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Research Article

# Cofinitely Semisimple (ss-) Lifting Modules

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**Abstract:** A $P-$module $N$ is named cofinitely semisimple lifting or briefly cofinitely $ss-$ lifting, if for each cofinite submodule $S$ of $N$, $N$ has a decomposition $N = U' \oplus V$ where $U' \subseteq S$ and $S \cap V \subseteq Soc_s(V)$. In this study, identical conditions to this definition are given. In addition, the basic features of this concept defined in this article are examined.

## Dual Sonlu Yarı Basit Yükseltilebilir Modüller

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**Anahtar Kelimeler**
Dual sonlu ss-yükseltilebilir modül,
ss-tümlenmiş modül,
Yarı basit modül

**Öz:** $N, P-$modülünün dual sonlu her $S$ alt modülü $N = U' \oplus V, U' \subseteq S$ ve $S \cap V \subseteq Soc_s(V)$ olacak şekilde bir ayrışıma sahip ise $N$ modülüne dual sonlu yarı basit yükseltilebilir modül veya kısaca dual sonlu ss-yükseltilebilir modül denir. Bu çalışmada, bu tanıma denk koşullar verilmiştir. Buna ek olarak, makalede tanımlanan bu kavramın basit özellikleri incelenmiştir.

## 1. Introduction

In this article, whole rings must be with identity and whole modules are taken as unitary left modules. Let $P$ and $N$ be a ring and a module with the given property, respectively. By the notation $S \leq N$, we will imply that $S$ is a submodule of $N$. The submodule $S$ of $N$ is named *small* in $N$ if $N \neq S + T$ for any proper submodule $T$ of $N$, indicated by $S \ll N$, and we point the sum of whole small submodules of it with $Rad(N)$. Dual to this concept, the submodule $S$ of $N$ is named *essential* in $N$, indicated by $S \trianglelefteq N$, if the intersection of $S$ with the other submodules of $N$ is non-zero except for $\{0\}$. A supplement submodule $T$ of $S$ in $N$ is a minimal element of the set $\{Y \leq N \mid N = S + Y\}$ that is equivalent to $N = S + T$ and $S \cap T \ll T$. Any module $N$ is named *supplemented*, if each submodule of $N$ has a supplement in $N$ (Wisbauer, 1991). Otherwise, the module $N$ is *amply supplemented*, if for each submodules $S$ and $S'$ of $N$ with $N = S + S'$, it can be found a supplement $T$ of $S$ where $T \leq S'$ (Wisbauer, 1991).

According to Mohamed & Müller (1990), any module $N$ is named *lifting* if each submodule $S$ of $N$ lies over a direct summand, i.e., there is a decomposition $N = T_1 \oplus T_2$ such that $T_1 \leq S$ and $S \cap T_2 \ll T_2$. By Wisbauer (1991), $N$ is lifting if and ony if $N$ is amply supplemented and each supplement submodule of $N$ is a direct summand of $N$. Lifting modules have been studied by many authors such as (Keskin, 2000; Tribak, 2008; Wang & Wu, 2010).

Following Zhou & Zhang (2011), the sum of whole simple submodules of $N$ which are small in $N$ is indicated by $Soc_s(N)$, i.e., $Soc_s(N) = \sum\{S \ll N \mid S \text{ is simple}\}$. Obviously $Soc_s(N) \subseteq Rad(N)$ and $Soc_s(N) \subseteq Soc(N)$.

