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A New Analytic Solution Method for a Class of Generalized Riccati Differential Equations

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Abstract

We give a useful and practicable solution method for the general Riccati differential equation of the form $w'(x) = p(x) + q(x)w(x) + r(x)w^2(x)$. In order to get the general solution many authors have been interested this type equation. They show that if there exists some relation about the coefficients $p(x)$, $q(x)$, and $r(x)$ then the general solution of this equation can be given in a closed form. We also determine some relations between these coefficients and find the general solutions to the given equation. Finally, we give some examples to illustrate the importance of the presented method.

1. Introduction

The general Riccati differential equation (GRDE) is a well-known first-order nonlinear type of differential equation that arises not only a whole range of mathematics but also physics and have many applications in different areas of science. Riccati differential equation was named after the Italian mathematician Jacopo Francesco Riccati [1]. In particular, the GRDE is given by

$$w'(x) = p(x) + q(x)w(x) + r(x)w^2(x), \tag{1.1}$$

where we assume that $w,\ p,\ q,\ r \in C(\mathbb{R},\mathbb{R})$ are real functions and the integral $\int q(x)dx$ exists. In case $r(x) = 0$, the GRDE reduces a first-order linear ordinary differential equation of the form

$$w'(x) = p(x) + q(x)w(x)$$

and its general solution can be expressed in closed form as

$$w(x) = \exp\left(\int q(x)dx\right)\left[\int p(x)\exp\left(-\int q(x)dx\right)dx + \text{ constant}\right].$$

Similarly in case $p(x) = 0$, the GRDE reduces a first-order ordinary differential equation and called Bernoulli differential equation of the form

$$w'(x) = q(x)w(x) + r(x)w^2(x),$$

and general solution can be expressed in closed form as

$$w(x) = \exp\left(-\int q(x)dx\right)\left[-\int r(x)\exp\left(\int q(x)dx\right)dx + \text{ constant}\right]^{-1}.$$

Thus, in this paper, we consider the case $p(x)r(x) \neq 0$ for all x. Because the GRDE has many application areas in fields of applied science, the solutions of the GRDE play a significant role see [2]. For instance, optimal control, random processes, diffusion problems, stochastic

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realization theory, robust stabilization, network synthesis, and more recently, financial mathematics [3–5], Kalman filtering systems such as orbiting satellites [3,6]. Additionally, it is well known that the GRDE of the form

$$w'(x)+p(x)+w^2(x)=0 \tag{1.2}$$

plays an important role in studying qualitative analysis of the second order linear differential equation of the form

$$\phi''(x)+p(x)\phi(x)=0. \tag{1.3}$$

In fact, if Eq. (1.3) has a positive solution $\phi(x)$ on an interval I, then the function $w(x)=\phi'(x)/\phi(x)$ is a solution of Eq. (1.2). The substitution $w(x)=\phi'(x)/\phi(x)$ for the Eq. (1.3) is embedded in the Picone identity and it can be considered a link between the so-called Riccati technique and variational technique in the oscillation theory of Eq. (1.3) [7–9].
It is well known that there is not a general method for solving method for the GRDE, but recently, there have been several papers which have presented methods for solving of the GRDE under certain conditions [10–14].
Let $w_0=w_0(x)$ be a particular solution of the GRDE, then the general solution of the Eq. (1.1) can be written as:

$$w(x)=w_0(x)+\Phi(x)\left[C-\int r(x)\Phi(x)\,dx\right]^{-1},$$

where

$$\Phi(x)=\exp\left(\int[2r(x)w_0(x)+q(x)]\,dx\right),$$

and C is an arbitrary constant, see [12].
The aim of this paper is to find a general solution to the GRDE by using the relations between the coefficients $p(x)$, $q(x)$, and $r(x)$ for which the Eq. (1.1) can be solved in closed form.
It is well known that if $r(x)\neq 0$ for all x, the substitution

$$w(x)=-\frac{y'(x)}{r(x)y(x)} \tag{1.4}$$

into the GRDE, Eq. (1.1) can always be reduced to the second-order linear ordinary differential equation of the form

