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Research Article

Parameterized Differential Transform Method and Its Application to Boundary Value Transmission Problems

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Abstract: In this study, we developed a new modification of the well-known differential transform method (DTM) that differs from the classical one by the algorithm for calculating the coefficients of an approximate solution given as a series. The proposed new algorithm we will call α-parameterized DTM ($\alpha - p$ DTM, for short). By using the proposed $\alpha - \mathrm{p}$ DTM, we solve the boundary value transmission problem for the third-order differential equation. For the same problem, an approximate solution is also found using also the classical DTM. The solutions obtained were compared graphically.

Parametreli Diferansiyel Dönüşüm Yöntemi ve Sınır Değer Geçiş Problemlerine Uygulanması

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Anahtar Kelimeler
Diferensiyel dönüşüm yöntemi,
Geçiş şartları,
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Öz: Bu çalışmada literatürden iyi bilinen diferansiyel dönüşüm yönteminin (DDY) yeni bir modifikasyonunu geliştirdik. Seri biçiminde verilmiş yaklaşık çözümde serinin katsayılarının hesaplanmasında uyguladığımız algoritma geleneksel DDY'den farklıdır. Önerdiğimiz yeni algoritmayı $\alpha -$parametreli diferansiyel dönüşüm yöntemi ($\alpha - \mathrm{p}$ DDY) olarak adlandırıyoruz. Geliştirdiğimiz $\alpha - \mathrm{p}$ DD yöntemini uygulayarak 3. mertebeden diferensiyel denklem için sınır değer geçiş probleminin yaklaşık çözümleri bulunmuştur. Aynı problemin başka bir yaklaşık çözümü, geleneksel DD yöntemi ile de bulunmuştur. Bulduğumuz sonuçlar grafiksel olarak karşılaştırılmıştır.

1. Introduction

Boundary value problems (BVPs, for short) for differential equations arise as a model of broad class of physical processes across of all areas of natural sciences. Different analytical methods are used to find the exact solutions to a particular type of linear differential equation. However, not all differential equations can be solved analytically. As a rule, purely analytical methods are not available for finding exact solutions to non-classical and singular boundary value problems. Therefore, various approximate and numerical techniques, such as Adomian Decomposition Method (ADM), Finite Difference Method (FDM), Differential Transform Method (DTM), Variational Iterative Method (VIM), Homotopy Perturbation Method (HPM) etc. are effective tools to investigate and understand the qualitative properties of many differential equations with exact unknown solutions.

Many approximate and numerical methods are also not effective enough in solving various non-classical and/or singular problems since they may require complex algebraic calculations. However, the differential transform technique can provide useful approximate solutions for most regular BVPs and some non-classical singular BVPs because it is based on a simple algorithm. This simplest algorithm differs from others in that it does not require large computations of high-order derivatives of data functions.

Zhou (1986) first developed the differential transformation method to solve initial value problems when modelling electrical circuits. This method allows us to find an approximate, series or analytical solution in a closed form of various linear and non-linear differential equations. The usefulness is that this method reduces the given differential equation to simple iterative equations that are more convenient to study. Inspired by the works of Zhou, interest has increased in the development of various modifications and generalizations of DTM for solving a new type of boundary value problems that arise in modern problems of physics and other natural sciences (see, for example (Cakir & Arslan, 2015; Lal & Ahlawat, 2015; Mohamed & Gepreel, 2017; Ghazaryan et al., 2018; Pratiksha, 2019; Mukhtarov et al., 2019; Liu et al., 2020; Mukhtarov & Yücel, 2020; Hussein Msmali et al., 2021; Mukhtarov et al., 2021; Sowmya & Gireesha, 2022) and references therein).

In recent years, the application of DTM and its various modifications to the solutions of nonclassical initial and/or boundary value problems of a new type, which arise in solving many specific problems of physics and technology, has attracted great interest. Bekiryazici et al. (2020) developed a modification of the random DTM and demonstrated its applications to some models. Jena & Chakraverty (2018) proposed a new version of DTM to study free vibration of nanobeams. Khudair et al. (2016) investigated some second-order random differential equations by DTM. Ünal & Gökdoğan (2017) generalized DTM to solve not only classical differential equations but also fractional differential equations. Mirzaaghaian & Ganji (2016) examined the applicability of DTM to some physical problems such as heat transfer through permeable walls and micropolar fluid flow. Elsaid & Helal (2020) developed a new modification of the DTM to calculate partial derivatives of nonlinear functions of two variables. They also used a different algoritm that does not require any intermediate calculations, which eliminates some of the shortcomings of many previous algorithms (Elsaid & Helal, 2020). Odibat et al. (2008) adapted the classical DTM in such a way that it could be applied not only to ordinary and partial differential equations but also fractional differential equations. Sepasgozar et al. (2017) applied DTM to find approximate solutions of momentum and heat transfer in a porous channel in a non-Newtonian fluid flow. Biazar & Eslami (2010), using differential transformation, numerically solved the telegraph equation. Karakoç & Bereketoğlu (2019) proposed a different version of DTM for solving delay differential equations. Ayaz (2004) defined a three-dimensional differential transformation for solving a system of differential equations. Mukhtarov et al. (2020) suggested a new generalization of DTM to investigate some spectral properties of a new type of boundary-value problem. Namely, they solved a new type of boundary value problem consisting of 2-interval differential equations and boundary-transmission conditions by developing a new differential transform (Mukhtarov et al., 2020). Biswas & Roy (2018) devoted the intuitionistic (fuzzy) differential transform method to solving Volterra-type fuzzy integrodifferential equations. Nazari & Shakmorad (2010) used the fractional DTM to solving fractional integro-differential equations under nonlocal BC's.