Besides these, semisimple supplemented ($ss-$) and semisimple ($ss-$) lifting modules are introduced by Kaynar et al. (2020) and Eryılmaz (2021), respectively. Let $N$ be a module and $S, B \leq N$. If $N = S + B$ and $S \cap B \subseteq Soc_s(B)$, then $B$ is a *semisimple supplement* of $S$ in $N$. Any module $N$ is called *semisimple* (or briefly $ss-$) *supplemented*, if each submodule $S$ of $N$ have a semisimple supplement $B$ in $N$. As a result of this definition, a finitely generated module $N$ is semisimple supplemented iff $N$ is supplemented and $Rad(N) \subseteq Soc(N)$. According to Eryılmaz (2021), a module $N$ is called *semisimple* (or briefly $ss-$) *lifting*, if each submodule $S$ of $N$, we have a decomposition $N = T_1 \oplus T_2$ where $T_1 \leq S$, $S \cap T_2 \ll T_2$ and $S \cap T_2$ is semisimple. Some fundamental properties of semisimple lifting modules are examined in this paper. Another version of these modules is studied in Türkmen & Türkmen (2020) and Sözen (2022).

Based on the definitions found in the literature, we can define cofinitely semisimple lifting modules. A module $N$ is called a cofinitely semisimple lifting (or briefly cofinitely $ss-$) if for each cofinite submodule $S$ of $N$, there is a direct summand $T$ of $N$ where $T \leq S$ and $\frac{S}{T} \subseteq Soc_s\left(\frac{N}{T}\right)$. Equivalent conditions to our definition are given in Theorem 3. It is proved that each $\pi-$projective and cofinitely semisimple supplemented module is cofinitely semisimple lifting. We will show that for a ring $P$, ${}_PP$ is cofinitely semisimple lifting if and only if $P$ is semiperfect with $Rad(P) \subseteq Soc({}_PP)$.

## 2. Results

### 2.1. Cofinitely semisimple (ss-) lifting modules

Now, we will examine some new properties of cofinitely semisimple lifting modules.

**Definition 1.** A module $N$ is named cofinitely semisimple lifting if for each cofinite submodule $S$ of $N$, $N$ has a decomposition $N = U' \oplus V$ where $U' \subseteq S$, $V \leq N$ and $S \cap V \subseteq Soc_s(V)$.

Since each submodule of a finitely generated module is cofinite, the concepts of semisimple lifting and cofinitely semisimple lifting modules overlap in finitely generated modules. Each semisimple lifting module is cofinitely semisimple lifting. However, the following example emphasizes that the converse is not valid.

**Example 2**. Assume that $P$ is a Dedekind domain and is not a field. Let $X$ be the field of fractions of $P$, $J$ be an infinite index set and $N$ be the $P-$module $X^{(J)}$. Then, $N$ has not a maximal submodule, and so $N$ is the only cofinite submodule of $N$. Therefore it is cofinitely semisimple lifting. By Theorem 2.4 and 3.1 in Zöschinger (1974), $N$ is not semisimple lifting.

**Theorem 3.** Let $N$ be a module. For any cofinite submodule $S$ of $N$, the following conditions are equivalent:

(1) There is a direct summand $T$ of $N$ where $T \leq S$ and $\frac{S}{T} \subseteq Soc_s\left(\frac{N}{T}\right)$.

(2) It can be found a submodule $U$ of $N$ and a direct summand $T$ of $N$ where $T \leq S$, $S = T + U$ and $U \subseteq Soc_s(N)$.

(3) There is a decomposition $N = T \oplus S'$ with $T \subseteq S$ and $S' \cap S \subseteq Soc_s(N)$.

(4) $S$ has a semisimple supplement $S'$ in $N$ where $S' \cap S$ is a direct summand in $S$.

(5) There is a homomorphism $f : N \to N$ with $f^2 = f$ where $f(N) \subseteq S$ and $(1-f)(S) \subseteq Soc_s(1-f)(N)$.

