CONSTRUCTIVE MATHEMATICAL ANALYSIS
5 (2022), No. 3, pp. 134-140
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

On the Poisson equation in exterior domains

WERNER VARNHORN*

ABSTRACT. We construct a solution of the Poisson equation in exterior domains $\Omega \subset \mathbb{R}^n,\ n \geq 2$, in homogeneous Lebesgue spaces $L^{2,q}(\Omega), ; 1 < q < \infty$, with methods of potential theory and integral equations. We investigate the corresponding null spaces and prove that its dimensions are equal to $n+1$ independent of q.

Keywords: Poisson equation, potential theory, homogeneous Lebesgue spaces.

2020 Mathematics Subject Classification: 31B10, 31B30, 35C15, 35J05.

1. INTRODUCTION

Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be an exterior domain with a smooth boundary ∂G of class C^2. We consider Poisson's equation concerning some scalar function u:

$$-\Delta u = f \text{ in } G,\ u_{|\partial G} = \Phi. \tag{1.1}$$

Here f is given in G and Φ is the boundary value prescribed on ∂G. As usual, Δ denotes the Laplacian in $\mathbb{R}^n$.

It is well-known that in unbounded domains the treatment of partial differential equations causes special difficulties, and that the usual Sobolev spaces $W^{m,q}(G)$ are not adequate in this case: Even for the Laplacian in $\mathbb{R}^n$ we find [6] that the operator $\Delta : W^{m,q}(\mathbb{R}^n) \to W^{m-2,q}(\mathbb{R}^n)$ is not a Fredholm operator in general, as it is in the case of bounded domains [16]. Thus in exterior domains, the equations (1.1) have mostly been studied in connection with weight functions: Either (1.1) has been solved in weighted Sobolev spaces directly [7, 12, 14] or it has first been multiplied by some weights and then been solved in standard Sobolev spaces [17].

It is the aim of the present note to prove the solvability of (1.1) in homogeneous spaces $L^{2,q}(G)$ $(1 < q < \infty)$ of the following type [5, 11]: Let $L^q(G)$ be the space of functions defined almost everywhere in G such that the norm

$$\|f\|_{q,G} = \left(\int_G |f(x)|^q \,\mathrm{d}x \right)^{1/q}$$

is finite. Then $L^{2,q}(G)$ is the space of all functions being locally in $L^q(\overline{G})$ and having all second order distributional derivatives in $L^q(G)$. We show that for f given in $L^q(G)$ and some boundary value $\Phi \in W^{2-1/q,q}(\partial G)$ (see the notations below) there exists always a solution $u \in L^{2,q}(G)$. Concerning the uniqueness of this solution we prove that the space of all $u \in L^{2,q}(G)$ satisfying (1.1) with $f = 0$ and $\Phi = 0$ has the dimension $n+1$, independent of q. This result also holds for the case $n = 2$.

Received: 14.06.2022; Accepted: 02.08.2022; Published Online: 12.08.2022
*Corresponding author: Werner Varnhorn; varnhorn@mathematik.uni-kassel.de
DOI: 10.33205/cma.1143800

Throughout this paper $G \subset \mathbb{R}^n$ $(n \geq 2)$ is an exterior domain, i.e. a domain whose complement is compact. Let $\overline{G}$ denote its closure in $\mathbb{R}^n$ and ∂G its boundary, which we assume to be of class C^2 [1, p. 67].

In the following, all function spaces contain real valued functions. Let $D \subset \mathbb{R}^n$ be any domain with a compact boundary ∂D of class C^2, or let $D = \mathbb{R}^n$. Besides the spaces $L^q(D)$ we need the well-known function spaces $C^\infty(D)$, $C_0^\infty(D)$, and the space $C_0^\infty(\overline{D})$, containing the restrictions $f_{|\overline{D}}$ of functions $f \in C_0^\infty(\mathbb{R}^n)$.

We call a function u locally in $L^q(\overline{D})$ $(1 < q < \infty)$ and write $u \in L^q_{\mathrm{loc}}(\overline{D})$ if $u \in L^q(D \cap B)$ for every ball $B \subset \mathbb{R}^n$. Note that this space does not coincide with the usual space $L^q_{\mathrm{loc}}(D)$ in general (except for $D = \mathbb{R}^n$).