$$y''(x)-\left(\frac{r'(x)}{r(x)}+q(x)\right)y'(x)+p(x)r(x)y(x)=0. \tag{1.5}$$

As we mentioned above, in general, for any real functions $p(x)$, $q(x)$, and $r(x)$ the Eq. (1.1) cannot be solved in closed form. However, if there exist some specified relations between these coefficient functions, then Eq. (1.1) can be transformed into a second order linear ordinary differential equation, which can be easily solved, for example see [15–17].
In this paper, we treat a special case of the GRDE Eq. (1.1) where the functions $p(x)$ and $r(x)$ are not identically zero for all x. More precisely, we consider the case where the functions have the following relations for all $x\geq x_0$

$$r(x)\exp\left(\int q(x)\,dx\right)=\alpha,\quad -p(x)\exp\left(-\int q(x)\,dx\right)=\beta,$$

where α and β are some real constants. We shall also use the obtained results to provide the solution of the linear second order ordinary differential equation corresponding to the considered GRDE. As far as the author is aware, the explicit solution of the class of ordinary differential equations considered here does not exist in the literature.

2. Solution Method

In order to be able to solve the GRDE there are some concepts which need to be introduced as given in [18].
In this section, we give the general solution of a class of GRDE. The following theorem gives a relationship between the GRDE and the homogeneous systems of first order differential equations.

Theorem 2.1. *Assume that p, q, and r are real functions and the integral $\int q(x)\,dx$ exists. Then the GRDE Eq. (1.1) has a solution $u(x)$, without zeros for $x\geq x_0$ iff the homogeneous system of first order differential equations*

$$z'(x)=\mathbf{A}(x)\cdot z(x) \tag{2.1}$$

has a solution $z(x)$. Where $z(x)=\begin{pmatrix} y(x)\\ \xi(x)\end{pmatrix}$ and

$$\mathbf{A}(x)=\begin{pmatrix} 0 & r(x)\exp\left(\int q(x)\,dx\right)\\ -p(x)\exp\left(-\int q(x)\,dx\right) & 0\end{pmatrix}. \tag{2.2}$$

Proof. Let $w(x)$ be a solution of Eq. (1.1) and let $w(x) = -\frac{y'(x)}{r(x)y(x)}$. Then $y(x)$ satisfies the second order linear differential equation Eq. (1.5)

$$y''(x) - \left(\frac{r'(x)}{r(x)} + q(x)\right) y'(x) + p(x)\, r(x)\, y(x) = 0.$$

Multiplying Eq. (1.5) by the integrating factor $(r(x))^{-1}\exp\left(-\int q(x)\,dx\right)$ for the first two term, we obtain

$$\left[(r(x))^{-1}\exp\left(-\int q(x)\,dx\right) y'(x)\right]' + p(x)\exp\left(-\int q(x)\,dx\right) y(x) = 0.$$

Hence, if we let $(r(x))^{-1}\exp\left(-\int q(x)\,dx\right) y'(x) = \xi(x)$ and $z(x) = \begin{pmatrix} y(x) \\ \xi(x) \end{pmatrix}$, then $z(x)$ is a solution of the homogeneous system of first order differential equations, Eq. (2.1) with $\mathbf{A}(x)$ is given (2.2). □

The following theorem summarizes the present study:

Theorem 2.2. *Assume that $p(x)$, $q(x)$, and $r(x)$ hold the relations*

$$r(x)\exp\left(\int q(x)\,dx\right) = \alpha, \quad -p(x)\exp\left(-\int q(x)\,dx\right) = \beta. \tag{2.3}$$

Then the general solution of Eq. (1.1) is given

$$w(x) = \begin{cases} \dfrac{\sqrt{\alpha\beta}}{r(x)}\left(\dfrac{1 - C\exp\left(2\sqrt{\alpha\beta}x\right)}{1 + C\exp\left(2\sqrt{\alpha\beta}x\right)}\right) & ;\ if\ \alpha\beta > 0, \\ \dfrac{\sqrt{-\alpha\beta}}{r(x)}\left(\dfrac{\sin\left(\sqrt{-\alpha\beta}x - C\cos\left(\sqrt{-\alpha\beta}x\right)\right)}{\cos\left(\sqrt{-\alpha\beta}x\right) + C\sin\left(\sqrt{-\alpha\beta}x\right)}\right) & ;\ if\ \alpha\beta < 0, \end{cases}$$

where C is any real constant.