In this study, we proposed a new transform method (called α-parameterized DTM) to solve boundary value transmission problems for two-interval differential equations.

2. Material and Methods

2.1. Basic properties of differential transformation method

Let $\mathrm{b} = \mathrm{b}(\mathrm{x})$ be any analytic function in some around of the point $x = x_0$. Then this function can be expanded in Taylor's series as

$$b(x) = \sum_{l=0}^{\infty} B_{x_0}(l)\,(x - x_0)^l \tag{1}$$

where $B(l)$ is Taylor's coefficient defined by

$$B_{x_0}(l) = \frac{1}{l!}\left[\frac{d^l}{dx^l} b(x)\right]_{x=x_0}, \quad l = 0,1,2,\dots \tag{2}$$

Definition 1. The sequence $B_{x_0}(0),\ B_{x_0}(1),\ B_{x_0}(2),\dots$ is said to be the differential transform of the analytic function $b(x)$, where $B_{x_0}(l),\ \ l = 0,1,2,\dots$ is defined by Equation (2). The differential inverse transformation of the sequence $(B_{x_0}(l))$ is defined by Equation (1). Here $b(x)$ is said to be the original function and the sequence $(B_{x_0}(l))$ is said to be the T-transform of $b(x)$.

Let us denote the T- transform of the original function $b(x)$ by $T_{x_0}(b)$, and the differential inverse transform of $(B_{x_0}(l))$ by $T_{x_0}^{-1}(B_{x_0}(l))$. From the definition of the T- transform it follows easily the following properties:

i. $T_{x_0}(a_1 + a_2) = T_{x_0}(a_1) + T_{x_0}(a_2)$
ii. $T_{x_0}(\gamma \mathrm{b}) = \gamma T_{x_0}(\mathrm{b})$ for any $\gamma \in \mathrm{R}$
iii. If $T_{x_0}(\mathrm{b}) = \left(B_{x_0}(\mathrm{l})\right)$, then $T_{x_0}\left(\frac{\mathrm{db}}{\mathrm{dx}}\right) = \left((\mathrm{l}+1)B_{x_0}(\mathrm{l}+1)\right)$ and $T_{x_0}\left(\frac{\mathrm{d}^2\mathrm{b}}{\mathrm{dx}^2}\right) = \left((\mathrm{l}+1)(\mathrm{l}+2)B_{x_0}(\mathrm{l}+2)\right)$
iv. If $T_{x_0}(\mathrm{a}) = \left(A_{x_0}(\mathrm{l})\right)$, $T_{x_0}(\mathrm{b}) = \left(B_{x_0}(\mathrm{l})\right)$ and $T_{x_0}(\mathrm{ab}) = \left(C_{x_0}(\mathrm{l})\right)$, then $C_{x_0}(\mathrm{l}) = (A_{x_0}(\mathrm{l}) * B_{x_0}(\mathrm{l}))$, where $A_{x_0}(l) * B_{x_0}(l)$ is denoted the convolution of the sequences $A_{x_0}(l)$ and $B_{x_0}(l)$.

In a real application, the differential inverse transform $T_{x_0}^{-1}(B_{x_0}(l))$ is defined by a finite sum

$$T_{x_0}^{-1}\left(B_{x_0}(l)\right) = \sum_{l=0}^{s} B_{x_0}(l)(x - x_0)^l \tag{3}$$

for sufficiently large s.

2.2. Definition and basic properties of the $\alpha -$ parameterized DTM

Let $\mathrm{g}: [\mathrm{a},\mathrm{b}] \to \mathrm{R}$ be a real-valued analytic function and $\alpha \in [0,1]$ is a real number.

Definition 1. We say the sequence $\left(T_\alpha(a,b)\right)_n(g)$ is the $\alpha -$ parameterized differential transform of the original function g(x) if

$$\left(T_\alpha(a,b)\right)_n(g) := \alpha\left(T_a(g)\right)_n + (1-\alpha)\left(T_b(g)\right)_n, \tag{4}$$

where

$$\left(\mathrm{T_a(g)}\right)_\mathrm{n} := \frac{\mathrm{d^n g(a)}}{\mathrm{n!}}, \qquad \left(\mathrm{T_b(g)}\right)_\mathrm{n} := \frac{\mathrm{d^n g(b)}}{\mathrm{n!}}. \tag{5}$$

Definition 2. We say the function $\mathrm{g(x)}$ is the $\alpha -$inverse differential transform if

$$\mathrm{g}_\alpha(\mathrm{x}) := \sum_{\mathrm{n}=0}^{\infty}\left(T_\alpha(a,b)\right)_n(g)\left(\mathrm{x} - (\alpha\mathrm{a} + (1-\alpha)\mathrm{b})\right)^\mathrm{n} \tag{6}$$

provided that the series is convergent. The $\alpha-$inverse differential transforms we shall denote by $(T_{\alpha}^{-1}(a,b)_n)(g)$.

