b([-\tau,+\infty);X)} \le [\frac{\omega}{\omega - l}(\frac{1}{e^{\omega a} - 1} + \frac{l}{\omega}) + 1]\|\psi\|_{C([-\tau,0];X)} + \frac{m}{\omega - l}. \tag{3.1}$$

Theorem 3.2. *Under the hypothesis of Lemma* 3.1*, the C^0-solution of problem* (1.1) *is unique if (f_2) in (H_f), and (c_2) in (H_c) are satisfied. Moreover, u is globally asymptotically stable.*

Proof. Proving its uniqueness by contradiction, if $u, v \in C([-\tau, +\infty); X)$ are two distinguish C^0-solutions of problem (1.1), then applying (2.3), (f_1), (f_2), and (p_2) for each $t \in [0, +\infty)$ we have

$$\|u(t) - v(t)\|_X \le e^{-\omega t}\|u(0) - v(0)\|_X + \int_0^t e^{-\omega(t-s)}\|f(s, u_s) - f(s, v_s)\|_X \mathrm{d}s$$

$$\le e^{-\omega t}\|u - v\|_{C_b([\alpha,+\infty);X)} + \frac{1 - e^{-\omega t}}{\omega} l\|u - v\|_{C([\alpha,+\infty);X)}.$$

For any $t \ge \alpha$, it follows from the last inequality above that

$$(1 - \frac{l}{\omega})\|u(t) - v(t)\|_X(1 - e^{-\omega t}) \le 0,$$

since (c_1), $(1-\frac{l}{\omega})(1-e^{-\omega t})>0$ is easily obtained, it follows that $\|u(t)-v(t)\|_X=0$. Further, we can infer $\|u(t)-v(t)\|_X=0$ for each $t\in[0,+\infty)$. Next we consider the situation when $t\in[-\tau,0]$. By (p_2), we get

$$\|u(t)-v(t)\|_X\le\|u-v\|_{C([\alpha,+\infty);X)}=0.$$

In conclusion, we have proved that the C^0-solution of problem (1.1) is unique.

Taking the unique C^0-solution of problem (1.1) as $u\in C_b([-\tau,+\infty);\overline{D(A)})$. For an arbitrary fixed $\varphi\in C([-\tau,0];\overline{D(A)})$, denote $v\in C_b([-\tau,+\infty);\overline{D(A)})$ as the unique C^0-solution to problem

$$\begin{cases} v'(t)=Av(t)+f(t,v_t), & t\in\mathbb{R}_+,\\ v(t)=\varphi(t), & t\in[-\tau,0]. \end{cases} \tag{3.2}$$

Using (f_1) and (2.3), we can get

$$\|u(t)-v(t)\|_X\le e^{-\omega t}\|u(0)-v(0)\|_X+\int_0^t le^{-\omega(t-s)}\|u_s-v_s\|_{C([-\tau,0];X)}\mathrm{d}s$$

for every $t\in[0,+\infty)$. Notes

$$e^{\omega s}\|u_s-v_s\|_{C([-\tau,0];X)}\le e^{\omega\tau}\sup_{\theta\in[-\tau,0]}\{e^{\omega(s+\theta)}\|u(s+\theta)-v(s+\theta)\|_X\}$$

for all $s\ge 0$. Then we can deduce

$$e^{\omega t}\|u(t)-v(t)\|_X\le\|u(0)-v(0)\|_X+\int_0^t le^{-\omega\tau}\sup_{\theta\in[-\tau,0]}\{e^{\omega(s+\theta)}\|u(s+\theta)-v(s+\theta)\|_X\}\mathrm{d}s$$

for each $t\in[0,+\infty)$. By Lemma 4.3 in [6] it has

$$e^{\omega t}\|u(t)-v(t)\|_X\le\|u(0)-v(0)\|_X+\|u-v\|_{C([-\tau,0];X)}+$$
$$le^{\omega\tau}(\|u(0)-v(0)\|_X+\|u-v\|_{C([-\tau,0];X)})\int_0^t e^{le^{\omega\tau}(t-s)}\mathrm{d}s$$