Proof. (1) $\Rightarrow$(2) Since $T$ is a direct summand of $N$, it can be found a submodule $S'$ of $N$ with $N = T \oplus S'$ Taking the intersection of both sides of the equation with $S$, we get that $S = T + (S' \cap S)$. Since $\frac{S}{T} \cong \frac{N}{T}$ and $\frac{S}{T}$ is semisimple, it can be written that $\Psi\left(\frac{S}{T}\right) = S' \cap T \cong T$ and $S' \cap S$ is semisimple where $\Psi : N \to T$ is the canonical projection.
(2)$\Rightarrow$(3) According to the assumption, we have $N = T \oplus S'$ for some submodule $S'$ of $N$. Then $S'$ is semisimple supplement of $T$ in $N$ and so the semisimple supplement of $S = T + U$ in $N$ by Lemma 22 in Kaynar et al. (2020). Therefore $S' \cap S \subseteq Soc_s(S')$.
(3) $\Rightarrow$(4) Taking the intersection of both sides of the equation $N = T \oplus S'$ with $S$, we can write $S = T \oplus (S' \cap S)$. Hence $S'$ is a semisimple supplement of $S$ in $N$.
(4)$\Rightarrow$(5) By the assumption, we have $N = S + S'$, $S' \cap S \subseteq Soc_s(S')$ and $S = (S' \cap S) \oplus T$ for some $T \subseteq S$. Then $N = S + S' = (S' \cap S) + T + S' = T + S'$, $(S' \cap S) \cap T = 0$ and $N = T + S'$. If we take $f : N \to N$ as a projection where $f(n) = t$, $n = t + s$, $t \in T$, $s \in S'$, then $f(N) \subseteq T \subseteq S$. Since $(1-f)(N) = S'$, we can easily get that $(1-f)(S) = S' \cap S \cong S' = (1-f)(N)$ and $S' \cap S$ is semisimple.
(5) $\Rightarrow$(1) Suppose that $T = f(N)$. Then we can obtain that $N = f(N) \oplus (1-f)(N)$ because $f$ is an idempotent. Thus $N = T \oplus (1-f)(N)$ with $T \subseteq S$. If we consider the isomorphism $\Phi : \frac{N}{T} \to (1-f)(N)$, then we get $\Phi\left(\frac{S}{T}\right) = (1-f)(S) \cong (1-f)(N) = \Phi\left(\frac{N}{T}\right)$. As $\Phi^{-1}$ is an isomorphism and $\frac{S}{T} \cong \frac{N}{T}$, it is easy to see that $\Phi^{-1}((1-f)(S)) = \frac{S}{T}$ is semisimple. Therefore $\frac{S}{T} \subseteq Soc_s\left(\frac{N}{T}\right)$.

Recall from Türkmen & Kılıç (2022) that a module $N$ is said *(amply) cofinitely semisimple supplemented* if each cofinite submodule of $N$ has (ample) semisimple supplements in $N$.

**Lemma 4.** If $N$ is a cofinitely semisimple lifting module, then it is cofinitely semisimple supplemented.

**Proof.** From the hypothesis, for a cofinite submodule $S$ of $N$, we can find submodules $T$ and $T'$ of $N$ where $N = T \oplus T', T \leq S$ and $T' \cap S \subseteq Soc_s(T')$. If we take the intersection of both sides with $S$, then we can obtain that $S = T \oplus (S \cap T')$ and $N = T' + S$. Hence $T'$ is a semisimple supplement of $S$ in $N$ and so $N$ is cofinitely semisimple supplemented.

**Theorem 5.** If $N$ is a $\pi-$projective and cofinitely semisimple supplemented module, then $N$ is cofinitely semisimple lifting.

**Proof.** Since $N$ is cofinitely semisimple supplemented, it can be found a submodule $T$ of $N$ where $N = S + T$ and $S \cap T \subseteq Soc_s(T)$ for every cofinite submodule $S$ of $N$. Moreover, there exists a submodule $S'$ of $N$ where $N = S' + T$, $S' \subseteq S$ and $S' \cap T \subseteq Soc_s(S')$ because $N$ is amply cofinitely semisimple supplemented by Proposition 3 in Türkmen & Kılıç (2022). Hence $S'$ and $T$ are mutually supplemented by 41.14(2) in Wisbauer (1991), $S' \cap T = 0$ and $N = S' \oplus T$. As a result, $N$ is cofinitely semisimple lifting.

**Theorem 6.** If $N$ is a cofinitely semisimple lifting module, then each cofinite direct summand of $N$ is cofinitely semisimple lifting.