Definition 3. The N-th partial sum of the series defined by Equation (6) is said to be N-th α-parametrized approximation of the original function g(x) and is denoted by $g_{\alpha,N}(x)$, that is

$$g_{\alpha,N}(x) := \sum_{n=0}^{N} \big(T_{\alpha}(a,b)\big)_n (g)\big(x-(\alpha a+(1-\alpha)b)\big)^n. \tag{7}$$

by using Definition 1 we can show that the α- parameterized differential transform has the following properties:

1. $\big(T_{\alpha}(a,b)\big)_n(cg) = c(T_{\alpha}(a,b)(g))_n$
2. $\big(T_{\alpha}(a,b)\big)_n(f \pm g) = \big(T_{\alpha}(a,b)\big)_n(f) \pm \big(T_{\alpha}(a,b)\big)_n(g)$
3. $\big(T_{\alpha}(a,b)\big)_n\left(\frac{d^m g}{dx^m}\right) = \frac{(n+m)!}{n!}\big(T_{\alpha}(a,b)\big)_n(g)$.

3. Results

3.1. Solution of transmission problems by using modified differential transformation method

Let us consider the 3rd-order differential equation,

$$\frac{d^3u}{dx^3} - 2x\frac{du}{dx} - 2u = 0, \qquad x \in \left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right] \tag{8}$$

together with the boundary conditions,

$$u(0) = 1, \quad u(1) = 1, \quad \frac{du(0)}{dx} = 1 \tag{9}$$

and with the additional transmission conditions at the singular point $x = \frac{1}{2}$, given by

$$u\left(\tfrac{1}{2}+0\right) = u\left(\tfrac{1}{2}-0\right), \qquad \frac{du(\frac{1}{2}+0)}{dx} = \frac{du(\frac{1}{2}-0)}{dx}, \ \frac{d^2u(\frac{1}{2}+0)}{dx^2} = \frac{1}{2}\,\frac{d^2u(\frac{1}{2}-0)}{dx^2}. \tag{10}$$

Denote by $U_0^-(k)$ and $U_1^+(k)$ the T-transforms of the function $u(x)$ at the end-points $x=0$ and $x=1$, respectively. If DTM is applied to the differential equation in the left interval, at the point $x=\frac{1}{2}$, we have

$$(l+1)(l+2)(l+3)U_0^-(l+3) - 2\sum_{r=0}^{l} U_0^-(l-r+1)(l-r+1)\delta(r-1) - 2U_0^-(l) = 0 \tag{11}$$

where $T^-(m) = \frac{1}{m!}\left[\frac{d^m}{dx^m}u(x)\right]_{x=x_0}$. The differential inverse transform in the left interval has the following form:

$$u^-(x) = \sum_{k=0}^{n} x^k U_0^-(k) = U_0^-(0) + xU_0^-(1) + x^2U_0^-(2) + \cdots + x^{11}U_0^-(11). \tag{12}$$

The first boundary condition $u(0) = 1$, becomes $U_0^-(0) = 1$, and the second condition $\frac{du(0)}{dx} = 1$, becomes $U_0^-(1) = 1$. Let $U_0^-(2) = a$.
Now proceed with the iteration using Equation (11); we can calculate the other terms of the T- transform as

$$\begin{gathered} U_0^-(3) = \frac{1}{3}, \quad U_0^-(4) = \frac{1}{6}, \quad U_0^-(5) = \frac{a}{10}, \quad U_0^-(6) = \frac{1}{45}, \quad U_0^-(7) = \frac{1}{126}, \\ U_0^-(8) = \frac{a}{280}, \quad U_0^-(9) = \frac{1}{1620}, \quad U_0^-(10) = \frac{1}{5670}, \quad U_0^-(11) = \frac{a}{15400}. \end{gathered} \tag{13}$$

If we carry out the iteration up to n = 7, then we have the following approximation of the left solution:

$$u^-(x) = 1 + x + ax^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \left(\frac{a}{10}\right)x^5 + \frac{1}{45}x^6 + \frac{1}{126}x^7. \tag{14}$$

Secondly, let us get the solution for the problem in the right interval $(\frac{1}{2}, 1]$. If the differential transform method is applied to the differential equation, in the around of the point $x_0 = 1$, we have

$$(l+1)(l+2)(l+3)U_1^+(l+3) - 2(l+1)U_1^+(l+1) - 2lU_1^+(l) - 2U_1^+(l) = 0. \tag{15}$$

The differential inverse transform in the right interval $(\frac{1}{2}, 1]$ has the following form:

$$u^+(x) = U_1^+(0) + (x-1)U_1^+(1) + (x-1)^2U_1^+(2) + \cdots + (x-1)^{11}U_1^+(11) \tag{16}$$

The third boundary condition $u(1) = 1$, becomes $U_1^+(0) = 1$. Let us $U_1^+(1) = b$ and $U_1^+(2) = c$. By using Equation (15), we find

$$\begin{gathered} U_1^+(3) = \frac{1}{3}(b+1), \quad U_1^+(4) = \frac{b}{6} + \frac{c}{6}, \quad U_1^+(5) = \frac{1}{10}\left(\frac{1}{3}(b+1) + c\right), \\ U_1^+(6) = \frac{1}{120}\left(\frac{4c}{3} + 4b + \frac{8}{3}\right), \quad U_1^+(7) = \frac{1}{21}\left(\frac{6b + 8c + 1}{30}\right), \quad \ldots \end{gathered} \tag{17}$$