Proof. If the conditions of (2.3) are fulfilled, the homogeneous system of first order differential equations Eq. (2.1) becomes a first order homogeneous system with constant coefficients

$$z'(x) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \cdot z(x). \tag{2.4}$$

Then, the eigenvalues of the coefficient matrix $\mathbf{A} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ are $r_1 = -\sqrt{\alpha\beta}$ and $r_2 = \sqrt{\alpha\beta}$. If $\alpha\beta > 0$ the eigenvalues of the coefficient matrix $\mathbf{A}$ are real constants such as $r_1 = -\sqrt{\alpha\beta}$ and $r_2 = \sqrt{\alpha\beta}$. Similarly if $\alpha\beta < 0$ the eigenvalues of the coefficient matrix $\mathbf{A}$ are complex constants such as $r_1 = -i\sqrt{-\alpha\beta}$ and $r_2 = i\sqrt{-\alpha\beta}$. In case, $\alpha\beta > 0$, $\Phi(x) = \begin{pmatrix} \exp\left(-\sqrt{\alpha\beta}x\right) & \exp\left(\sqrt{\alpha\beta}x\right) \\ -\sqrt{\frac{\beta}{\alpha}}\exp\left(-\sqrt{\alpha\beta}x\right) & \sqrt{\frac{\beta}{\alpha}}\exp\left(\sqrt{\alpha\beta}x\right) \end{pmatrix}$ is a fundamental matrix of Eq. (2.4). Then general solution of Eq. (2.4)

$$z(x) = \Phi(x)\cdot\mathbf{C} = \begin{pmatrix} \exp\left(-\sqrt{\alpha\beta}x\right) & \exp\left(\sqrt{\alpha\beta}x\right) \\ -\sqrt{\frac{\beta}{\alpha}}\exp\left(-\sqrt{\alpha\beta}x\right) & \sqrt{\frac{\beta}{\alpha}}\exp\left(\sqrt{\alpha\beta}x\right) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where c_1 and c_2 are real constants. Thus the general solution of Eq. (1.5) is

$$y(x) = c_1\exp\left(-\sqrt{\alpha\beta}x\right) + c_2\exp\left(\sqrt{\alpha\beta}x\right).$$

Therefore,

$$w(x) = -\frac{y'(x)}{r(x)\,y(x)} = \frac{\sqrt{\alpha\beta}}{r(x)}\left(\frac{c_1\exp\left(-\sqrt{\alpha\beta}x\right) - c_2\exp\left(\sqrt{\alpha\beta}x\right)}{c_1\exp\left(-\sqrt{\alpha\beta}x\right) + c_2\exp\left(\sqrt{\alpha\beta}x\right)}\right).$$

When $c_1 = 0$, the function $w(x) = -\frac{\sqrt{\alpha\beta}}{r(x)}$ is a solution of the Eq. (1.1). When $c_1 \neq 0$ we can divide the numerator and denominator by $c_1 e^{-\sqrt{\alpha\beta}x}$ to get that

$$w(x) = \frac{\sqrt{\alpha\beta}}{r(x)}\left(\frac{1 - C\exp\left(2\sqrt{\alpha\beta}x\right)}{1 + C\exp\left(2\sqrt{\alpha\beta}x\right)}\right),$$

is general solution of the Eq. (1.1), where $C = \dfrac{c_2}{c_1}$ is any real constant. Thus the proof of the first part is complete. Similarly if $\alpha\beta < 0$

$$\Phi(x) = \begin{pmatrix} \cos\left(\sqrt{-\alpha\beta}x\right) & \sin\left(\sqrt{-\alpha\beta}x\right) \\ -\dfrac{\sqrt{-\alpha\beta}}{\beta}\sin\left(\sqrt{-\alpha\beta}x\right) & \dfrac{\sqrt{-\alpha\beta}}{\alpha}\cos\left(\sqrt{-\alpha\beta}x\right) \end{pmatrix}$$

is a fundamental matrix of Eq. (2.4). Then general solution of Eq. (2.4)

