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ANNIHILATOR CONDITIONS WITH GENERALIZED SKEW DERIVATIONS AND LIE IDEALS OF PRIME RINGS

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Abstract. Let R be a prime ring, Q_r its right Martindale quotient ring, L a non-central Lie ideal of R, $n \geq 1$ a fixed integer, F and G two generalized skew derivations of R with the same associated automorphism, $p \in R$ a fixed element. If $p\big(F(x)F(y) - G(y)x\big)^n = 0$, for any $x, y \in L$, then there exist $a, c \in Q_r$ such that $F(x) = ax$ and $G(x) = cx$, for any $x \in R$, with $pa = pc = 0$, unless when R satisfies the standard polynomial identity $s_4(x_1, \ldots, x_4)$.

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1. Introduction

This work is devoted to consider some related problems concerning annihilators of power values of some appropriate identities involving additive maps in prime rings. Throughout this paper R always denotes a prime ring, $Z(R)$ the center of R, Q_r the right Martindale quotient ring of R and $C = Z(Q_r)$, the center of Q_r (C is usually called the extended centroid of R). We introduce on R an additive mapping d which satisfies the following rule:

$$d(xy) = d(x)y + \alpha(x)d(y)$$

for all $x, y \in R$. The map d is said to be a *skew derivation* of R and α is called the *associated automorphism* of d. Consequently, let us also define the concept of a *generalized skew derivation* F of R, that is an additive mapping F such that

$$F(xy) = F(x)y + \alpha(x)d(y)$$

for all $x, y \in R$, where d is a skew derivation of R and α is the associated automorphism of d. The map d is called an *associated skew derivation* of F. The automorphism α is called the *associated automorphism* of F.

Nilpotent values of skew derivations and generalized skew derivations of prime rings were recently studied by several authors.

In [2], J.-C. Chang shows that if F is a generalized skew derivation of R, L is a non-commutative Lie ideal of R and $n \geq 1$ a fixed integer such that $F(x)^n = 0$, for all $x \in L$, then $F(x) = 0$, for all $x \in R$. Later, in [20], a generalization of the previous cited result involving an annihilator condition is given. More precisely, the main result in [20] proves that if F is a generalized skew derivation of R, L is a non-commutative Lie ideal of R, $n \geq 1$ a fixed integer and $a \in R$ is a fixed element such that $aF(x)^n = 0$, for all $x \in L$, then $aF(x) = 0$, for all $x \in R$, unless R satisfies the standard identity s_4.

This last result has recently been further improved as follows: let $0 \neq p$ be an element of R, F and G generalized skew derivations with the same associated skew derivation d of a prime ring R, L a non-commutative Lie ideal of R, $l_1, \dots, l_k, n$ nonnegative integers with $l_1 \neq 0$ and $n > 0$. If

$$p\Big(F(u)^{l_1}G(u)^{l_2}F(u)^{l_3}G(u)^{l_4}\cdots G(u)^{l_k}\Big)^n = 0 \quad \forall u \in L,$$

then $d = 0$ and there exist $a, c \in Q_r$ such that $F(x) = ax$ and $G(x) = cx$, for any $x \in R$. Moreover either $pa = 0$ or $c = 0$, unless R satisfies s_4 (see [14, Main Theorem]).

Further nil-power conditions have been investigated in another recent paper (see [19]) and the following result was proved: If R is a prime ring, F is a generalized skew derivation of R, L is a non-central Lie ideal of R and $n \geq 1$ is a fixed integer such that $(F(x)F(y) - yx)^n = 0$, for any $x, y \in L$, then $char(R) = 2$ and $R \subseteq M_2(C)$, the 2×2 matrix ring over C.

Following this line of investigation, the aim of this paper is to generalize the result in [19] to the case when two different generalized skew derivations act on the non-central Lie ideal L, also introducing an annihilating condition. To be more precise, we will prove the following:

Theorem 1.1. *Let R be a prime ring, Q_r its right Martindale quotient ring, L a non-central Lie ideal of R, $n \geq 1$ a fixed integer, F and G two generalized skew derivations of R with the same associated automorphism, $p \in R$ a fixed element. If $p\big(F(x)F(y) - G(y)x\big)^n = 0$, for any $x, y \in L$, then there exist $a, c \in Q_r$ such that $F(x) = ax$ and $G(x) = cx$, for any $x \in R$, with $pa = pc = 0$, unless when R satisfies the standard polynomial identity $s_4(x_1, \dots, x_4)$.*

Let us recall some well known results and notations which will be useful in the sequel.

We will denote by $SDer(Q_r)$ the set of all skew-derivations of Q_r and by SD_{int} the C-subspace of $SDer(Q_r)$ consisting of all inner skew-derivations of Q_r.

Two different skew derivations d and δ are said to be $C-$linearly dependent modulo SD_{int}, if there exist $\lambda, \mu \in C$, $a \in Q_r$ and $\alpha \in Aut(Q)$ such that $\lambda d(x) + \mu\delta(x) = ax - \alpha(x)a$ for all $x \in R$.

If d and δ are $C-$linearly independent skew derivations modulo SD_{int}, associated with the same automorphism α, such that $\Phi(x_i, d(x_j), \delta(x_k))$ is a skew-differential identity on R, then $\Phi(x_i, y_j, z_k)$ is a generalized polynomial identity of R, where x_i, y_j, z_k are distinct indeterminates (it follows from main results in [4,5,6]).