Now, applying differential inverse transform for n = 7, we have

$$\begin{aligned} u^+(x) = 1 + (x-1)b + (x-1)^2c + (x-1)^3\frac{1}{3}(b+1) + (x-1)^4(\frac{b}{6} + \frac{c}{6}) \\ + (x-1)^5\frac{1}{10}(\frac{1}{3}(b+1) + c) + (x-1)^6\frac{1}{120}(\frac{4c}{3} + 4b + \frac{8}{3}) \\ + (x-1)^7\frac{1}{21}\left(\frac{6b + 8c + 1}{30}\right). \end{aligned} \tag{18}$$

Substituting Equations (14)-(18) in the transmission conditions

$$u\left(\frac{1}{2}+0\right) = u\left(\frac{1}{2}-0\right), \quad \frac{du(\frac{1}{2}+0)}{dx} = \frac{du(\frac{1}{2}-0)}{dx}, \quad \frac{d^2u(\frac{1}{2}+0)}{dx^2} = \frac{1}{2}\,\frac{d^2u(\frac{1}{2}-0)}{dx^2}, \tag{19}$$

we can find, $a = -1.49383$, $b = -0.426463$, $c = -0.0391322$.

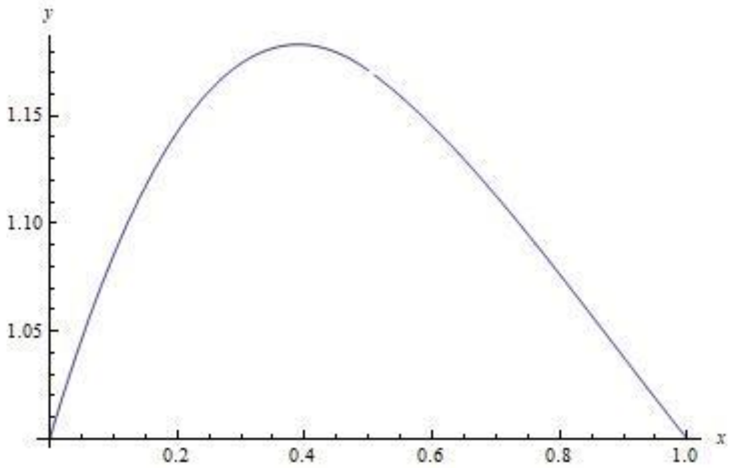

Figure 1. Graph of the classical DTM solution in the $[0,\frac{1}{2}) \cup (\frac{1}{2},1]$.

3.2. Application of the $\alpha - p\ DTM$ and comparision with the classical DTM

Consider the differential equation

$$\frac{d^3u}{dx^3} - 2x\frac{du}{dx} - 2u = 0, \quad x \in \left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right] \tag{20}$$

subject to the boundary conditions

$$u(0) = 1, \quad u(1) = 1, \quad \frac{du(0)}{dx} = 1 \tag{21}$$

and additional transmission conditions at the point of interaction $x = \frac{1}{2}$, given by

$$u\left(\frac{1}{2}+0\right) = u\left(\frac{1}{2}-0\right), \quad \frac{du(\frac{1}{2}+0)}{dx} = \frac{du(\frac{1}{2}-0)}{dx}, \quad \frac{d^2u(\frac{1}{2}+0)}{dx^2} = \frac{1}{2}\,\frac{d^2u(\frac{1}{2}-0)}{dx^2}. \tag{22}$$

Taking the $\alpha - p$ DT of the Equation (20) on the interval $[0,\frac{1}{2})$ we obtain

$$T(u,\alpha;m+3) = \frac{1}{(m+1)(m+2)(m+3)}[2\sum_{r=0}^{m} T(u,\alpha;m-r+1)(k-r+1)\delta(r-1) + 2T(u,\alpha;m)] \tag{23}$$

where $T(u,\alpha;m)$ is defined by $T(u,\alpha;m) \coloneqq T_\alpha\left(0,\frac{1}{2}\right)_m(u)$.
Using the boundary conditions becomes

$$u_{l,\alpha,N}(0) = \sum_{k=0}^{N} T(u,\alpha;k)\left(\frac{\alpha-1}{2}\right)^k = 1 \tag{24}$$

$$u'_{l,\alpha,N}(0) = \sum_{k=0}^{N} T(u,\alpha;k)k\left(\frac{\alpha-1}{2}\right)^{k-1} = 1. \tag{25}$$

Denoting $T(u,\alpha;0) = \mathcal{A}$, $T(u,\alpha;1) = \mathcal{B}$ and $T(u,\alpha;2) = \mathcal{C}$ then substituting in the recursive Equation (23), we get

$$T(u,\alpha;3)=\frac{2\,\mathcal{A}}{3!} \tag{26}$$

Now proceeding the iteration using Equation (23) we can calculate the other terms of the α −parametrized sequence $T(u,\alpha;n)$ as

$$\begin{gathered}T(u,\alpha;4)=\frac{\mathcal{B}}{3!},\qquad T(u,\alpha;5)=\frac{\mathcal{C}}{10},\qquad T(u,\alpha;6)=\frac{\mathcal{A}}{45},\\ T(u,\alpha;7)=\frac{\mathcal{B}}{126},T(u,\alpha;8)=\frac{\mathcal{C}}{280},\qquad \dots\end{gathered} \tag{27}$$