for each $t\in[0,+\infty)$. Then we shall be able to infer

$$\|u(t)-v(t)\|_X\le e^{-(\tau-b)t}(\|u(0)-v(0)\|_X+\|u-v\|_{C([-\tau,0];X)})$$

for each $t\in[0,+\infty)$. As $le^{\omega\tau}<\omega$, which suggests

$$\lim_{t\to+\infty}\|u(t)-v(t)\|_X=0.$$

This implies the second assertion of this theorem is valid. □

Remark 3.3. It should be pointed out that Banach's fixed-point theorem also can be used to prove the existence and uniqueness of the C^0-solution to problem (1.1) (see [23]).

The next theorem sates that the C^0-solution u of problem (1.1) has closed compact orbit and is pseudo-almost periodic.

Theorem 3.4. *Suppose $(I-A)^{-1}$ is compact. Then under assumptions (H_A), (H_f), $(H_p)$and (H_c), the unique C^0-solution has closed compact orbit to problem (1.1) for which it is pseudo-almost periodic.*

To prove the theorem, the following theorem is required.

Lemma 3.5. *Assume that (H_A), (f_1), (f_2) and (f_3) in (H_f), (H_c), $(H_p)$and $(H_{\psi,p})$ hold. Then for problem* (3.2) *the unique C^0-solution, v, over $[-\tau,+\infty)$ is uniformly continuous.*

Proof. It is sufficient to prove the uniform continuity of u since it is continuous.

$$\|v(t+h)-v(t)\|_X\le e^{-\omega t}\|v(h)-v(0)\|_X+\int_0^t e^{-\omega(t-s)}\|f(s+h,v_{s+h})-f(s,v_s)\|_X\mathrm{d}s$$

hold for every $h > 0$ and each $t \in [0, +\infty)$. Suppose that $\gamma : [0, +\infty) \to [0, +\infty)$ is the modulus for the family

$$F(\omega, l) := \{t \mapsto f(t, v) : v \in B(\omega, l)\},$$

with equi-uniform continuity, where

$$B(\omega, l) := \{v \in C([-\tau, 0]; \overline{D(A)}) : \|v\|_{C([-\tau,0];X)} \leq [\frac{\omega}{\omega - l}(\frac{1}{e^{\omega a} - 1} + \frac{l}{\omega}) + 1]\|\psi\|_{C([-\tau,0];X)} + \frac{m}{\omega - l}\}.$$

Namely, for each $h > 0$ we have

$$\gamma(h) := sup\{\|f(t + \theta, v) - f(t, v)\|_X : \theta \in (0, h], t \in [0, +\infty), v \in B(\omega, l)\}.$$

Then $\lim_{h \downarrow 0} \gamma(h) = 0$ since, on $[0, +\infty)$, $F(\omega, l)$ is equi-uniformly continuous beneath assumption (f_3) in (H_f). Therefore, for every $h > 0$ and each $t \in [0, +\infty)$, we have

$$\begin{aligned}\|v(t + h) - v(t)\|_X &\leq e^{-\omega t}\|v(h) - v(0)\|_X + \int_0^t e^{-\omega(t-s)}[l\|v_{s+t} - v_s\|_{C([-\tau,0];X)} + \gamma(h)]\mathrm{d}s \\ &\leq e^{-\omega t}\|v(h) - v(0)\|_X + \int_0^t e^{-\omega(t-s)} l\|v_{s+t} - v_s\|_{C([-\tau,0];X)}\mathrm{d}s + \frac{\gamma(h)}{\omega}.\end{aligned}$$