It is known that, if I is a two-sided ideal I of R, then I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [3]). Furthermore, I, R, and Q_r satisfy the same generalized polynomial identities with automorphisms (see [5, Theorem 1]).

2. The result for inner generalized derivations

We start by proving the main theorem in case both F and G are generalized inner derivations of R and $[R, R] \subseteq L$. In this sense we assume that there are $a, b, c, q \in Q_r$ such that $F(x) = ax + xb$ and $G(x) = cx + xq$, for any $x \in R$. Hence, by our assumption, R satisfies the generalized polynomial identity

$$\Psi(x_1, x_2, y_1, y_2) = \\ p\Big\{(a[x_1, x_2] + [x_1, x_2]b)(a[y_1, y_2] + [y_1, y_2]b) - (c[y_1, y_2] + [y_1, y_2]q)[x_1, x_2]\Big\}^n. \quad (1)$$

For brevity we denote $X = [x_1, x_2]$, $Y = [y_1, y_2]$ and

$$\Psi(X, Y) = \\ p\Big\{(aX + Xb)(aY + Yb) - (cY + Yq)X\Big\}^n. \quad (2)$$

Lemma 2.1. *Assume $p \neq 0$. Either $\Psi(X, Y)$ is a non-trivial generalized polynomial identity for R or $b, q \in C$ with $p(a + b) = p(c + q) = 0$.*

Proof. Assume that $\Psi(X, Y)$ is a trivial generalized polynomial identity for R. Let $T = Q_r *_C C\{X\}$ be the free product over C of the C-algebra Q_r and the free C-algebra $C\{X\}$, with X the set consisting of non-commuting indeterminates x_1, x_2, y_1, y_2.

Now consider the generalized polynomial $\Psi(X, Y) \in Q_r *_C C\{X\}$.

By our hypothesis,

$$\begin{aligned}\Psi(X,Y) &= p\Big\{(aX+Xb)(aY+Yb)-(cY+Yq)X\Big\}^n \\ &= p\Big\{(aX+Xb)(aY+Yb)-(cY+Yq)X\Big\}^{n-1}\cdot \\ &\quad\cdot\Big\{(aX+Xb)aY-(cY+Yq)X+(aX+Xb)Yb\Big\} \\ &= 0\in T.\end{aligned} \tag{3}$$

Suppose firstly $b\notin C$, that is $\{b,1\}$ is linearly C-independent. Therefore, since $\Psi(X,Y)=0\in T$,

$$p\Big\{(aX+Xb)(aY+Yb)-(cY+Yq)X\Big\}^{n-1}\cdot(aX+Xb)Yb=0\in T$$

implying

$$p\Big\{(aX+Xb)(aY+Yb)-(cY+Yq)X\Big\}^{n-1}\cdot(aX+Xb)=0\in T$$

that is

$$p\Big\{(aX+Xb)(aY+Yb)-(cY+Yq)X\Big\}^{n-1}\cdot Xb=0\in T.$$

Thus

$$p\Big\{(aX+Xb)(aY+Yb)-(cY+Yq)X\Big\}^{n-1}=0\in T.$$

Continuing this process, we get

$$p\Big\{(aX+Xb)(aY+Yb)-(cY+Yq)X\Big\}=0\in T$$

which means that

$$p(aX+Xb)Yb=0\in T.$$

Hence the contradiction $pXb=0$ follows. Thus $\{b,1\}$ is linearly C-dependent, that is $b\in C$.

Analogously, by (3) and $a'=a+b$, it follows that

$$\begin{aligned}&p\Big\{a'Xa'Y-(cY+Yq)X\Big\}^{n-1}\cdot(a'Xa')Y- \\ &p\Big\{a'Xa'Y-(cY+Yq)X\Big\}^{n-1}\cdot(cY+Yq)X=0\in T\end{aligned}$$

that is, both

$$p\Big\{a'Xa'Y-(cY+Yq)X\Big\}^{n-1}\cdot(a'Xa')Y=0\in T$$

and

$$p\Big\{a'Xa'Y-(cY+Yq)X\Big\}^{n-1}\cdot(cY+Yq)X=0\in T. \tag{4}$$

In particular, (4) implies

$$p\Big\{a'Xa'Y-(cY+Yq)X\Big\}^{n-1}\cdot(cY+Yq)=0\in T.$$

Hence, if we suppose $q\notin C$, it follows that

$$p\Big\{a'Xa'Y-(cY+Yq)X\Big\}^{n-1}\cdot Yq=0\in T$$

which implies

$$p\Big\{a'Xa'Y-(cY+Yq)X\Big\}^{n-1}=0\in T.$$

As above, continuing this process we get

$$p\Big\{a'Xa'Y-(cY+Yq)X\Big\}=0\in T$$

and arrive at the contradiction $pYq=0$.
Therefore $q\in C$ and, for $c'=c+q$, we write relation (3) as follows

$$p\Big\{a'Xa'Y-c'YX\Big\}\cdot\Big\{a'Xa'Y-c'YX\Big\}^{n-1}=0\in T. \tag{5}$$

It is easy to see that if either $pa'=0$ or $pc'=0$ then both pa' and pc' must be zero. Then we finally assume $pa'\neq 0$ and $pc'\neq 0$. In case there exists $0\neq\lambda\in C$ such that $pc'=\lambda pa'\neq 0$, then (5) implies