Hence, the left α −parametrized series solution $u_l(x,\alpha)$ is evaluated up to $N=7$:

$$\begin{aligned}u_l(x,\alpha)&=\sum_{k=0}^{7}D_l(u,\alpha;k)(x-(\frac{1-\alpha}{2}))^k\\ &=\mathcal{A}+\left(x-(\frac{1-\alpha}{2})\right)\mathcal{B}+\left(x-(\frac{1-\alpha}{2})\right)^2\mathcal{C}+\frac{1}{3}\left(x-\left(\frac{1-\alpha}{2}\right)\right)^3\mathcal{A}\\ &+\frac{1}{6}\left(x-\left(\frac{1-\alpha}{2}\right)\right)^4\mathcal{B}+\frac{1}{10}\left(x-\left(\frac{1-\alpha}{2}\right)\right)^5\mathcal{C}+\frac{1}{45}\left(x-\left(\frac{1-\alpha}{2}\right)\right)^6\mathcal{A}\\ &+\frac{1}{126}\left(x-\left(\frac{1-\alpha}{2}\right)\right)^7\mathcal{B}+\frac{1}{280}\left(x-\left(\frac{1-\alpha}{2}\right)\right)^8\mathcal{C}+\frac{1}{1620}\left(x-\left(\frac{1-\alpha}{2}\right)\right)^9\mathcal{A}\\ &+\frac{1}{5670}\left(x-\left(\frac{1-\alpha}{2}\right)\right)^{10}\mathcal{B}+\cdots\end{aligned} \tag{28}$$

where $x_\alpha=\frac{1-\alpha}{2}$ and $T(u,\alpha;0)=\mathcal{A}$, $T(u,\alpha;1)=\mathcal{B}$ and $T(u,\alpha;2)=\mathcal{C}$.
Now we will consider the problem Equations (20)-(22) on the right interval $\left(\frac{1}{2},1\right]$. Being in a similar way we obtain

$$\begin{aligned}D_r(u,\alpha;k+3)=&\frac{1}{(k+1)(k+2)(k+3)}[2\sum_{r=0}^{k}D_r(u,\alpha;k-r+1)(k-r+1)\delta(r-1)\\ &+2D_r(u,\alpha;k)]\end{aligned} \tag{29}$$

Denoting $D_r(u,\alpha;0)=\mathcal{D}$, $D_r(u,\alpha;1)=\mathcal{E}$ and $D_r(u,\alpha;2)=\mathcal{F}$ then substituting in the recursive Equation (29), we have

$$D_r(u,\alpha;3)=\frac{\mathcal{D}}{3} \tag{30}$$

Now proceeding the iteration using Equation (29) we can calculate the other terms of the α −parameterized sequence $D_r(u,\alpha;n)$ as

$$\begin{gathered}D_r(u,\alpha;4)=\frac{\mathcal{E}}{3!},D_r(u,\alpha;5)=\frac{\mathcal{F}}{10},\quad D_r(u,\alpha;6)=\frac{\mathcal{D}}{45},\quad D_r(u,\alpha;7)=\frac{\mathcal{E}}{126},\\ D_r(u,\alpha;8)=\frac{\mathcal{F}}{280},\quad D_r(u,\alpha;9)=\frac{\mathcal{D}}{1620},\qquad D_r(u,\alpha;10)=\frac{\mathcal{E}}{5670},\dots\end{gathered} \tag{31}$$

Hence the α −parametrized series solution $u_r(x,\alpha)$ is evaluated up to $N=7$:

$$\begin{aligned}u_r(x,\alpha) &= \sum_{k=0}^{7} D_r(u,\alpha;k)\left(x-\left(1-\frac{\alpha}{2}\right)\right)^k \\ &= \mathcal{D}+\left(x-\left(1-\frac{\alpha}{2}\right)\right)\mathcal{E}+\left(x-\left(1-\frac{\alpha}{2}\right)\right)^2\mathcal{F}+\frac{1}{3}\left(x-\left(1-\frac{\alpha}{2}\right)\right)^3\mathcal{D} \\ &+\frac{1}{3!}\left(x-\left(1-\frac{\alpha}{2}\right)\right)^4\mathcal{E}+\frac{1}{10}\left(x-\left(1-\frac{\alpha}{2}\right)\right)^5\mathcal{F}+\frac{1}{45}\left(x-\left(1-\frac{\alpha}{2}\right)\right)^6\mathcal{D} \\ &+\frac{1}{126}\left(x-(1-\frac{\alpha}{2})\right)^7\mathcal{E}+\frac{1}{280}\left(x-(1-\frac{\alpha}{2})\right)^8\mathcal{F}+\frac{1}{1620}\left(x-(1-\frac{\alpha}{2})\right)^9\mathcal{D} \\ &+\frac{1}{5670}\left(x-(1-\frac{\alpha}{2})\right)^{10}\mathcal{E}\end{aligned} \tag{32}$$

where $x_\alpha = (1-\frac{\alpha}{2})$ and $D_r(y,\alpha;0)=\mathcal{D}, D_r(y,\alpha;1)=\mathcal{E},\ D_r(y,\alpha;2)=\mathcal{F}$.
Using the boundary condition $u(1) = 1$ becomes