By $W^{m,q}(D)$ $(m = 0, 1, 2; W^{0,q}(D) = L^q(D))$ we mean the usual Sobolev space of functions u such that $D^\alpha u \in L^q(D)$ for all multiindices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n = \{0, 1, \dots\}^n$ with $\alpha_1 + \dots + \alpha_n \leq m$ [1]. Here we use

$$D^\alpha u = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} u,\ D_i = \partial/\partial x_i\ (i = 1, \dots, n;\ x = (x_1, \dots, x_n) \in \mathbb{R}^n).$$

The spaces $W^{m,q}_{\mathrm{loc}}(D)$ and $W^{m,q}_{\mathrm{loc}}(\overline{D})$ are defined analogously.

We need the fractional order space $W^{2-1/q,q}(\partial D)$, which contains the trace $u_{|\partial D}$ of all $u \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^n)$ [1, p. 216]. The norm in $W^{2-1/q,q}(\partial D)$ is denoted by $\| \cdot \|_{2-1/q,q,\partial D}$.

The term $\nabla u = (D_j u)_{j=1,\dots,n}$ represents the gradient of u and $\nabla^2 u = (D_i D_j u)_{i,j=1,\dots,n}$ means the system of all second order derivatives of u. For these terms we define the seminorms

$$\|\nabla u\|_{q,D} = \left(\sum_{k=1}^n \|D_k u\|_{q,D}^q \right)^{1/q},\ \|\nabla^2 u\|_{q,D} = \left(\sum_{j,k=1}^n \|D_j D_k u\|_{q,D}^q \right)^{1/q},$$

and introduce for $m = 1, 2$ and $1 < q < \infty$ the homogeneous spaces

$$L^{m,q}(D) = \left\{ u \in L^q_{\mathrm{loc}}(\overline{D}) \mid \|\nabla^m u\|_{q,D} < \infty \right\}. \tag{1.2}$$

Finally, concerning the norms and seminorms, we sometimes omit the domain of definition if it is obvious and use $\| \cdot \|_q$ or $\| \cdot \|_{2-1/q,q}$ instead of $\| \cdot \|_{q,G}$ or $\| \cdot \|_{2-1/q,q,\partial G'}$ for example.

2. POTENTIAL THEORY

Besides the Poisson equation (1.1) we also consider the special case of Laplace' equation with Dirichlet boundary condition

$$-\Delta u = 0 \text{ in } G, \quad u_{|\partial G} = \Phi. \tag{2.3}$$

These equations have mostly been studied with methods of potential theory (see for example [8, 15]). We collect some well-known facts in this section.

Let E_n $(n \geq 2)$ in the following denote the fundamental solution of the Laplacian such that $-\Delta E_n(x) = \delta(x)$ where δ is Dirac's distribution in $\mathbb{R}^n$. It is well-known that

$$E_2(x) = -\frac{\ln|x|}{\omega_2}\ (n = 2), \quad E_n(x) = \frac{|x|^{2-n}}{(n-2)\omega_n}\ (n \geq 3), \tag{2.4}$$

where ω_n is the area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ $(n \geq 2)$.

Proposition 2.1. *Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be an exterior domain with boundary ∂G of class C^2, and let $\Phi \in W^{2-1/q,q}(\partial G)$ be given $(1 < q < \infty)$. Then there exists a unique function $u \in L^{2,q}(G)$ satisfying*

(2.3) *in G, if we require the following decay conditions as $|x| \to \infty$:*

$$u(x) - a\ln|x| = 0(1)\ (n=2), \quad u(x) = 0(|x|^{2-n})\ (n \geq 3), \tag{2.5}$$
$$\nabla^m u(x) = O(|x|^{2-n-m})\ (n \geq 2;\ m = 1,2).$$

Here $a \in \mathbb{R}$ is a fixed prescribed constant.