Further, for every fixed $h > 0$ and each $t \in [0, +\infty)$,

$$\begin{aligned}\|v(t + h) - v(t)\|_X &\leq \frac{\gamma(h)}{\omega - le^{\omega\tau}} + e^{-(\omega - le^{\omega\tau})t}[\|v(h) - v(0)\|_X + \|v_h - v_0\|_{C([-\tau,0];X)}] \\ &\leq \frac{\gamma(h)}{\omega - le^{\omega\tau}} + \|v_h - v_0\|_{C([-\tau,0];X)}\end{aligned}$$

can be obtained from Lemma 4.2 in [23]. Finally, it follows that u is uniformly continuous from right on $[0, +\infty)$ by $\lim_{h\downarrow 0} \gamma(h) = \lim_{h\downarrow 0} \|v_h - v_0\|_{C([-\tau,0];X)} = 0$. □

Now we can go ahead and prove the last theorem.

Proof of Theorem 3.4. For each $t \in [-\tau, 0]$, and a unique C^0-solution $u \in C_b([-\tau, +\infty); \overline{D(A)})$ to problem (1.1), write $\varphi = p(u) + \psi$. By Lemma 3.5 we can claim that u is uniformly continuous from the right on $[0, +\infty)$. Denote $J_\lambda = (I - \lambda A)^{-1}$, where $\lambda > 0$, the contraction semigroup generated by the operator A on $\overline{D(A)}$ is denoted as $\{S(t) : t \geq 0\}$. Then

$$\|J_\lambda u(t) - u(t)\|_X \leq \frac{4}{\lambda}\int_0^\lambda [\|u(t + s) - u(t)\|_X + \|S(s)u(t) - u(t + s)\|_X]\mathrm{d}s \tag{3.3}$$

is obtained based on Lemma 2.2 in [23]. On the other hand, for each $t, s \in [0, +\infty)$,through Lemma 2.1 in [23], there is

$$\|S(s)u(t) - u(t + s)\|_X \leq \int_t^{t+s} \|f(\theta, u_\theta)\|_X \mathrm{d}\theta \tag{3.4}$$

established. Let $\delta : [0, +\infty) \to [0, +\infty)$ be defined as

$$\delta(\lambda) := \sup\{\|u(t + s) - u(t)\|_X : s \in [0, \lambda]; t \in [-\tau, +\infty)\}$$

which is the modulus of uniform continuity from the right of u on $[-\tau, +\infty)$, further, $\lim_{\lambda\downarrow 0} \delta(\lambda) = 0$.

Then through (3.3), (3.4) and (f_1), we have

$$\begin{aligned}\|J_\lambda u(t) - u(t)\|_X &\leq \frac{4}{\lambda}\int_0^\lambda \{[\frac{lm}{\omega - l} + \|\psi\|_{C([-\tau,0];X)}((\frac{1}{e^{\omega a} - 1} + \frac{l}{\omega})\frac{l\omega}{\omega - l} + l) + m]s + \delta(\lambda)\}\mathrm{d}s \\ &\leq 2\lambda[\frac{lm}{\omega - l} + \|\psi\|_{C([-\tau,0];X)}((\frac{1}{e^{\omega a} - 1} + \frac{l}{\omega})\frac{l\omega}{\omega - l} + l) + m] + 4\delta(\lambda)\end{aligned}$$

for each $t \in [0, +\infty)$ and $\lambda > 0$. Hence

$$\lim_{\lambda\downarrow 0} \|J_\lambda u(t) - u(t)\|_X = 0$$

uniformly for $t \in [0,+\infty)$. Due to the boundedness of u and the compactness of J_λ is compact for each $\lambda > 0$, the positive trajectory of u can be deduced to be relatively compact. The proof of the first assertion of the Theorem 3.2 has been completed, and then we show that u is pseudo-almost periodic. This requires proving that there exists $v \in AP(\mathbb{R}_+;X)$ and $w \in PAP_0(\mathbb{R}_+;X)$ to enable u to be factored into $u = v + w$, and v is C^0-solution to