$$\Big\{a'Xa'Y-c'YX\Big\}^{n-1}=0\in T \tag{6}$$

which is a contradiction, since $a'\neq 0$ and $c'\neq 0$.
Hence $\{pc',pa'\}$ is linearly independent and by (5) we get

$$pa'Xa'Y\cdot\Big\{a'Xa'Y-c'YX\Big\}^{n-1}=0\in T.$$

Once again relation (6) holds and we are done. □

Lemma 2.2. *Assume that R is a primitive ring, which is isomorphic to a dense ring of linear transformations on some vector space V over a division ring D, $dim_D V\geq 2$, $f\in End(V)$ and $a\in R$. If $av=0$, for any $v\in V$ such that $\{v,f(v)\}$ is linearly D-independent, then $a=0$, unless $dim_D V=2$ and $char(R)=2$.*

Proof. We fix a vector $v \in V$ such that $\{v, f(v)\}$ is linearly D-independent, then $av = 0$. Let $w \in V$ be such that $\{w, v\}$ is linearly D-dependent. Then both $aw = 0$ and $w \in Span\{v, f(v)\}$ follow trivially.
Let now $w \in V$ such that $\{w, v\}$ is linearly D-independent and $aw \neq 0$. By the hypothesis it follows that $\{w, f(w)\}$ is linearly D-dependent, as are $\{w+v, f(w+v)\}$ and $\{w - v, f(w - v)\}$. Therefore there exist $\lambda_w, \lambda_{w+v}, \lambda_{w-v} \in D$ such that

$$f(w) = w\lambda_w, \quad f(w + v) = (w + v)\lambda_{w+v}, \quad f(w - v) = (w - v)\lambda_{w-v}.$$

In other words we have

$$w\lambda_w + f(v) = w\lambda_{w+v} + v\lambda_{w+v} \tag{7}$$

and

$$w\lambda_w - f(v) = w\lambda_{w-v} - v\lambda_{w-v}. \tag{8}$$

Assume $dim_D V \geq 3$. It is easy to see that $w \in Span\{v, f(v)\}$, otherwise (7) forces a contradiction. Therefore, for any choice of $w \in V$, we have $w \in Span\{v, f(v)\}$, that is $V = Span\{v, f(v)\}$, a contradiction.
In order to complete the proof, we then consider the case $dim_D V = 2$ and assume $char(R) \neq 2$, if not we are finished.
By comparing (7) with (8) we get both

$$w(2\lambda_w - \lambda_{w+v} - \lambda_{w-v}) + v(\lambda_{w-v} - \lambda_{w+v}) = 0 \tag{9}$$

and

$$2f(v) = w(\lambda_{w+v} - \lambda_{w-v}) + v(\lambda_{w+v} + \lambda_{w-v}). \tag{10}$$

By (9) and since $\{w, v\}$ is D-independent and $char(R) \neq 2$, we have $\lambda_w = \lambda_{w+v} = \lambda_{w-v}$. Thus by (10) it follows $2f(v) = 2v\lambda_w$. Since $\{f(v), v\}$ is D-independent, the conclusion $\lambda_w = \lambda_{w+v} = 0$ follows, that is $f(w) = 0$ and $f(w + v) = 0$, which implies the contradiction $f(v) = 0$. Thus, if $dim_D V = 2$ and $char(R) \neq 2$, it follows that $aw = 0$, for any choice of $w \in V$, that is $aV = (0)$. Therefore $a = 0$ follows. □

Proposition 2.3. *If R satisfies (1) then $b, q \in C$ and $p(a + b) = p(c + q) = 0$, unless when $char(R) = 2$ and R satisfies s_4.*

Proof. We of course suppose $p \neq 0$. In light of Lemma 2.1, we may assume that the generalized polynomial $\Psi(x_1, x_2, y_1, y_2)$ is a non-trivial generalized polynomial identity for R. By [3] it follows that $\Psi(x_1, x_2)$ is a non-trivial generalized polynomial identity for Q_r. In view of [13, Theorem 2.5 and Theorem 3.5], we know that both Q_r and $Q_r \bigotimes_C \overline{C}$ are centrally closed, where $\overline{C}$ is the algebraic closure of C. We may

replace Q_r by itself or $Q_r \bigotimes_C \overline{C}$ according as C is finite or infinite. Therefore we may assume that Q_r is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem [18], Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. In light of Jacobson's theorem [16, page 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C. Since R is not commutative, we have $dim_C V \geq 2$. Moreover, if $dim_C V = 2$ we would assume $char(R) \neq 2$, if not we are done.
We divide the proof in several steps.