Proof. To prove uniqueness let $u = u^1 - u^2$ be the difference of two solutions u^1 and u^2 with the required decay properties above. Define the bounded domain $G_r = G \cap B_r(0)$ where $B_r(0) \subset \mathbb{R}^n$ denotes an open ball with center at zero and radius r such that $\partial G \subset B_r(0)$. From the local regularity theory we find $D_j u \in L^2_{\text{loc}}(\overline{G})$ $(j = 1, \ldots, n)$. Thus in G_r we may apply Greens first identity, obtaining

$$\int_{G_r} |\nabla u|^2 \,\mathrm{d}x = \int_{\partial B_r} (\partial_N u) u \,\mathrm{d}o, \tag{2.6}$$

because the boundary integral over ∂G vanishes. Here N denotes the outward (with respect to G_r) unit normal vector on the boundary $\partial B_r = \partial B_r(0)$ and $\partial_N u$ is the normal derivative of u. Now do to the decay properties of u, the right hand side in (2.6) tends to zero as $r \to \infty$. This is obvious if $n \geq 3$. For $n = 2$, using the expansion theorem for harmonic functions at infinity [15, p. 523], we find $u(x) = 0(1)$ and $\nabla u(x) = 0(|x|^{-2})$ as $|x| \to \infty$, which implies the assertion above, too. It follows $\nabla u = o$ in G, hence $u = 0$ in G because u vanishes on the boundary ∂G. This proves the uniqueness.

To show the existence of a solution with the required properties we use the boundary integral equations method: Let us define the simple layer potential

$$(E^n\Theta)(x) = \int_{\partial G} E_n(x-y)\Theta(y)\,\mathrm{d}o_y, \ (x \notin \partial G),$$

the double layer potential

$$(D^n\Theta)(x) = -\int_{\partial G} \partial_{N(y)} E_n(x-y)\Theta(y)\,\mathrm{d}o_y \ (x \notin \partial G),$$

and the normal derivative of the simple layer potential

$$(H^n\Theta)(x) = -\int_{\partial G} \partial_{N(x)} E_n(x-y)\Theta(y)\,\mathrm{d}o_y \ (x \notin \partial G).$$

Here and in the following, $N = N(z)$ is the outward (with respect to the bounded domain $G_b = \mathbb{R}^n/\overline{G}$) unit normal vector in $z \in \partial G$, and $\Theta \in W^{2-1/q,q}(\partial G)$ is the unknown source density. Then we have the continuity relation

$$(E^n\Theta)^e = (E^n\Theta)^i = E^n\Theta \quad \text{on } \partial G \tag{2.7}$$

and the jump relations

$$D^n\Theta - (D^n\Theta)^e = (D^n\Theta)^i - D^n\Theta = 1/2\Theta \quad \text{on } \partial G, \tag{2.8}$$

$$H^n\Theta - (H^n\Theta)^e = (H^n\Theta)^i - H^n\Theta = -1/2\Theta \quad \text{on } \partial G. \tag{2.9}$$

The index e stands for the limit from outside, and the index i for the limit from inside. Now let us first assume $n \geq 3$. Following [3, 10] (here for the case of Helmholtz' equation), for the solution of (2.3) we choose in G the ansatz

$$u = D^n\Theta - \alpha E^n(\Theta) \quad (0 < \alpha \in \mathbb{R}).$$

Then by means of (2.7), (2.8) we obtain the second kind Fredholm boundary integral equation

$$\Phi = -1/2\Theta + D^n\Theta - \alpha E^n\Theta \quad \text{on } \partial G \tag{2.10}$$

for the unknown source density $\Theta \in W^{2-1/q,q}(\partial G)$. To see that (2.10) is uniquely solvable for all boundary values $\Phi \in W^{2-1/q,q}(\partial G)$, let $0 \neq \Psi$ be a solution of the homogeneous adjoint integral equation

$$0 = -1/2\Psi + H^n\Psi - \alpha E^n\Psi \quad \text{on } \partial G. \tag{2.11}$$

By (2.7) and (2.9), this implies $\alpha(E^n\Psi)^i = (H^n\Psi)^i = -(\partial_N E^n\Psi)^i$, and Green's first identity yields $\int_{G_b} |\nabla(E^n\Psi)|^2 \, dx = \int_{\partial G}(E^n\Psi)^i(\partial_N E^n\Psi)^i \, do = -\alpha\int_{\partial G}|E^n\Psi|^2 \, do$, hence $E^n\Psi = 0$ in $\overline{G}_b$. This implies $(E^n\Psi)^e = 0$ using (2.7), and the uniqueness statement above yields $E^n\Psi = 0$ in G, too. Thus $E^n\Psi = 0$ in the whole $\mathbb{R}^n$, and we obtain $\Psi = 0$ by (2.9), as asserted. This proves the existence in the case $n \geq 3$.