$$\begin{cases} v'(t) \in Av(t) + g(t, v_t), & t \in \mathbb{R}_+, \\ v(t) = p(v)(t) + \psi(t), & t \in [-\tau, 0], \end{cases} \tag{3.5}$$

as well as w is C^0-solution to

$$\begin{cases} w'(t) \in Aw(t) + h(t, w_t), & t \in \mathbb{R}_+, \\ w(t) = [p(v+w) - p(v)](t), & t \in [-\tau, 0], \end{cases} \tag{3.6}$$

where $g + h = f$, with $g(\cdot, v(\cdot + s)) \in AP(\mathbb{R}_+;X)$, $h(\cdot, w(\cdot + s)) \in PAP_0(\mathbb{R}_+;X)$ for each $s \in [-\tau, 0]$. Via Remark 2.2, it suffices to show for every sequence $(r_n)_n \in S_\infty$ that there is

$$\lim_n v(t + r_n) = \widetilde{v}(t)$$

uniformly for $t \in [0,+\infty)$ one can show that show that v is almost periodic. By the boundedness of u, which is a consequence of Theorem 3.1, we can obtain boundedness of $\mathcal{C}= \{v_t : t \in [0,+\infty)\}$ on $C([-\tau,0];X)$. And there is a function $\widetilde{g} : [0,+\infty) \times \mathcal{C} \to X$ such that, for each $v \in \mathcal{C}$,

$$\lim_n g(t + r_n, v) = \widetilde{g}(t, v)$$

uniformly for $t \in [0,+\infty)$ on at least one subsequence. In particular,

$$\lim_n g(s + r_n, v_s) = \widetilde{g}(s, v_s) \tag{3.7}$$

uniformly for $s \in [0,+\infty)$. In addition, by the uniform continuity of v, from $[-\tau,+\infty)$ to X, $\{s \mapsto v(s + r_n) : n \in \mathbb{N}\}$ can be shown to be equi-continuous. Moreover, A relatively compact trajectory for u leads us to infer a relatively compact cross section in X for the above family. We infer from Arzelà-Ascoli's Theorem [22] that $\{s \mapsto v(s + r_n) : n \in \mathbb{N}\}$ is relatively compact in $C_b([-\tau,+\infty);X)$ is given uniform convergence over a compact topology. Thus, if necessary, another subsequence can be chosen so that there is $\widetilde{v} \in C_b([-\tau,+\infty);X)$ making

$$\lim_n v(t + r_n) = \widetilde{v}(t) \tag{3.8}$$

holds uniformly for $t \in [-\tau, k]$, $k \in \mathbb{N}$. For the completion of this proof it is necessary to state, on the sub-subsequence, $(v(t + r_n))_n$ convergence to under the norm of $C_b([0,+\infty);X)$. For this reason, without losing generality, the sequence $(v(\cdot + r_n))_n$ can be assumed to satisfy the above properties. According the continuity of g and (3.8), it is clear that

$$\lim_n g(s, v_{r_n+s}) = g(s, \widetilde{v}_s)$$

uniformly for $0 \le s \le k$, $k \in \mathbb{N}$. We conclude from the boundedness of $(s \mapsto g(s, v_{r_n+s}))_n$ along with (3.7) and (3.8), that $\widetilde{v}$ is the unique C^0-solution of problem

$$\begin{cases} \widetilde{v}'(t) \in A\widetilde{v}(t) + g(t, \widetilde{v}_t), & t \in \mathbb{R}_+, \\ \widetilde{v}(t) = \lim_n v(t + r_n), & t \in [-\tau, 0]. \end{cases}$$