$$\begin{aligned}\mathcal{D}+\frac{\alpha}{2}\mathcal{E}+\left(\frac{\alpha}{2}\right)^2\mathcal{F}+\left(\frac{\alpha}{2}\right)^3\frac{\mathcal{D}}{3}+\left(\frac{\alpha}{2}\right)^4\frac{\mathcal{E}}{3!}+\left(\frac{\alpha}{2}\right)^5\frac{\mathcal{F}}{10}+\left(\frac{\alpha}{2}\right)^6\frac{\mathcal{D}}{45}+\left(\frac{\alpha}{2}\right)^7\frac{\mathcal{E}}{126}+\left(\frac{\alpha}{2}\right)^8\frac{\mathcal{F}}{280} \\ +\left(\frac{\alpha}{2}\right)^9\frac{\mathcal{D}}{1620}+\left(\frac{\alpha}{2}\right)^{10}\frac{\mathcal{E}}{5670}=1.\end{aligned} \tag{33}$$

By using the transmission conditions,

$$u_l\left(\frac{1}{2}\right)=u_r\left(\frac{1}{2}\right), \qquad \frac{du_l(\frac{1}{2})}{dx}=\frac{du_r(\frac{1}{2})}{dx}, \quad \frac{d^2u_l(\frac{1}{2})}{dx^2}=2\frac{d^2u_r(\frac{1}{2})}{dx^2}. \tag{34}$$

We can obtain the values of the auxiliary parameters $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E}\ and\ \mathcal{F}$.

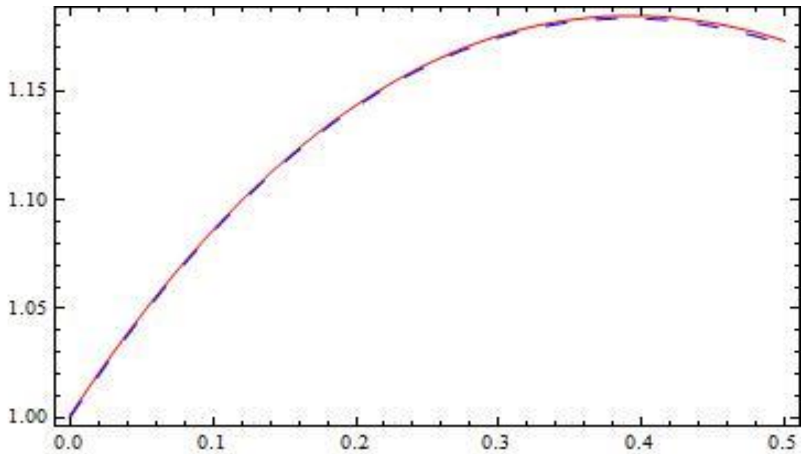

Figure 2. Comparison of the DTM solution (blue line) and the α −parameterized DT solution for $\alpha = \frac{999}{1000}$ (red line) on the left interval $[0,\frac{1}{2})$.

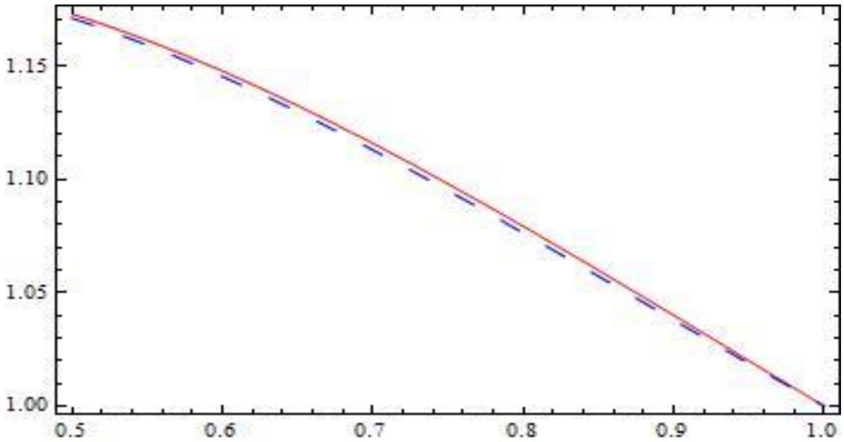

Figure 3. Comparison of the DTM solution (blue line) and the α −parameterized DT solution for $\alpha = \frac{999}{1000}$ (red line) on the left interval $(\frac{1}{2}, 1]$.

3.3. Solution of nonlinear boundary value problem by $\alpha - p\ DTM$ and comparision with the exact solution

Let us consider the following illustrative nonlinear boundary value problem,

$$u'' + (u')^2 = 0, \qquad x \in [0,1] \tag{35}$$

$$u(0) = 1, \quad u(1) = 0. \tag{36}$$

By applying the α −p DTM to the nonlinear differential Equation (18) we have

$$(T_\alpha(0,1))_{m+2}(u) = \frac{-1}{(m+1)(m+2)}[\sum_{r=0}^{m} \alpha(r+1)(T_0\ (u))_{r+1}\ (m-r+1)(T_0\ (u))_{m-r+1}$$
$$+(1-\alpha)\ (r+1)\ (T_1\ (u))_{r+1}(m-r+1)(T_1\ (u))_{m-r+1}] \tag{37}$$