$$z(x)=\Phi(x)\cdot\mathbf{C}=\begin{pmatrix}\cos\left(\sqrt{-\alpha\beta}x\right) & \sin\left(\sqrt{-\alpha\beta}x\right)\\ -\frac{\sqrt{-\alpha\beta}}{\beta}\sin\left(\sqrt{-\alpha\beta}x\right) & \frac{\sqrt{-\alpha\beta}}{\alpha}\cos\left(\sqrt{-\alpha\beta}x\right)\end{pmatrix}\cdot\begin{pmatrix}c_1\\ c_2\end{pmatrix},$$

where c_1 and c_2 are real constants. Thus the general solution of Eq. (1.5) is

$$y(x)=c_1\cos\left(\sqrt{-\alpha\beta}x\right)+c_2\sin\left(\sqrt{-\alpha\beta}x\right).$$

Therefore,

$$w(x)=-\frac{y'(x)}{r(x)y(x)}=\frac{\sqrt{-\alpha\beta}}{r(x)}\left(\frac{c_1\sin\left(\sqrt{-\alpha\beta}x\right)-c_2\cos\left(\sqrt{-\alpha\beta}x\right)}{c_1\cos\left(\sqrt{-\alpha\beta}x\right)+c_2\sin\left(\sqrt{-\alpha\beta}x\right)}\right).$$

When $c_1=0$, the function $w(x)=-\frac{\sqrt{-\alpha\beta}}{r(x)}\tan\left(\sqrt{-\alpha\beta}x\right)$ is a solution of the Eq. (1.1). When $c_1\neq 0$

$$w(x)=\frac{\sqrt{\alpha\beta}}{r(x)}\left(\frac{\sin\left(\sqrt{-\alpha\beta}x\right)-C\cos\left(\sqrt{-\alpha\beta}x\right)}{\cos\left(\sqrt{-\alpha\beta}x\right)+C\sin\left(\sqrt{-\alpha\beta}x\right)}\right),$$

is general solution of the Eq. (1.1), where C is any real constant. Thus the proof is complete. □

Remark 2.3. *If the functions $p(x)$, $q(x)$, and $r(x)$ are constants such as $p(x)=a$, $q(x)=b$, and $r(x)=c$. Then, the conditions of (2.3) are fulfilled as $\alpha=c$ and $\beta=-a$ for $b=0$ and we can use the Theorem 2.2 for the general solution of Eq. (1.1). But when $b\neq 0$, the conditions of (2.3) not satisfied. In general case $a,b,c\in\mathbb{R}$ and $ac\neq 0$, the Eq. (1.1) becomes a first-order separable ordinary differential equation which is defined by*

$$\frac{dw}{a+bw+cw^2}=dx.$$

Based on the integral involving the rational algebraic functions of the form

$$\int\frac{dw}{a+bw+cw^2}=\begin{cases}\frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2cw+b}{\sqrt{4ac-b^2}}\right) & ;\ if\ 4ac-b^2>0,\\ \frac{2}{\sqrt{b^2-4ac}}\ln\left|\frac{2cw+b-\sqrt{b^2-4ac}}{2cw+b+\sqrt{b^2-4ac}}\right| & ;\ if\ 4ac-b^2<0,\\ -\frac{2}{2cw+b} & ;\ if\ 4ac-b^2=0,\end{cases}$$

in view of this, the general solution of Eq. (1.1) is given in a closed form by

$$w(x)=\begin{cases}\frac{1}{2c}\left[-b+\sqrt{4ac-b^2}\tan\left(\frac{1}{2}\sqrt{4ac-b^2}x+C\right)\right] & ;\ if\ 4ac-b^2>0,\\ \frac{\sqrt{b^2-4ac}}{2c}\left(\frac{1+C\exp\left(\frac{\sqrt{b^2-4ac}}{2}x\right)}{1-C\exp\left(\frac{\sqrt{b^2-4ac}}{2}x\right)}\right) & ;\ if\ 4ac-b^2<0,\\ -\frac{1}{2c}\left(b+\frac{2}{x+C}\right) & ;\ if\ 4ac-b^2=0,\end{cases}$$

where C is an arbitrary constant.

3. Some Examples

Here, we illustrate some examples to consider some special cases. In these examples, we assume that the above conditions are satisfied and the general solutions of the GRDE are obtained easily.