Step 1. $b \in C$**:**

Suppose $b \notin C$ and let $v \in V$ be such that $\{v, bv\}$ is linearly C-independent. Since $dim_C V \geq 2$ and by the density of Q_r, there exist $r_1, r_2, s_1, s_2 \in Q_r$ such that

$$r_1 v = 0 \quad r_2 v = v \quad r_1(bv) = -v \quad r_2(bv) = 0$$

$$s_1 v = 0 \quad s_2 v = v \quad s_1(bv) = -v \quad s_2(bv) = 0.$$

By (1) we get

$$0 =$$
$$p\Big\{(a[r_1, r_2] + [r_1, r_2]b)(a[s_1, s_2] + [s_1, s_2]b) - (c[s_1, s_2] + [s_1, s_2]q)[r_1, r_2]\Big\}^n v = pv.$$

Hence we have proved that $pv = 0$ for any vector $v \in V$ such that $\{v, bv\}$ is linearly independent. By Lemma 2.2, $p = 0$ follows. This contradiction says that b must be a central element of Q_r and (1) reduces to

$$p\Big\{a'[x_1, x_2]a'[y_1, y_2] - (c[y_1, y_2] + [y_1, y_2]q)[x_1, x_2]\Big\}^n \tag{11}$$

where $a' = a + b$.

Step 2. $q \in C$**:**

Assume now $q \notin C$ and let $v \in V$ be such that $\{v, qv\}$ is linearly C-independent. As above, there are $r_1, r_2, s_1, s_2 \in Q_r$ such that

$$r_1 v = 0 \quad r_2 v = qv \quad r_1(qv) = v$$

$$s_1 v = 0 \quad s_2 v = v \quad s_1(qv) = v \quad s_2(qv) = 0.$$

By (11) we get

$$0 = p\Big\{a'[r_1, r_2]a'[s_1, s_2] - (c[s_1, s_2] + [s_1, s_2]q)[r_1, r_2]\Big\}^n v = pv.$$

Thus $pv = 0$ for any vector $v \in V$ such that $\{v, qv\}$ is linearly independent. As above the contradiction $p = 0$ follows.
Therefore both $b \in C$ and $q \in C$, that is Q_r satisfies

$$p\Big\{a'[x_1, x_2]a'[y_1, y_2] - c'[y_1, y_2][x_1, x_2]\Big\}^n \tag{12}$$

where $a' = a + b$ and $c' = c + q$.

Step 3. Either $pa' = 0$ or $a' \in C$:

If $a' \notin C$ then there is $v \in V$ such that $\{v, a'v\}$ is linearly C-independent. By the density of Q_r, there are $r_1, r_2, s_1, s_2 \in Q_r$ such that

$$r_1(a'v) = 0 \quad r_2(a'v) = v \quad r_1v = v$$

$$s_1v = 0 \quad s_2v = v \quad s_1(a'v) = -v \quad s_2(a'v) = 0.$$

By (12) it follows

$$0 = p\Big\{a'[r_1, r_2]a'[s_1, s_2] - c'[s_1, s_2][r_1, r_2]\Big\}^n a'v = pa'v.$$

Thus $pa'v = 0$ for any vector $v \in V$ such that $\{v, a'v\}$ is linearly independent, implying $pa' = 0$.

Step 4. Let $dim_C V \geq 3$, then either $pc' = 0$ or $c' \in C$:

If $c' \notin C$ then there is $v \in V$ such that $\{v, c'v\}$ is linearly C-independent. Moreover, since $dim_C V \geq 3$, there exists $w \in V$ such that $\{v, c'v, w\}$ is linearly C-independent. Again by the density of Q_r, there are $r_1, r_2, s_1, s_2 \in Q_r$ such that

$$r_1(c'v) = 0 \quad r_2(c'v) = v \quad r_1v = v$$

$$s_1v = 0 \quad s_2v = w \quad s_1w = v \quad s_1(c'v) = 0 \quad s_2(c'v) = c'v.$$

Relation (12) implies

$$0 = p\Big\{a'[r_1, r_2]a'[s_1, s_2] - c'[s_1, s_2][r_1, r_2]\Big\}^n c'v = (-1)^n pc'v.$$

Hence, $pc'v = 0$ for any vector $v \in V$ such that $\{v, c'v\}$ is linearly independent, that is $pc' = 0$.

Step 5. Let $dim_C V \geq 3$. If $pa' = 0$, then $pc' = 0$:

If $c' \notin C$ the conclusion follows from Step 4. Moreover, if $a' \in C$ then $p = 0$, which is not possible. Hence we assume $c' \in C$ and $a' \notin C$. Therefore Q_r satisfies

$$pc'[y_1, y_2][x_1, x_2]\Big\{a'[x_1, x_2]a'[y_1, y_2] - c'[y_1, y_2][x_1, x_2]\Big\}^{n-1}. \tag{13}$$

Since $a' \notin C$, there is $v \in V$ such that $\{v, a'v\}$ is linearly C-independent. Moreover, since $dim_C V \geq 3$, there exists $w \in V$ such that $\{v, a'v, w\}$ is linearly C-independent. By the density of Q_r, there are $r_1, r_2, s_1, s_2 \in Q_r$ such that

$$r_1 v = 0 \quad r_2 v = a'v \quad r_1(a'v) = a'v$$

$$s_1 v = 0 \quad s_2 v = v \quad s_1(a'v) = 0 \quad s_2(a'v) = w \quad s_1 w = v.$$

Relation (13) implies

$$0 = pc'[s_1, s_2][r_1, r_2]\Big\{a'[r_1, r_2]a'[s_1, s_2] - c'[s_1, s_2][r_1, r_2]\Big\}^{n-1} v = (-c')^n pv.$$

Hence $(-c')^n pv = 0$ for any vector $v \in V$ such that $\{v, a'v\}$ is linearly independent, that is $pc' = 0$ (we remark that, since we assume $p \neq 0$, this implies $c' = 0$).