**Proof.** Assume that $S$ is a cofinite direct summand of $N$ and $L$ is a cofinite submodule of $S$. Then, we can find a submodule $S'$ of $N$ where $N = S \oplus S'$. By the way, we can write $\frac{N}{L} = \frac{S}{L} \oplus \frac{S'+L}{L}$ and $\frac{N}{S} \cong \frac{S'+L}{L} \cong S'$ where are finitely generated. Here, $L$ is a cofinite submodule of $N$ because

$$\frac{\frac{N}{L}}{\frac{S'+L}{L}} \cong \frac{S}{L}. \tag{1}$$

Since $N$ is cofinitely semisimple lifting, it can be found submodules $T$ and $T'$ of $N$ such as $N = T \oplus T', T \leq L$ and $L \cap T' \subseteq Soc_s(T')$. If we take intersection with $S$, then we can write that $S = T \oplus (T' \cap S)$ and $L \cap (S \cap T') = (L \cap S) \cap T' = L \cap T' \subseteq Soc_s(S \cap T')$. Thus, $S$ is cofinitely semisimple lifting.

According to Wisbauer (1991), if $f(S)$ is included in $S$ for any $P-$endomorphism $f$ of $N$, then a submodule $S$ of $N$ is called *fully invariant* and if each submodule of $N$ is fully invariant, then the module is named a *duo module* (Özcan et al., 2006).

**Theorem 7**. Let $N$ be a cofinitely semisimple lifting module. If $S$ is a fully invariant submodule of it, then $\frac{N}{S}$ is cofinitely semisimple lifting.

**Proof.** Let's take $\frac{K}{S}$ as a cofinite submodule of $\frac{N}{S}$. Since $\frac{\frac{N}{S}}{\frac{K}{S}} \cong \frac{N}{K}$, $K$ is a cofinite submodule of $N$. By using the hypothesis, we can find submodules $T$ and $T'$ which satisfy $N = T \oplus T'$, $T \leq K$ and $K \cap T' \subseteq Soc_s(T')$. Using Proposition 26 in Kaynar et al. (2020), we can say that $\frac{T'+S}{S}$ is a semisimple supplement of $\frac{K}{S}$ in $\frac{N}{S}$. Moreover, we have $S = (S \cap T) \oplus (S \cap T')$ by Lemma 2.1 of Özcan et al. (2006). Therefore, we obtain $\frac{N}{S} = \frac{T+S}{S} \oplus \frac{T'+S}{S}$ and $\frac{T+S}{S} \leq \frac{K}{S}$. Since $K \cap T'$ is semisimple, it follows from Kasch (1982) that $\frac{(K \cap T')+S}{S} = \frac{K}{S} \cap \frac{T'+S}{S}$ is semisimple. Finally, $\frac{N}{S}$ is cofinitely semisimple lifting.

**Corollary 8**. If $N$ is a cofinitely semisimple lifting and duo module, then each factor module of $N$ is cofinitely semisimple lifting.

Since $Rad(N)$ and $Soc_s(N)$ are fully invariant submodules of $N$, we have the following result.

**Corollary 9.** For any cofinitely semisimple lifting module $N$, $\frac{N}{Rad(N)}$ and $\frac{N}{Soc_s(N)}$ are cofinitely semisimple lifting.

**Theorem 10.** Let $N$ be a module with $Rad(N) \ll N$. $N$ is cofinitely semisimple lifting if and only if $N$ is cofinitely lifting and $Rad(N) \subseteq Soc(N)$.

**Proof.** The proof is done in Theorem 4 in Eryılmaz (2021).

Since projective modules have small radicals, we can get the next result.

**Corollary 11.** (1) Let $N$ be a finitely generated module. $N$ is semisimple lifting if and only if $N$ is cofinitely semisimple lifting.
(2) Let $N$ be a projective module. $N$ is cofinitely semisimple lifting if and only if $N$ is lifting and its radical is semisimple.