Now let $n = 2$. As in [9] (for the case of Stokes' equations) we use in G the ansatz

$$u = -a\omega_2 E^2 1 + D^2\Theta - \alpha E^2\Theta^* - \beta b_\Theta \quad (0 < \alpha \in \mathbb{R},\ 0 \neq \beta \in \mathbb{R}).$$

Here $a \in \mathbb{R}$ is the prescribed constant from (2.5), $E^2 1$ is the simple layer potential with constant density $\Psi = 1$,

$$b_\Theta = \int_{\partial G} \Theta(y) \, do_y$$

is some constant, and the source density Θ^* is defined by

$$\Theta^*(x) = \Theta(x) - b_\Theta/(\text{meas}(\partial G)), \tag{2.12}$$

which implies $b_{\Theta^*} = \int_{\partial G} \Psi^*(y) \, do_y = 0$. Note that the decay properties (2.5) are fulfilled in this case. Here again, (2.7) and (2.8) lead to the second kind Fredholm boundary integral equation

$$\Phi + a\omega_2 E^2 1 = -1/2\Theta + D^2\Theta - \alpha E^2\Theta^* - \beta b_\Theta \quad \text{on } \partial G. \tag{2.13}$$

To see that (2.13) has a unique solution $\Theta \in W^{2-1/q,q}(\partial G)$ for all boundary values $\Phi \in W^{2-1/q,q}(\partial G)$ and all $a \in \mathbb{R}$, let $0 \neq \Psi$ solve the homogeneous adjoint integral equation

$$0 = -1/2\Psi + H^2\Psi - \alpha E^2\Psi^* - \beta b_\Psi \quad \text{on } \partial G.$$

Because for any constant $c \in \mathbb{R}$ we have $-1/2c + D^2 c = 0$ [15, p. 511] and $E^2 c^* = 0$ (see (2.12) for the definition of c^*), we find

$$0 = \langle c, -1/2\Psi + H^2\Psi - \alpha E^2\Psi^* - \beta b_\Psi\rangle = -\beta\langle c, b_\Psi\rangle,$$

where here $\langle\psi,\varphi\rangle = \int_{\partial G}\psi(y)\varphi(y) \, do$ denotes the corresponding duality. It follows $b_\Psi = 0$ and $\Psi^* = \Psi$, hence Ψ is a solution of

$$0 = -1/2\Psi + H^2\Psi - \alpha E^2\Psi \quad \text{on } \partial G,$$

too. Now the same arguments as for (2.11) in the case $n \geq 3$ yield the assertion and the proposition is proved. □

3. The Poisson equation

The first theorem ensures the solvability of Poisson's equation (1.1) in the space $L^{2,q}(G)$, defined by (1.2).

Theorem 3.1. *Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be an exterior domain with boundary ∂G of class C^2, and let $1 < q < \infty$. Then for every $f \in L^q(G)$ and $\Phi \in W^{2-1/q,q}(\partial G)$ there exists some $u \in L^{2,q}(G)$ satisfying the Poisson equation (1.1) in G.*

Proof. Setting $f = 0$ in $\mathbb{R}^n/G$ we obtain some function $\tilde{f} \in L^q(\mathbb{R}^n)$ with $\tilde{f}_{|G} = f$ in G. Let $\tilde{f}_i \in C_0^\infty(\mathbb{R}^n)$ denote a sequence such that $\tilde{f}_i \to \tilde{f}$ in $L^q(\mathbb{R}^n)$ as $i \to \infty$. Consider now for fixed i the equation $-\Delta \tilde{u}_i = \tilde{f}_i$ in $\mathbb{R}^n$. We can solve it by convolution with E_n (see (2.4)), obtaining $x \in \mathbb{R}^n$ the representation

$$\tilde{u}_i(x) = (E_n * \tilde{f}_i)(x) = \int_{\mathbb{R}^n} E_n(x-y)\tilde{f}_i(y)\,\mathrm{d}y.$$

Moreover, by the theorem of Calderon-Zygmund [4], for the second order derivatives we obtain the estimate $\|\nabla^2 \tilde{u}_i\|_q \leq c\|\tilde{f}_i\|_q$ with some constant c independent of $i \in \mathbb{N}$, which implies $\|\nabla^2(\tilde{u}_i - \tilde{u}_k)\|_q \to 0$ as $i, k \to \infty$.