Example 3.1. *Consider the first-order nonlinear differential equation for $x\geq x_0>0$*

$$w'(x)=\frac{4}{x^2}-\frac{2}{x}w(x)+x^2w^2(x). \tag{3.1}$$

For this equation the conditions of (2.3) are satisfied with $p(x)=\frac{4}{x^2}$, $q(x)=-\frac{2}{x}$, and $r(x)=-x^2$. Thus, by Theorem 2.2 $\alpha\beta=-4<0$ and the general solution of Eq. (3.1) is obtained as

$$w(x)=\frac{2}{x^2}\left(\frac{\sin 2x-C\cos 2x}{\cos 2x+C\sin 2x}\right),$$

where C is an arbitrary constant.

Example 3.2. *Consider the first-order nonlinear differential equation*

$$w'(x)=\frac{9e^{x\arctan x}}{\sqrt{x^2+1}}+(\arctan x)\,w(x)+\left(4e^{-x\arctan x}\sqrt{x^2+1}\right)w^2(x). \tag{3.2}$$

For this equation the conditions of (2.3) *are satisfied with* $p(x)=\frac{9e^{x\arctan x}}{\sqrt{x^2+1}}, q(x)=\arctan x$, *and* $r(x)=4e^{-x\arctan x}\sqrt{x^2+1}$. *Thus, by Theorem* 2.2 $\alpha\beta=-36<0$ *and the general solution of Eq.* (3.2) *is obtained as*

$$w(x)=\frac{3e^{x\arctan x}}{2\sqrt{x^2+1}}\left(\frac{\sin 6x-C\cos 6x}{\cos 6x+C\sin 6x}\right),$$

where C *is an arbitrary constant.*

Example 3.3. *Consider the first-order nonlinear differential equation for* $x\geq x_0>0$

$$w'(x)=x^xe^{-x}+(\ln x)\,w(x)-\left(x^{-x}e^x\right)w^2(x). \tag{3.3}$$

Note that the conditions of (2.3) *are satisfied with* $\alpha=\beta=-1$, $p(x)=x^xe^{-x}$, $q(x)=\ln x$, *and* $r(x)=-x^{-x}e^x$. *Thus, by Theorem* 2.2 $\alpha\beta=1>0$ *and general solution of Eq.* (3.3) *is*

$$w(x)=-x^xe^{-x}\left(\frac{1-Ce^{2x}}{1+Ce^{2x}}\right),$$

where C *is any constant.*

Example 3.4. *Consider the first-order nonlinear differential equation*

$$w'(x)=1+5w(x)+9w^2(x). \tag{3.4}$$

For this equation $p(x)=1$, $q(x)=5$, *and* $r(x)=9$ *are constant functions and conditions of* (2.3) *not satisfied. Thus, we can not use the Theorem* 2.2 *for the general solution of the Eq.* (3.4). *But we can use the Remark* 2.3 *for the general solution of equation, Eq.* (3.4) *and the general solution obtained as*

$$w(x)=\frac{1}{6}\left(\frac{1+C\exp\left(\frac{3}{2}x\right)}{1-C\exp\left(\frac{3}{2}x\right)}\right),$$

where $4ac-b^2=-9<0$ *and* C *is any real constant.*

Example 3.5. *Consider the first-order nonlinear differential equation*

$$w'(x)=1+4w(x)+4w^2(x). \tag{3.5}$$

If we use the Remark 2.3 *for the equation Eq.* (3.5) *we get the general solution as*

$$w(x)=-\frac{1}{4}\left(2+\frac{1}{x+C}\right),$$

where $4ac-b^2=0$ *and* C *is any real constant.*

4. Conclusion

In this paper, we have obtained the general solution of a class of first-order nonlinear ordinary differential equation, which called GRDE. We have converted the GRDE into a homogeneous system of first-order differential equations. In order to do this we use two well-known transformations as explained above. The first transformation converts the nonlinear first-order ordinary differential equation Eq. (1.1) to a linear second-order ordinary differential equation Eq. (1.5). The second one converts Eq. (1.5) to a homogeneous system of first order differential equations Eq. (2.1). Then, by using the fact of the Section 2, we give general solution of a particular class of GRDE. Examples were given here for each case demonstrate the present method.

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