**Example 12.** The local $\mathbb{Z}-$module $\mathbb{Z}_8$ is cofinitely lifting but not cofinitely semisimple lifting. Because $Rad(\mathbb{Z}_8) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \not\subset Soc(\mathbb{Z}_8) = \{\bar{0}, \bar{4}\}$.

For any $R-$ module $N$, if the sum of two direct summands of $N$ is again a direct summand of it, then $N$ is said to have the *Summand Sum Property (SSP)*, (Wisbauer, 1991).

**Proposition 13.** If $N$ is a cofinitely semisimple lifting module with the property (SSP) and $S$ is a direct summand of $N$, then $\frac{N}{S}$ is a cofinitely semisimple lifting module.

Proof. Assume that $\frac{L}{S}$ is a cofinite submodule of $\frac{N}{S}$. Since $\frac{\frac{N}{S}}{\frac{L}{S}} \cong \frac{N}{L}$, it is easy to see that $L$ is a cofinite submodule of $N$. By the assumption, it can be found submodules $V$ and $V'$ which satisfy $N = V \oplus V'$, $V \leq L$ and $L \cap V' \subseteq Soc_s(V')$. If we use Proposition 26 in Kaynar et al. (2020), then we can say that $\frac{V'+S}{S}$ is a semisimple supplement of $\frac{L}{S}$ in $\frac{N}{S}$. Moreover, we have $N = (V+S) \oplus X$ for a submodule $X$ of $N$ because $N$ has (SSP). Therefore, we obtain that $\frac{N}{S} = \frac{V+S}{S} \oplus \frac{X+S}{S}$ and $\frac{V+S}{S} \leq \frac{L}{S}$. Moreover $\frac{L}{S} \cap \frac{X+S}{S} \subseteq Soc_s\left(\frac{X+S}{S}\right)$. Finally, $\frac{N}{S}$ is cofinitely semisimple lifting.

**Lemma 14.** If $N$ is a cofinitely lifting module with $Rad(N) \subseteq Soc(N)$, then $N$ is cofinitely semisimple lifting.

**Proof.** The evidence is obvious.

Recall from Keskin (2000) that for any module $N$, let $T \leq S \leq N$. A $T$ is named a *co-essential submodule* of $S$ in $N$, if $\frac{S}{T} \ll \frac{N}{T}$. If $S$ has no proper co-essential submodule, then a submodule $S$ of $N$ is called *co-closed*. If $T$ is a co-essential submodule of $S$ and $T$ is co-closed in $N$, then $T$ is named as *co-closure (or s-closure)* of $S$.

**Proposition 15.** Let $N$ be a module whose each cofinite submodule has a co-closure in $N$ and $Rad(N) \subseteq Soc(N)$. Then $N$ is cofinitely semisimple lifting if and only if each cofinitely co-closed submodule of $N$ is a direct summand of $N$.

**Proof.** It follows from Lemma 14 and Proposition 3.6 in Wang & Wu (2010).

Since each submodule of an amply supplemented module $N$ has a co-closure in $N$ and each amply semisimple supplemented module is amply supplemented, we have the next result.

**Corollary 16.** Suppose that $N$ is an amply semisimple supplemented module with $Rad(N) \subseteq Soc(N)$. Then, $N$ is cofinitely semisimple lifting if and only if each cofinitely co-closed submodule of $N$ is a direct summand of it.

**Theorem 17.** Suppose that $\{N_j\}_{j \in J}$ is a collection of cofinite semisimple lifting modules and $N = \bigoplus_{j \in J} N_j$. If each submodule of $N$ is fully invariant, then $N$ is cofinitely semisimple lifting.