At the moment, it can be noted that

$$\begin{aligned} \|v(t + r_n) - \widetilde{v}(t)\|_X &\le e^{-\omega t}\|v(r_n) - \widetilde{v}(0)\|_X + \int_0^t e^{-\omega(t-s)}\|g(s + r_n, v_{s+r_n}) - g(s, \widetilde{v}_s)\|_X \mathrm{d}s \\ &\le e^{-\omega t}\|v(r_n) - \widetilde{v}(0)\|_X + \int_0^t e^{-\omega(t-s)}\|g(s + r_n, v_{s+r_n}) - g(s + r_n, \widetilde{v}_s)\|_X \mathrm{d}s \\ &\quad + \int_0^t e^{-\omega(t-s)}\|g(s + r_n, \widetilde{v}_s) - g(s, \widetilde{v}_s)\|_X \mathrm{d}s \end{aligned}$$

for each $t \in [0,+\infty)$ and $n \in \mathbb{N}$. Then apply similar reasoning as in [23], for each $n \in \mathbb{N}$, denote $a_n := \|v(r_n) - v(0)\|_X$ satisfies $\lim_n a_n = 0$, and $b_n := sup\{\|g(t+r_n, v) - \widetilde{g}(s, v)\|_X : s \in [0,+\infty), v \in \mathcal{C}\}$ satisfies $\lim_n b_n = 0$. Form the last inequality we get

$$\|v(t+r_n) - \widetilde{v}(t)\|_X \le e^{-\omega t} a_n + \frac{1-e^{-\omega t}}{\omega} b_n + \int_0^t l e^{-\omega(t-s)} \|v_{s+r_n} - \widetilde{v}_s\|_{C([-\tau,0];X)} \mathrm{d}s$$

$$\le e^{-\omega t} a_n + \frac{b_n}{\omega} + \int_0^t l e^{-\omega(t-s)} \|v_{s+r_n} - \widetilde{v}_s\|_{C([-\tau,0];X)} \mathrm{d}s$$

for each $t \in [0,+\infty)$ and $n \in \mathbb{N}$. Applying the Lemma 4.2 in [23], we can get

$$\|v(t+r_n) - \widetilde{v}(t)\|_X \le (a_n + \|v_{r_n} - v_0\|_{C([-\tau,0];X)}) e^{-(\omega-b)t} + \frac{b_n}{\omega - b}$$

$$\le a_n + \frac{b_n}{\omega - b} + \|v_{r_n} - v_0\|_{C([-\tau,0];X)}$$

for each $t \in [0,+\infty)$ and $n \in \mathbb{N}$, where $b = le^{\omega\tau}$. Further, the almost periodicity of v can be obtained from

$$\lim_n \|v(t+r_n) - \widetilde{v}(t)\|_X = 0$$

uniformly for $t \in [0,+\infty)$, which is a consequence of $\lim_n a_n = \lim_n b_n = \lim_n \|v_{r_n} - v_0\|_{C([-\tau,0];X)} = 0$. Next, we will demonstrate that $w(t) \in PAP_0(\mathbb{R}_+; X)$. In reality, according (2.3), we can obtain that

$$\|w(t)\|_X \le \|u(t) - v(t)\|_X \le e^{-\omega(t-s)} \|u(s) - v(s)\|_X + \int_s^t e^{-\omega(t-\theta)} \|f(\theta, u_\theta) - g(\theta, v_\theta)\|_X \mathrm{d}\theta,$$

consequently,

$$\frac{1}{r} \int_0^r \|w(t)\|_X \mathrm{d}t \le I + J,$$

where

$$I = \frac{1}{r} \int_0^r e^{-\omega(t-s)} \|u(s) - v(s)\|_X \mathrm{d}t,$$

$$J = \frac{1}{r} \int_0^r \int_s^t e^{-\omega(t-\theta)} \|h(\theta, w_\theta)\|_X \mathrm{d}\theta \mathrm{d}t.$$