Step 6. Let $dim_C V \geq 3$. If $pc' = 0$, then $pa' = 0$:

The proof of this step is quite similar to the previous one and we omit it for brevity.

Step 7. If $dim_C V \geq 3$, then both $pa' = 0$ and $pc' = 0$:

In light of the previous argument, to complete the proof of this step we may assume both $a' \in C$ and $c' \in C$. In this case Q_r satisfies

$$p\Big\{a'^2[x_1, x_2][y_1, y_2] - c'[y_1, y_2][x_1, x_2]\Big\}^n. \tag{14}$$

Let $\{v, w\}$ be a set of linearly independent vectors of V and $r_1, r_2, s_1, s_2, r'_1, r'_2, s'_1, s'_2 \in Q_r$ such that

$$r_1 v = 0 \quad r_2 v = v \quad s_1 v = 0 \quad s_2 v = w \quad s_1 w = w \quad r_1 w = v \quad r_2 w = 0$$

and

$$s'_1 v = 0 \quad s'_2 v = v \quad r'_1 v = 0 \quad r'_2 v = w \quad r'_1 w = w \quad s'_1 w = v \quad s'_2 w = 0.$$

Thus (14) implies both

$$p\Big\{a'^2[r_1, r_2][s_1, s_2] - c'[s_1, s_2][r_1, r_2]\Big\}^n v = (-1)^n pa'^{2n} v$$

and

$$p\Big\{a'^2[r'_1, r'_2][s'_1, s'_2] - c'[s'_1, s'_2][r'_1, r'_2]\Big\}^n v = pc'^n v.$$

As above we may conclude that $pa' = 0$ and $pc' = 0$, as required.

Finally, in all that follows we assume $dim_C V = 2$, that is $Q_r \cong M_2(C)$, with $char(C) \neq 2$. Firstly we notice that (12) reduces to

$$p\Big\{a'[x_1, x_2]a'[y_1, y_2] - c'[y_1, y_2][x_1, x_2]\Big\}^2. \tag{15}$$

We resume our proof starting from the Step 3, so we know that either $pa' = 0$ or $a' \in C$.

Step 8. If $Q_r \cong M_2(C)$ and $pa' = 0$ then $pc' = 0$:

Under this assumption Q_r satisfies

$$pc'[y_1, y_2][x_1, x_2]\Big\{a'[x_1, x_2]a'[y_1, y_2] - c'[y_1, y_2][x_1, x_2]\Big\}. \tag{16}$$

Of course we may assume that a' is not a scalar matrix, if not $p = 0$ follows.
We firstly suppose C is an infinite field. By [9, Lemma 1] there exists an C-automorphism φ of $M_2(C)$ such that $\varphi(a')$ has all non-zero entries. Clearly $\varphi(a')$, $\varphi(c')$ and $\varphi(p)$ must satisfy the condition (16) that is

$$\varphi(pc')[y_1, y_2][x_1, x_2]\Big\{\varphi(a')[x_1, x_2]\varphi(a')[y_1, y_2] - \varphi(c')[y_1, y_2][x_1, x_2]\Big\} \tag{17}$$

is an identity for $M_2(C)$. Let e_{ij} denote the matrix unit with 1 in (i, j)-entry and zero elsewhere. Thus, for $[x_1, x_2] = e_{12}$ and $[y_1, y_2] = e_{21}$ in (17), and right multiplying by e_{11} we get

$$\varphi(pc')e_{22}\varphi(a')e_{12}\varphi(a')e_{21} = 0.$$

Since $\varphi(a')$ has all non-zero entries, it follows that both $(1, 2)$-entry and $(2, 2)$-entry of the matrix $\varphi(pc')$ must be zero. Similarly, for $[x_1, x_2] = e_{21}$ and $[y_1, y_2] = e_{12}$ in (17), and right multiplying by e_{22} we have that both $(2, 1)$-entry and $(1, 1)$-entry of the matrix $\varphi(pc')$ must be zero. Therefore $\varphi(pc') = 0$, that is $pc' = 0$.
Now let K be an infinite field which is an extension of the field C and let $\overline{Q_r} = M_2(K) \cong Q_r \otimes_C K$. Consider the generalized polynomial

$$P(x_1, x_2, x_3, x_4) = pc'[x_3, x_4][x_1, x_2]\Big\{a'[x_1, x_2]a'[x_3, x_4] - c'[x_3, x_4][x_1, x_2]\Big\}$$

which is a generalized polynomial identity for Q_r. Moreover it is multi-homogeneous of multi-degree $(2, 2, 2, 2)$ in the indeterminates x_1, x_2, x_3, x_4.
Hence the complete linearization of $P(x_1, x_2, x_3, x_4)$ is a multilinear generalized polynomial $\Theta(x_1, \ldots, x_4, z_1, \ldots, z_4)$ in 8 indeterminates, moreover

$$\Theta(x_1, \ldots, x_4, z_1, \ldots, z_4) = 2^4 P(x_1, x_2, x_3, x_4).$$

Clearly the multilinear polynomial $\Theta(x_1, \ldots, x_4, z_1, \ldots, z_4)$ is a generalized polynomial identity for Q_r and $\overline{Q_r}$ too. Since $char(C) \neq 2$ we obtain $P(r_1, r_2, r_3, r_4) = 0$, for all $r_1, \ldots, r_4 \in \overline{Q_r}$, and the conclusion $pc' = 0$ follows from the above argument.