**Proof.** For any cofinite submodule $S$ of $N$, we have $S = \bigoplus_{j \in J} (S \cap N_j)$ because $S$ is fully invariant. Moreover, we have $\frac{N}{S} = \bigoplus_{j \in J} \left[\frac{(S+N_j)}{S}\right]$ and $\frac{(S+N_j)}{S} \cong \frac{N_j}{S \cap N_j}$. Therefore, $S \cap N_j$ is a cofinite submodule of $N_j$ for each $j \in J$. By the assumption, since $N_j$ is a cofinitely semisimple lifting module for each $j \in J$, there exist submodules $A_j, B_j \leq N_j$ such that $A_j \leq S \cap N_j$, $S \cap B_j = \left(S \cap N_j\right) \cap B_j \subseteq Soc_s\left(B_j\right)$ and $N_j = A_j \oplus B_j$. It follows that $N_j = A_j \oplus B_j \subseteq \left(S \cap N_j\right) + B_j$ and $N_j = \left(S \cap N_j\right) + B_j$. Let $A = \bigoplus_{j \in J} A_j$ and $B = \bigoplus_{j \in J} B_j$. Since $A_j \subseteq S \cap N_j \subseteq S$ and $N_j = A_j + B_j$ for every $j \in J$, we can obtain that $A \leq S$ and $N = \bigoplus_{j \in J} N_j = \bigoplus_{j \in J} \left(A_j + B_j\right) = A \oplus B$. Hence $S \cap B \subseteq Soc_s\left(B\right)$ and so $N$ is cofinitely semisimple lifting.

**Corollary 18**. Let $\{N_i\}_{i \in I}$ be the collection of cofinitely semisimple lifting modules. If $\bigoplus_{j \in J} N_j = N$ is a duo module, then $N$ is a cofinitely semisimple lifting module.

A module with the lattice of its submodules linearly ordered under inclusion, is called a *uniserial module*. If a module can be written as a direct sum of uniserial modules, then it is called *serial*. A ring $P$ is called *left (right) serial* if ${}_P P$ $(P_P)$ is a serial module. A ring $P$ is called *left (right) artinian*, if it satisfies the descending chain condition (dcc) on left (right) ideals. An artinian ring is a ring which is left and right artinian. A ring $P$ is said to be *artinian serial ring* if it is both left, right artinian and left, right serial. For more detailed information on artinian and serial rings see Wisbauer (1991) chapters 4 and 55.

**Theorem 19.** $P$ is a left and right artinian serial ring with $Rad(P) \subseteq Soc({}_P P)$ iff each left $P-$module is cofinitely semisimple lifting.

**Proof.** By Theorem 6 in Eryılmaz (2021) and Lemma 14.

**Example 20**. Consider the local ring $P = \mathbb{Z}_4$. Note that $Rad(\mathbb{Z}_4) = \left\{\bar{0}, \bar{2}\right\} = Soc({}_{\mathbb{Z}_4}\mathbb{Z}_4)$ and it is left and right artinian serial ring. By using the previous theorem, each left $P-$module is cofinitely semisimple lifting.

Recall from Kasch (1982) that an epimorphism $g: S \to N$ is named as *small cover* if $Ker\left(g\right) \ll S$. A projective module $S$ together with a small cover $g: S \to N$ is called a *projective cover* of $N$. If each finitely generated left (or right) $P-$module has a projective cover, a ring $P$ is called *semiperfect*.

**Theorem 21.** For a ring $P$, the following statements are equivalent:
(1) ${}_P P$ is cofinitely semisimple lifting.
(2) ${}_P P$ is cofinitely semisimple supplemented.
(3) Each left $P-$module is cofinitely semisimple supplemented.
(4) $P$ is semiperfect with $Rad(P) \subseteq Soc({}_P P)$.

**Proof.** (1)⇒ (2) By Lemma 4.
(2)⇒ (3) and (3) ⇒ (4) By Theorem 3 in Türkmen & Kılıç (2022).
(4)⇒ (1) By Theorem 10.

## 3. Discussion and Conclusion

In this study, the concept of “cofinitely ss-lifting module” is defined based on the known concepts which can be found in the literature. Using this study, the concept of “(cofinitely) $\delta-$lifting module” can be defined and the similar features can be examined easily.

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