Next consider a sequence of open balls $(B_j)_j$ with $B_j \subset B_{j+1} \bigcup_{j=1}^\infty B_j = \mathbb{R}^n$. Let us define the space

$$\mathbb{P} = \{P : x \to P(x) = a + b \cdot x \mid b, x \in \mathbb{R}^n, a \in \mathbb{R}\} \tag{3.14}$$

of linear functions $P : \mathbb{R}^n \to \mathbb{R}$. Then by the generalized Poincaré inequality (compare [11, p. 22] or [13, p. 112]) we obtain for every $v \in L^{2,q}(\mathbb{R}^n)$ the estimate

$$\|v\|_{L^q(B_j)/\mathbb{P}} := \inf_{P \in \mathbb{P}} \|v + P\|_{L^q(B_j)} \leq c_j \|\nabla^2 v\|_{L^q(B_j)^{n^2}} \tag{3.15}$$

with some constants $c_j > 0$. Because $\tilde{u}_i \in L^{2,q}(\mathbb{R}^n)$ we conclude that $(\tilde{u}_i)_i$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{L^q(B_1)/\mathbb{P}}$ on the left hand side of (3.15) for fixed $j = 1$. This implies the existence of linear functions $P_i \in \mathbb{P}$ such that $(\tilde{u}_i + P_i)_i$ is a Cauchy sequence in $L^q(B_1)$. Repeating this argument now for $j = 2$, there exist linear functions $Q_i \in \mathbb{P}$ such that $\tilde{u}_i + Q_i$ is a Cauchy sequence in $L^q(B_2)$, hence in $L^q(B_1)$, because $B_1 \subset B_2$. Thus the difference $(P_i - Q_i)_i$ is a Cauchy sequence in $L^q(B_1)$, and using the representation

$$P_i(x) = \alpha_i + B_i \cdot x,\ Q_i(x) = \gamma_i + \delta_i \cdot x,$$

we obtain that $(\alpha_i - \gamma_i)_i$ and $(\beta_i - \delta_i)_i$ are Cauchy sequences in $\mathbb{R}$ and in $\mathbb{R}^n$, respectively. From this we find that $(P_i - Q_i)_i$ is a Cauchy sequence in $L^q(B_2)$, and thus also $(\tilde{u}_i + P_i)_i = (\tilde{u}_i + Q_i)_i + (P_i - Q_i)_i$. Repeating this procedure it follows that $(\tilde{u}_i + P_i)_i$ is a Cauchy sequence in $L^q(B_j)$ for all $j = 1, 2, \ldots$. Thus we can find some $\tilde{u} \in L^{2,q}(\mathbb{R}^n)$ such that $(\tilde{u}_i + P_i) \to \tilde{u}$ in $L^q_{\text{loc}}(\mathbb{R}^n)$ and $\|\nabla^2(\tilde{u} - \tilde{u}_i)\|_{q,\mathbb{R}^n} \to 0$ as $i \to \infty$. Moreover, $\tilde{u}$ satisfies $-\Delta\tilde{u} = \tilde{f}$ in $\mathbb{R}^n$ and the estimate $\|\nabla^2\tilde{u}\|_q \leq c\|\tilde{f}\|_q$. Since $\tilde{u} \in W^{2,q}_{\text{loc}}(\mathbb{R}^n)$ we conclude from the usual trace theorem [1, p. 217] that $\tilde{u}_{|\partial G} \in W^{2-1/q,q}(\partial G)$. Following Proposition 2.1 there is a function $w \in L^{2,q}(G)$ satisfying the equations

$$-\Delta w = 0 \text{ in } G, \quad w_{|\partial G} = \tilde{u}_{|\partial G} - \Phi,$$

where $\Phi \in W^{2-1/q,q}(\partial G)$ is the prescribed boundary value. Now setting $u = \tilde{u}_{|G} - w$ we obtain the desired solution and the theorem is proved. □