Step 9. If $Q_r \cong M_2(C)$ and $a' \in C$ then $a' = 0$ and $pc' = 0$:

In this final case Q_r satisfies

$$p\Big\{a'^2[x_1,x_2][y_1,y_2]-c'[y_1,y_2][x_1,x_2]\Big\}^2. \tag{18}$$

For $[x_1,x_2]=e_{12}$ and $[y_1,y_2]=e_{21}$ in (18), and right multiplying by e_{11} we get $a'^4pe_{11}=0$, implying that both $(2,1)$-entry and $(1,1)$-entry of the matrix a'^4p must be zero. Once again, for $[x_1,x_2]=e_{21}$ and $[y_1,y_2]=e_{12}$ in (18), and right multiplying by e_{22} we have $a'^4pe_{22}=0$, that is both $(2,2)$-entry and $(1,2)$-entry of the matrix a'^4p must be zero. Therefore $a'^4p=0$, that is $a'=0$. Hence (18) reduces to

$$p\Big\{c'[y_1,y_2][x_1,x_2]\Big\}^2. \tag{19}$$

Notice that, if $c'\in C$ it follows that $c'^2p[x_1,x_2]^4$ is an identity for Q_r. In this case it is well known that $c'^2p=0$, that is $c'=0$. On the other hand, if we assume that $c'\notin C$, there is $v\in V$ such that $\{v,c'v\}$ is linearly C-independent. By the density of Q_r, there are $r_1,r_2,s_1,s_2\in Q_r$ such that

$$r_1(c'v)=0 \quad r_2(c'v)=v \quad r_1v=v$$

$$s_1v=0 \quad s_2v=c'v \quad s_1(c'v)=v.$$

Thus, relation (19) implies

$$0=p\Big\{c'[s_1,s_2][r_1,r_2]\Big\}^2c'v=pc'v.$$

as above, this last relation implies $pc'=0$, as required. □

3. The case of inner generalized skew derivations

In this section we consider the case when the maps have the following forms:

$$F(x)=ax+\alpha(x)b, \qquad G(x)=cx+\alpha(x)u$$

for all $x\in R$, for suitable fixed elements $p,a,b,c,u\in Q_r$ and $\alpha\in Aut(Q_r)$. Moreover we suppose that Q_r satisfies

$$\begin{aligned} p\Big\{&\Big(a[x_1,x_2]+\alpha([x_1,x_2])b\Big)\Big(a[y_1,y_2]+\alpha([y_1,y_2])b\Big)\\ &-\Big((c[y_1,y_2]+\alpha([y_1,y_2])u)[x_1,x_2]\Big)\Big\}^n. \end{aligned} \tag{20}$$

In light of Proposition 2.3 we may always assume $\alpha\neq I_R$, the identity map on R.

Lemma 3.1. *Assume that R is isomorphic to a dense ring of linear transformations on some vector space V over a division ring D, containing non-zero linear transformations of finite rank. If R satisfies (20) then there exist $a', c' \in Q_r$ such that $F(x) = a'x$ and $G(x) = c'x$, for any $x \in R$, with $pa' = pc' = 0$, unless when $dim_D V \leq 2$.*

Proof. We suppose $dim_D V \geq 3$.
Since R is a primitive ring with non-zero socle, by [16, p. 79], there exists a semi-linear automorphism $T \in End(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in R$.
Hence, R satisfies

$$p\Big\{\Big(a[x_1,x_2] + T[x_1,x_2]T^{-1}b\Big)\Big(a[y_1,y_2] + T[y_1,y_2]T^{-1}b\Big) - \Big((c[y_1,y_2] + T[y_1,y_2]T^{-1}u)[x_1,x_2]\Big)\Big\}^n. \tag{21}$$

Assume there exists $v \in V$ such that $\{v, T^{-1}bv\}$ is linearly D-independent.
Since $dim_D V \geq 3$, there exists $w \in V$ such that $\{w, v, T^{-1}bv\}$ is linearly D-independent. Moreover, by the density of R, there exist $r_1, r_2, s_1, s_2 \in R$ such that

$$r_1v = 0 \quad r_2v = v \quad r_1w = T^{-1}v \quad r_1T^{-1}bv = 0 \quad r_2T^{-1}bv = w$$

$$s_1v = 0 \quad s_2v = v \quad s_1w = T^{-1}v \quad s_1T^{-1}bv = 0 \quad s_2T^{-1}bv = w$$

and we get

$$0 = p\Big\{\Big(a[r_1,r_2] + T[r_1,r_2]T^{-1}b\Big)\Big(a[s_1,s_2] + T[s_1,s_2]T^{-1}b\Big) - \Big((c[s_1,s_2] + T[s_1,s_2]T^{-1}u)[r_1,r_2]\Big)\Big\}^n v = pv.$$