Using the boundary conditions Equation (36) we have $(T_0(u))_0 = 1, \quad (T_1(u))_0 = 0$
Denoting $(T_0(u))_1$ and $(T_1(u))_1$ by A and B respectively and then substituting in the iterative Equation (37) we get

$$(T_\alpha(0,1))_2 = \alpha\left(\frac{-A^2}{2}\right) + (1-\alpha)\left(\frac{-B^2}{2}\right), (T_\alpha(0,1))_3 = \alpha\left(\frac{A^3}{3}\right) + (1-\alpha)\left(\frac{B^3}{3}\right),$$
$$(T_\alpha(0,1))_4 = \alpha\left(\frac{-A^4}{4}\right) + (1-\alpha)\left(\frac{-B^4}{4}\right), \qquad (T_\alpha(0,1))_5 = \alpha\left(\frac{A^5}{5}\right) + (1-\alpha)\left(\frac{B^5}{5}\right), \ldots \tag{38}$$

Proceeding the iteration in the similar way we can compute the other terms of the α − p DTM solution. Then we get the following α − p DTM solution in the form of a series. Hence the α −parametrized series solution $u(x, \alpha)$ is evaluated:

$$
\begin{aligned}
u_\alpha(x) = \sum_{k=0}^{\infty} (T_\alpha(0,1))_k (x - x_\alpha)^k \\
&= \alpha + (\alpha A + (1-\alpha)B)\big(x - (1-\alpha)\big)^1 \\
&+ \left[\alpha\left(\frac{-A^2}{2}\right) + (1-\alpha)\left(\frac{-B^2}{2}\right)\right]\big(x-(1-\alpha)\big)^2 \\
&+ \left[\alpha\left(\frac{A^3}{3}\right) + (1-\alpha)\left(\frac{B^3}{3}\right)\right]\big(x-(1-\alpha)\big)^3 + \cdots
\end{aligned} \tag{39}
$$

Substituting N-th partial sum of this series into the boundary conditions Equation (36) we can obtain the parameters A and B. It is easy to verify that the exact solution of the nonlinear boundary value problem Equations (35), (36) is

$$u(x) = 1 + \ln\left(1 - \frac{e-1}{e}x\right). \tag{40}$$

In the following figure we have compared the approximate $\alpha - \mathrm{p}$ DTM solution

$$u_{\alpha,N}(x) = \sum_{k=0}^{N} (T_\alpha(0,1))_k (x - (1-\alpha))^k \tag{41}$$

with the exact solution defined by Equation (40) for $N = 20$ and $\alpha = \frac{1}{1000}$.

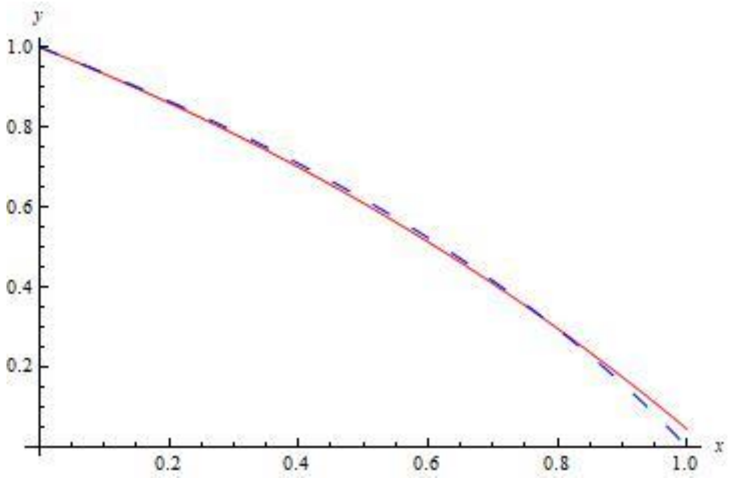

Figure 4. Comparison of the exact solution (blue line) and the α −parameterized DT solution for $\alpha = \frac{1}{1000}$ (red line).

4. Discussion and Conclusion

In this work, we proposed a new generalization of the classical DTM, which we called as α −parameterized DTM ($\alpha - \mathrm{p}$ DTM). Then by applying $\alpha - \mathrm{p}$ DTM we get an approximate $\alpha - \mathrm{p}$ DTM solutions of the third order linear boundary value transmission problem and the second order non-linear boundary value problem. The results obtained are illustrated graphically in Figures 3 and 4. Moreover the approximate $\alpha - \mathrm{p}$ DTM solution of the nonlinear BVP were compared graphically with the exact solution of the same problem. Note that the proposed $\alpha - \mathrm{p}$ DTM is reduced to the classical DTM for the special cases $\alpha = 0$ and $\alpha = 1$, so the proposed $\alpha - \mathrm{p}$ DTM is the generalization of the classical DTM.

References

Ayaz, F. (2004). Solution of the system of differential equations by differential transform method. *Applied Mathematics and Computation*, 147(2), 547-567. doi:10.1016/S0096-3003(02)00794-4

Bekiryazici, Z., Merdan, M., & Kesemen, T. (2021). Modification of the random differential transformation method and its applications to compartmental models. *Communications in Statistics-Theory and Methods*, 50(18), 4271-4292. doi:10.1080/03610926.2020.1713372

Biazar, J., & Eslami M. (2010). Analytic solution for telegraph equation by differential transform method. *Physics Letters A*, 374(29), 2904-2906. doi:10.1016/j.physleta.2010.05.012

Biswas, S., & Roy, T. K. (2018). Generalization of Seikkala derivative and differential transform method for fuzzy Volterra integro-differential equations. *Journal of Intelligent & Fuzzy Systems*, 34(4), 2795-2806. doi:10.3233/JIFS-17958