Because functions $u \in L^{2,q}(G)$ have no suitable decay properties at infinity, in general we cannot expect uniqueness for the solution of (1.1) constructed in Theorem 3.1. Thus we consider in G the homogeneous equations and defined the nullspace of (1.1) by

$$N_q(G) = \{u \in L^{2,q}(G) \mid -\Delta u = 0 \text{ in } G,\ u_{|\partial G} = 0\}. \tag{3.16}$$

Theorem 3.2. *Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be an exterior domain with boundary ∂G of class C^2, and let $1 < q < \infty$. Then for the dimension $\dim N_q(G)$ of the nullspace defined in (3.16) we have $\dim N_q(G) = n + 1$ independent of q.*

Proof. Consider the space $\mathbb{P}$ of linear functions defined in (3.14). Because for every $P \in \mathbb{P}$ we have $P(x) = a + b \cdot x$ with some $a \in \mathbb{R}$ and some vector $b \in \mathbb{R}^n$ we find $\dim \mathbb{P} = n + 1$. Let u^P denote the uniquely determined solution of the equation

$$-\Delta u = 0,\ u_{|\partial G} = -P_{|\partial G}$$

with $P \in \mathbb{P}$, according to Lemma 2.1. Here in the case $n = 2$ we require

$$u(x) - a \ln|x| = 0(1) \quad \text{as } |x| \to \infty, \tag{3.17}$$

where the constant a is choosen from $P(x) = a + b \cdot x$. Setting

$$Mq(G) = \{u^P + P_{|\overline{G}} \mid P \in \mathbb{P}\}$$

we obtain $M_q(G) \subset N_q(G)$, obviously. Furthermore, we have $\dim M_q(G) = \dim \mathbb{P} = n + 1$, which can be shown as follows: Let $p(x) = a + b \cdot x$ and let $u^P + P_{|\overline{G}} = 0$ in $\overline{G}$. Then from the decay properties of u^P and ∇u^P established in Lemma 2.1 we find $a = 0$ and $b = 0$, hence $P = 0$. Here in the case $n = 2$ we obtain $a = 0$ due to the special choice of the number a in (3.17). Together with the uniqueness statement in Lemma 2.1 this means that, if B is a basis of $\mathbb{P}$, then

$$B_q(G) = \{u^P + P_{|\overline{G}} \mid P \in B\}$$

is a basis of $M_q(G)$. Thus it remains to show

$$N_q(G) \subset M_q(G). \tag{3.18}$$

To do so, let us first determine the null space

$$N_q(\mathbb{R}^n) = \{u \mid u \in L^{2,q}(\mathbb{R}^n) \text{ with } -\Delta u = 0 \quad \text{in } \mathbb{R}^n\}.$$

From $\Delta u = 0$, hence $\Delta \nabla^2 u = 0$ with $D^2_{jk} u \in L^q(\mathbb{R}^n)$ $(j, k = 1, \ldots, n)$ we obtain $\nabla^2 u = 0$ in $\mathbb{R}^n$, which implies $u = P$ for some $P \in \mathbb{P}$. Thus we have shown that

$$N_q(\mathbb{R}^n) = \mathbb{P}. \tag{3.19}$$

Now let $u \in N_q(G)$. We extend u on the whole space obtaining a function $\tilde{u} \in L^{2,q}(\mathbb{R}^n)$ with $\tilde{u}_{|G} = u$ [1, p. 83]. Moreover, this function satisfies on $\mathbb{R}^n$ the identity $-\Delta \tilde{u} = \tilde{f} \in L^q(\mathbb{R}^n)$, where the function $\tilde{f}$ has a compact support in the bounded domain $\mathbb{R}^n \setminus \overline{G}$. Consider the equations