Hence, for any $v \in V$ such that $\{v, T^{-1}bv\}$ is linearly D-independent, it follows $pv = 0$. By Lemma 2.2 we get $p = 0$, which is a contradiction.
Therefore, for any $v \in V$, there exists $\lambda_v \in D$ such that $T^{-1}bv = v\lambda_v$. In this case, it is well known that there exists a unique $\lambda \in D$ such that $T^{-1}bv = v\lambda$, for all $v \in V$ (see for example Lemma 1 in [7]). Thus

$$\Big(ax + \alpha(x)b\Big)v = \Big(ax + TxT^{-1}b\Big)v = axv + T(xv\lambda) =$$

$$axv + T((xv)\lambda) = axv + T(T^{-1}bxv) =$$

$$axv + bxv = (a+b)xv.$$

Hence, for all $v \in V$,

$$\Big(ax + \alpha(x)b - (a+b)x\Big)v = 0$$

which implies $F(x) = ax + \alpha(x)b = (a + b)x$, for all $x \in R$, since V is faithful. Therefore we have that R satisfies

$$p\Big\{(a+b)[x_1,x_2](a+b)[y_1,y_2] - \Big((c[y_1,y_2] + T[y_1,y_2]T^{-1}u)[x_1,x_2]\Big)\Big\}^n. \quad (22)$$

Now assume there exists $v \in V$ such that $\{v, T^{-1}uv\}$ is linearly D-independent. As above there exists $w \in V$ such that $\{w, v, T^{-1}uv\}$ is linearly D-independent and there exist $r_1, r_2, s_1, s_2 \in R$ such that

$$r_1v = 0 \quad r_2v = w \quad r_1w = v$$

$$s_1v = 0 \quad s_2v = v \quad s_1w = T^{-1}v \quad s_1T^{-1}uv = 0 \quad s_2T^{-1}uv = w.$$

From (22) it follows that

$$0 = p\Big\{(a+b)[r_1,r_2](a+b)[s_1,s_2] - \Big((c[s_1,s_2] + T[s_1,s_2]T^{-1}u)[r_1,r_2]\Big)\Big\}^n v = (-1)^n pv.$$

Once again, since p is not zero, by Lemma 2.2 we obtain a contradiction. Thus, there exists a unique $\mu \in D$ such that $T^{-1}uv = v\mu$, for all $v \in V$. This implies $G(x) = cx + \alpha(x)u = (c + u)x$, for all $x \in R$.

Therefore, we have proved that, if $dim_D V \geq 3$, both F and G are inner generalized derivations. The required conclusion then follows from Proposition 2.3. □

Proposition 3.2. *If R satisfies (20) then there exist $a', c' \in Q_r$ such that $F(x) = a'x$ and $G(x) = c'x$, for any $x \in R$, with $pa' = pc' = 0$, unless when R satisfies s_4.*

Proof. Suppose firstly α is an X-inner automorphism of R. Thus assume $\alpha(x) = qxq^{-1}$, for all $x \in R$, that is

$$F(x) = ax + qxq^{-1}b, \qquad G(x) = cx + qxq^{-1}u$$

for all $x \in R$, where q is an invertible element of Q_r. Under our assumption, R satisfies

$$\begin{aligned} &p\Big\{\Big(a[x_1,x_2] + q[x_1,x_2]q^{-1}b\Big)\Big(a[y_1,y_2] + q[y_1,y_2]q^{-1}b\Big) \\ &- \Big((c[y_1,y_2] + q[y_1,y_2]q^{-1}u)[x_1,x_2]\Big)\Big\}^n. \end{aligned} \quad (23)$$

Since α is not the identity map on R, we consider the case $q \notin C$. Moreover, notice that if both $q^{-1}b \in C$ and $q^{-1}u \in C$, then F and G are inner generalized derivations defined respectively as follows

$$F(x) = (a + b)x, \quad G(x) = (c + u)x \quad \forall x \in R$$

and the conclusion follows again from Proposition 2.3.
On the other hand, if either $q^{-1}b \notin C$ or $q^{-1}u \notin C$, the identity (23) is a non-trivial generalized polynomial identity for R as well as for Q_r. In light of the same arguments set out in Proposition 2.3, we may assume that Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. Moreover Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C. By Lemma 3.1 we conclude that $dim_C V \leq 2$, that is Q_r satisfies s_4, as required.
Then we now consider the case α is not an inner automorphism of R. Since $\alpha \neq I_R$,by [4] R is a GPI-ring and Q_r is also GPI-ring by [3]. Once again Q_r is isomorphic to a dense ring of linear transformations on some vector space V and its associated division ring D is finite-dimensional over C. Thus, by Lemma 3.1, one of the following holds:

(1) there exist $a', c' \in Q_r$ such that $F(x) = a'x$ and $G(x) = c'x$, for any $x \in R$, with $pa' = pc' = 0$ (in this case we are done)

(2) $dim_D V \leq 2$.

To complete the proof we have to study this last case. Since $dim_D V \leq 2$ and by our main hypothesis, Q_r satisfies

$$p\Big\{\Big(a[x_1,x_2]+\alpha([x_1,x_2])b\Big)\Big(a[y_1,y_2]+\alpha([y_1,y_2])b\Big) - \Big((c[y_1,y_2]+\alpha([y_1,y_2])u)[x_1,x_2]\Big)\Big\}^2. \tag{24}$$

Here we divide the argument into the following three cases.