Hence we have to prove that I and J converge to 0 as $r \to +\infty$. In fact, as $r \to +\infty$ there are

$$I = \frac{1}{r} \int_0^r e^{-\omega(t-s)} \|u(s) - v(s)\|_X \mathrm{d}t$$

$$\le \frac{2}{r} [\frac{m}{\omega - l} + \|\psi\|_{C([-\tau,0];X)} ((\frac{1}{e^{\omega a} - 1} + \frac{l}{\omega}) \frac{\omega}{\omega - l} + 1)] \int_0^r e^{-\omega(t-s)} \mathrm{d}t$$

$$\le \frac{2}{r\omega} [\frac{m}{\omega - l} + \|\psi\|_{C([-\tau,0];X)} ((\frac{1}{e^{\omega a} - 1} + \frac{l}{\omega}) \frac{\omega}{\omega - l} + 1)] e^{\omega s} \to 0,$$

and

$$J = \frac{1}{r} \int_0^r \int_s^t e^{-\omega(t-\theta)} \|h(\theta, w_\theta)\|_X \mathrm{d}\theta \mathrm{d}t$$

$$\le \frac{1}{r} \int_0^r \int_0^t e^{-\omega(t-\theta)} \|h(\theta, w_\theta)\|_X \mathrm{d}\theta \mathrm{d}t$$

$$\le \frac{1}{r} \int_0^r \|h(t, w_t)\|_X \mathrm{d}t \int_0^{+\infty} e^{-\omega\delta} \mathrm{d}\delta$$

$$\le \frac{1}{r\omega} \int_0^r \|h(t, w_t)\|_X \mathrm{d}t \to 0,$$

Hence, we deduce that $\frac{1}{r}\int_0^r \|w(t)\|_X \mathrm{d}t \to 0$ as $r \to +\infty$, that is to say $w(t) \in PAP_0(\mathbb{R}_+; X)$. To sum up, u is a pseudo-almost periodic C^0-solution for (1.1). □

4. Application

To illustrate our main results, an example shall be presented now. Consider the existence and uniqueness of pseudo-almost periodic C^0-solution for the following problem:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = -a\frac{\partial u}{\partial x}(t,x) - \omega u(t,x) + f(t,u_t)(x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(t,x) = u(t,x+\pi), \quad t \in \mathbb{R}_+, \\ u(t,x) = p(u)(t)(x) + \psi(t,x), \quad t \in [-\tau,0], \quad x \in (0,\pi). \end{cases} \tag{4.1}$$

C_π is the space consisting of all π-periodic functions $u \in C(\mathbb{R};\mathbb{R})$, for each $u \in C_\pi$ endowed with the norm $\|\cdot\|_{C_\pi} := \|u\|_{([0,\pi];\mathbb{R})}$. For $D(A) = \{u \in C_\pi : u' \in C_\pi\}$, considering $Au = -a\frac{\partial u}{\partial x}(t,x) - \omega u(t,x)$, where $a \in \mathbb{R}\setminus\{0\}$ and $\omega > 0$. And obviously the operator A satisfies hypothesis (H_A), reference [23]. Let $f : [0,+\infty) \times C([-\tau,0];C_\pi) \to C_\pi$ be continuous, such as $f(t,u_t) = u_t \sin t + u_t \sin(\sqrt{2}t) + \frac{u_t}{1+t^2}$ with $u_t(s) \in C([-\tau,0];C_\pi)$ for $t \in [0,+\infty)$. It is clear that the function $f(t,u_t)$ satisfies hypothesis (H_f). Furthermore, define $p : ([-\tau,+\infty);C_\pi) \to C([-\tau,0];C_\pi)$ as $p(u)(t) = \int_\tau^{+\infty} k(\theta)u(t+\theta)\mathrm{d}\theta$ with $\int_\tau^{+\infty} |k(\theta)|\mathrm{d}\theta = 1$ for $k \in L^1([-\tau,+\infty);\mathbb{R})$. Obviously, this satisfies the hypothesis (H_p).