Cakir, M., & Arslan, D. (2015). The Adomian decomposition method and the differential transform method for numerical solution of multi-pantograph delay differential equations. *Applied Mathematics*, 6(8), 1332. doi:10.4236/am.2015.68126

Elsaid, A. & Helal, S. M. (2020). A new algorithm for computing the differential transform in nonlinear two-dimensional partial differential equations. *Journal of King Saud University-Science*, 32(1), 858-861. doi:10.1016/j.jksus.2019.04.004

Ghazaryan, D., Burlayenko, V. N., Avetisyan, A., & Bhaskar, A. (2018). Free vibration analysis of functionally graded beams with non-uniform cross-section using the differential transform method. *Journal of Engineering Mathematics*, 110(1), 97-121. doi:10.1007/s10665-017-9937-3

Hussein Msmali, A., Alotaibi, A. M., El-Moneam, M. A., Badr, B. S., & Ahmadini, A. A. H. (2021). A general scheme for solving systems of linear first-order differential equations based on the differential transform method. *Journal of Mathematics*, 2021, 8839201. doi:10.1155/2021/8839201

Jena, S. K., & Chakraverty, S. (2018). Free vibration analysis of Euler–Bernoulli nanobeam using differential transform method. *International Journal of Computational Materials Science and Engineering*, 7(3), 1850020. doi:10.1142/S2047684118500203

Karakoç, F., & Bereketoğlu, H. (2009). Solutions of delay differential equations by using differential transform method. *International Journal of Computer Mathematics*, 86(5), 914-923. doi:10.1080/00207160701750575

Khudair, A. R., Haddad, S. A. M., & Khalaf, S. L. (2016). Mean square solutions of second-order random differential equations by using the differential transformation method. *Open Journal of Applied Sciences*, 6(4), 287-297. doi:10.4236/ojapps.2016.64028

Lal, R., & Ahlawat, N. (2015). Axisymmetric vibrations and buckling analysis of functionally graded circular plates via differential transform method. *European Journal of Mechanics-A/Solids*, 52, 85-94. doi:10.1016/j.euromechsol.2015.02.004

Liu, J. G., Yang, X. J., Feng, Y. Y., & Cui, P. (2020). On the (N+ 1)-dimensional local fractional reduced differential transform method and its applications. *Mathematical Methods in the Applied Sciences*, 43(15), 8856-8866. doi:10.1002/mma.6577

Mirzaaghaian, A., & Ganji, D. D. (2016). Application of differential transformation method in micropolar fluid flow and heat transfer through permeable walls, *Alexandria Engineering Journal*, 55(3), 2183-2191. doi:10.1016/j.aej.2016.06.011

Mohamed, M. S., & Gepreel, K. A. (2017). Reduced differential transform method for nonlinear integral member of Kadomtsev–Petviashvili hierarchy differential equations. *Journal of the Egyptian Mathematical Society*, 25(1), 1-7. doi:10.1016/j.joems.2016.04.007

Mukhtarov, O., Çavuşoğlu, S., & Olğar, H. (2019). Numerical solution of one boundary value problem using finite difference method. *Turkish Journal of Mathematics and Computer Science*, 11, 85-89.

Mukhtarov, O. S., & Yücel, M. (2020). A study of the eigenfunctions of the singular Sturm–Liouville problem using the analytical method and the decomposition technique. *Mathematics*, 8(3), 415. doi:10.3390/math8030415

Mukhtarov, O. S., Yücel, M., & Aydemir, K. (2020). Treatment a new approximation method and its justification for sturm–Liouville problems. *Complexity*, 2020, 8019460. doi:10.1155/2020/8019460

Mukhtarov, O., Yücel, M., & Aydemir, K. (2021). A new generalization of the differential transform method for solving boundary value problems. *Journal of New Results in Science*, 10(2), 49-58.

Nazari, D., & Shahmorad, S. (2010). Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions. *Journal of Computational and Applied Mathematics*, 234(3), 883-891. doi:10.1016/j.cam.2010.01.053

Odibat, Z., Momani, S., & Ertürk, V. S. (2008). Generalized differential transform method: Application to differential equations of fractional order. *Applied Mathematics and Computation*, 197(2), 467-477. doi:10.1016/j.amc.2007.07.068

Pratiksha, A. G. (2019). A cumulative study on differential transform method. *International Journal of Mathematical, Engineering and Management Sciences*, 4(1), 170-181. doi:10.33889/IJMEMS.2019.4.1-015

Sepasgozar, S., Faraji, M., & Valipour, P. (2017). Application of differential transformation method (DTM) for heat and mass transfer in a porous channel. *Propulsion and Power Research*, 6(1), 41-48. doi:10.1016/j.jppr.2017.01.001

Sowmya, G., & Gireesha, B. J. (2022). Analysis of heat transfer through different profiled longitudinal porous fin by differential transformation method. *Heat Transfer*, 51(2), 2165-2180. doi:10.1002/htj.22394

Ünal, E., & Gökdoğan, A. (2017). Solution of conformable fractional ordinary differential equations via differential transform method. *Optik*, 128, 264-273. doi:10.1016/j.ijleo.2016.10.031

Zhou, J. K. (1986). Differential transformation and its applications for electrical circuits. Wuhan, China: Huazhong University Press.