$$-\Delta w = \tilde{f} \quad \text{in } \mathbb{R}^n. \tag{3.20}$$

Again, it can be solved by convolution with the fundamental solution E_n of the Laplacian: We obtain $w = E_n * \tilde{f}$ in $\mathbb{R}^n$ and the Calderon-Zygmund theorem implies $D^2_{jk} w \in L^r(\mathbb{R}^n)$ for all $1 < r \leq q$ $(j, k = 1, \ldots, n)$. Here we used $\tilde{f} \in L^r(\mathbb{R}^n)^n$ for all $1 < r \leq q$ due to its compact support. Now using a well-known estimate of Hardy-Littlewood-Sobolev-type [2, p. 242] we find $w \in L^s(\mathbb{R}^n)$ for some $s \geq q$, hence $w \in L^s_{\text{loc}}(\mathbb{R}^n) \subset L^q_{\text{loc}}(\mathbb{R}^n)$. Thus we have constructed some solution w of (3.20) such that $w \in L^{2,q}(\mathbb{R}^n)$. Setting $W = \tilde{u} - w$ we obtain $W \in N_q(\mathbb{R}^n)$, and (3.19) leads to $\tilde{u} = w + P$ for some $P \in \mathbb{P}$. Because $\tilde{u}_{|\partial G} = 0$ and since $\tilde{u}_{|G} = u$ we find $u \in M_q(G)$, which proves (3.18) and thus the theorem. □

References

[1] R. A. Adams: *Sobolev Spaces*, New York-San Francisco-London: Academic Press (1975).

[2] L. Bers, F. John and M. Schechter: *Partial differential equations*, Providence R.I.: Wiley (1979).

[3] H. Brakhage, P. Werner: *Über das Dirichlet'sche Außenraumproblem für die Helmholtz'sche Schwingungschleichung*, Arch. Math., **16** (1965), 325–329.

[4] A. P. Calderon, A. Zygmund: *On singular integrals*. Amer. J. Math., **78** (1956) 289–309.

[5] J. Deny, J.-L. Lions: *Les espaces du type de Beppo Levi*, Ann. Inst. Fourier, **5** (1953–1954) 305–370.

[6] D. Fortunato: *On the index of elliptic partial differential operators in* $\mathbb{R}^n$, Ann. Mat. Pura Appl., **119** (1979), 317–331.

[7] J. Giroire: *Etude de quelques problemes aux limites exterieurs et resolution par equations integrales. Thèse de doctorat d'etat es sciences mathematiques*. Paris 6: Universite Pierre et Marie Curie (1987).

[8] N. M. Günter: *Die Potentialtheorie und ihre Anwendungen auf Grundaufgaben der mathematischen Physik*, Leipzig: Verlagsgessellschaft (1957).

[9] G. C. Hsiao, R. Kress: *On an integral equation for the two-dimensional exterior Stokes problem*. NAM-Bericht 33. Universität Göttingen (1983).

[10] R. Leis: *Zur Eindeutigkeit der Randwertaufgaben der Helmholtz'schen Schwingungsgleichung*, Math. Z., **85** (1964) 141–153.

[11] V. G. Maz'ja: *Sobolev spaces*, Berlin Heidelberg New York Tokyo, Springer Verlag (1985).

[12] R. McOwen: *The behaviour of the Laplacian on weighted Sobolev spaces*, Comm. Pure Appl. Math., **32** (1979), 783–795.

[13] J. Nečas: *Les mèthodes directes en thèorie des èquations elliptiques*, Prague, Academia (1967).

[14] J. Saranen, K. J. Witsch: *Exterior boundary value problems for elliptic equations*. Ann. Acad. Sci. Fennicae Series A. I. Mathematica, **8** (1983) 3–42.

[15] W. I. Smirnow: *Lehrgang der höheren Mathematik 4*, Berlin: Deutscher Verlag der Wissenschaften (1979).

[16] C. G. Simader: *On Dirichlet's boundary value problem*, Berlin Heidelberg New York, Springer Lecture Notes 268 (1972).

[17] W. Varnhorn: *The Poisson equation with weights in exterior domains of* $\mathbb{R}^n$, Applic. Anal., **43** (1992), 135–145.

Werner Varnhorn
Kassel University
Institute of Mathematics, Faculty of Mathematics and Natural Sciences
34127 Kassel, Germany
ORCID: 0000-0001-9486-1319
E-mail address: varnhorn@mathematik.uni-kassel.de