Theorem 4.1. *Under the following assumptions, problem* (4.1) *has a unique globally asymptotically stable pseudo-almost periodic C^0-solution, which has compact orbit.*

(*a*) *For each $u, v \in C([-\tau,0];C_\pi)$ and $t, s \in [0,+\infty)$,*

$$\|f(t,u) - f(s,v)\|_{C_\pi} \le l[|t-s| + \|u-v\|_{C([-\tau,0];C_\pi)}],$$

$$\|f(t,0)\|_{C_\pi} \le m$$

where m, $l > 0$;

(*b*) *$le^{\omega\tau} < \omega$;*

(*c*) *for $t \in [0,+\infty)$, $\{t \mapsto f(t,u) : u \in \mathcal{C}\}$ is equi-uniformly continuous, where $\mathcal{C}$ is an arbitrary bounded of subset $C([-\tau,0];C_\pi)$;*

(*d*) *for $t \in [0,+\infty)$, $\{t \mapsto f(t,u) : u \in \mathcal{C}\}$ is uniformly pseudo-almost periodic, where $\mathcal{C}$ is an arbitrary bounded of subset $C([-\tau,0];C_\pi)$;*

(*e*) *there is $\alpha > 0$, for each $u, v \in C_b([-\tau,+\infty);C_\pi)$, it is valid to have*

$$\|p(u) - p(v)\|_{C([-\tau,0];C_\pi)} \le \|u-v\|_{C_b([\alpha,+\infty);C_\pi)},$$

in particularly,

$$\|p(u)\|_{C([-\tau,0];C_\pi)} \le \|u\|_{C_b([0,+\infty);C_\pi)}.$$

.

Proof. Define $B : D(B) \subseteq C_\pi \to C_\pi$ to be

$$\begin{cases} D(B) = \{u \in C_\pi : u' \in C_\pi\}, \\ Bu = -au', \quad u \in D(B). \end{cases}$$

To prove the example it is sufficient to write problem (4.1) in the form of (1.1) with $A = B - \omega I$ and f and p as above. The next step shows that $(I-A)^{-1}$ is compact.

For each $\xi \in C_\pi$, $x \in \mathbb{R}$ and $t \ge 0$, it may be inferred from [22] that a C^0-group of isometries $\{T(t) : t \in \mathbb{R}\}$ is generated by B, where

$$[T(t)\xi](x) = \xi(x - at).$$

Hence, $\{S(t) : t \ge 0\}$ is a contracted C^0-semigroup generated by A, where

$$S(t) = e^{-\omega t}T(t).$$

From Arzelà-Ascoli's Theorem for infinite-dimensional versions [22], it follow that, in C_π, the closures of

$$\{u \in D(A) : \|u\|_{C_\pi} + \|Au\|_{C_\pi} \le k\}$$

is compact, where $k \in \mathbb{R}_+$. Then, $(I - A)^{-1}$ is compact, which can be inferred from [22]. Overall, all the assumptions of Theorem 3.2 and Theorem 3.4 are held, from which we conclude. □

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References

[1] S. Aizicovici and H. Lee, *Nonlinear nonlocal cauchy problems in Banach spaces*, Applied Mathematics Letters, **18** (4), 401–407, 2005.

[2] G. Akagi and U. Stefanelli, *Periodic solutions for doubly nonlinear evolution equations*, Journal of Differential Equations, **251** (7), 1790–1812, 2011.

[3] E. Alvarez and C. Lizama, *Weighted pseudo almost periodic solutions to a class of semilinear integro-differential equations in Banach spaces*, Advances in Difference Equations, **2015** (1), 31, 2015.

[4] V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces*, Springer Science & Business Media2010.

[5] S. Bilal, O. Cârjă, T. Donchev, N. Javaid and A. I. Lazu, *Nonlocal evolution inclusions under weak conditions*, Advances in Difference Equations, **2018** (1), 1–14, 2018.

[6] M.-D. Burlică and D. Roşu, *A class of nonlinear delay evolution equations with nonlocal initial conditions*, Proceedings of the American Mathematical Society, **142** (7), 2445–2458, 2014.

[7] M.-D. Burlică, D. Roşu and I. I. Vrabie, *Abstract reaction–diffusion systems with nonlocal initial conditions*, Nonlinear Analysis: Theory, Methods & Applications, **94**, 107–119, 2014.

[8] Y. Chen, J. J. Nieto and D. O'Regan, *Anti-periodic solutions for evolution equations associated with maximal monotone mappings*, Applied Mathematics Letters, **24** (3), 302–307, 2011.

[9] Y. Cheng, F. Cong and H. Hua, *Anti-periodic solutions for nonlinear evolution equations*, Advances in Difference Equations, **2012** (1), 1–15, 2012.

[10] M. G. Crandall and T. M. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, American Journal of Mathematics, **93** (2), 265–298, 1971.

[11] C. Cuevas and M. Pinto, *Existence and uniqueness of pseudo almost periodic solutions of semilinear cauchy problems with nondense domain*, Nonlinear Analysis: Theory, Methods & Applications, **45** (1), 73–73, 2001.

[12] K. Deng, *Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions*, Journal of Mathematical Analysis and Applications, **179** (2), 630–637, 1993.

[13] T. Diagana, *Pseudo almost periodic functions in Banach spaces*, New York (NY): Nova Science Publishers Inc2007.

[14] J. K. Hale, *Functional differential equations*, Springer1971.

[15] C. Lizama and E. Alvarez-Pardo, *Pseudo asymptotic solutions of fractional order semilinear equations*, Banach Journal of Mathematical Analysis, **7** (2), 42–52, 2013.

[16] M. McKibben, *Discovering evolution equations with applications: Volume 1-Deterministic equations*, Chapman and Hall/CRC2010.

[17] B. Meknani, *The existence and uniqueness of integral solutions to some nonlinear reaction diffusion system with nonlocal retarded initial conditions*, Journal of Taibah University for Science, **14** (1), 569–578, 2020.

[18] B. Meknani, J. Zhang and T. Abdelhamid, *Pseudo-almost periodic C^0 solutions to the evolution equations with nonlocal initial conditions*, Applicable Analysis, (3), 1–11, 2021.

[19] A. Paicu and I. I. Vrabie, *A class of nonlinear evolution equations subjected to nonlocal initial conditions*, Nonlinear Analysis: Theory, Methods & Applications, **72** (11), 4091–4100, 2010.

[20] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, *Periodic solutions for a class of evolution inclusions*, Computers and Mathematics with Applications, **75** (8), 3047–3065, 2018, doi:https://doi.org/10.1016/j.camwa.2018.01.031.

[21] P. Sattayatham, S. Tangmanee and W. Wei, *On periodic solutions of nonlinear evolution equations in Banach spaces*, Journal of Mathematical Analysis and Applications, **276** (1), 98–108, 2002.

[22] I. I. Vrabie, *Compactness methods for nonlinear evolutions*, vol. 75, CRC Press1995.

[23] I. I. Vrabie, *Almost periodic solutions for nonlinear delay evolutions with nonlocal initial conditions*, Journal of Evolution Equations, **13** (3), 693–714, 2013.

[24] I. I. Vrabie, *Delay evolution equations with mixed nonlocal plus local initial conditions*, Communications in Contemporary Mathematics, **17** (02), 1350035, 2015.

[25] X. Xue and Y. Cheng, *Existence of periodic solutions of nonlinear evolution inclusions in Banach spaces*, Nonlinear Analysis: Real World Applications, **11** (1), 459–471, 2010.

[26] Z.-H. Zhao, Y.-K. Chang and W.-S. Li, *Asymptotically almost periodic, almost periodic and pseudo-almost periodic mild solutions for neutral differential equations*, Nonlinear Analysis: Real World Applications, **11** (4), 3037–3044, 2010.