Case 1: Assume $char(R) = 0$ or $char(R) = p \geq 3$.
By [5, Theorem 3] and (24), it follows that

$$p\Big\{\Big(a[x_1,x_2]+[t_1,t_2]b\Big)\Big(a[y_1,y_2]+[z_1,z_2]b\Big) - \Big((c[y_1,y_2]+[z_1,z_2]u)[x_1,x_2]\Big)\Big\}^2. \tag{25}$$

is a generalized polynomial identity for Q_r. In particular Q_r satisfies the blended component

$$p\Big\{[t_1,t_2]b[z_1,z_2]b\Big\}^2 \tag{26}$$

which implies easily $b = 0$, since we suppose $p \neq 0$.
Analogously, for $b = 0$ and $y_1 = y_2 = 0$ in (25), we have that Q_r also satisfies

$$p\Big\{[z_1,z_2]u[x_1,x_2]\Big\}^2 \tag{27}$$

that is $u = 0$. Therefore $F(x) = ax$ and $G(x) = cx$, for any $x \in R$, and $pa = pc = 0$ follows from Proposition 2.3, unless Q_r satisfies s_4.

Case 2: Assume the automorphism α is not Frobenius.
Also in this case, by (24) and [5, Theorem 2], one can see that Q_r satisfies (25), and we conclude as above.

Case 3: Assume the automorphism α is Frobenius and $char(R) = 2$.
Hence there exists a fixed integer h such that $\alpha(x) = x^{2^h}$, for all $x \in C$. In particular, there is $x \in C$ such that $x^{2^h} \neq x$. Moreover we assume C is infinite, otherwise D should be a finite division ring, that is D is a field and we are done. Let $0 \neq \lambda \in C$ be such that $\lambda^{2^h} \neq \lambda$. In (24) replace y_1 by λy_1 and get

$$\begin{aligned} p\Bigg\{ & \Big(a[x_1, x_2] + \alpha([x_1, x_2])b\Big)\Big(a[y_1, y_2] + \lambda^{2^h-1}\alpha([y_1, y_2])b\Big) \\ & - \Big((c[y_1, y_2] + \lambda^{2^h-1}\alpha([y_1, y_2])u)[x_1, x_2]\Big)\Bigg\}^2. \end{aligned} \tag{28}$$

If denote

$$\Phi_1(x_1, x_2, y_1, y_2) = a[x_1, x_2]a[y_1, y_2] + \alpha([x_1, x_2])ba[y_1, y_2] - c[y_1, y_2][x_1, x_2]$$

and

$$\Phi_2(x_1, x_2, y_1, y_2) = a[x_1, x_2]\alpha([y_1, y_2])b + \alpha([x_1, x_2])b\alpha([y_1, y_2])b - \alpha([y_1, y_2])u[x_1, x_2]$$

it follows that

$$p\Big\{\Phi_1(r_1, r_2, r_3, r_4) + \gamma\Phi_2(r_1, r_2, r_3, r_4)\Big\}^2 = 0$$

for all $r_1, r_2, r_3, r_4 \in Q_r$, with $\gamma = \lambda^{2^h-1} \neq 1$. Expanding the latter relation, we get

$$p\Big\{\Phi_1^2 + \gamma(\Phi_1\Phi_2 + \Phi_2\Phi_1) + \gamma^2\Phi_2^2\Big\} = 0.$$

For the sake of clearness, let us denote $t_0 = p\Phi_1^2$, $t_1 = p(\Phi_1\Phi_2 + \Phi_2\Phi_1)$ and $t_2 = p\Phi_2^2$. Then we can write

$$t_0 + \gamma t_1 + \gamma^2 t_2 = 0. \tag{29}$$

Replacing in the previous argument γ successively by $1, \gamma, \gamma^2$, the equation (29) gives the system of equations

$$\begin{aligned} & t_0 + t_1 + t_2 = 0 \\ & t_0 + \gamma t_1 + \gamma^2 t_2 = 0 \\ & t_0 + \gamma^2 t_1 + \gamma^4 t_2 = 0. \end{aligned} \tag{30}$$

Moreover, since C is infinite, there exist infinitely many $\lambda \in C$ such that $\lambda^{i(2^h-1)} \neq 1$ for $i = 1, \ldots, 4$, that is there exist infinitely many $\gamma = \lambda^{2^h-1} \in C$ such that $\gamma^i \neq 1$ for $i = 1, \ldots, 4$. Hence, the Vandermonde determinant (associated with the system (30))

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \gamma & \gamma^2 \\ 1 & \gamma^2 & \gamma^4 \end{vmatrix} = \prod_{0 \leq i < j \leq 4} (\gamma^i - \gamma^j)$$

is not zero. Thus, we can solve the above system (30) and obtain $t_i = 0$ $(i = 0, 1, 2)$. In particular $t_0 = 0$ and $t_2 = 0$, that is

$$p\Big\{a[x_1, x_2]a[y_1, y_2] + \alpha([x_1, x_2])ba[y_1, y_2] - c[y_1, y_2][x_1, x_2]\Big\}^2 \tag{31}$$

and

$$p\Big\{a[x_1, x_2]\alpha([y_1, y_2])b + \alpha([x_1, x_2])b\alpha([y_1, y_2])b - \alpha([y_1, y_2])u[x_1, x_2]\Big\}^2 \tag{32}$$

are satisfied by Q_r.

In (31) replace x_1 by λx_1 and get

$$p\Big\{a[x_1, x_2]a[y_1, y_2] + \lambda^{2^h-1}\alpha([x_1, x_2])ba[y_1, y_2] - c[y_1, y_2][x_1, x_2]\Big\}^2. \tag{33}$$

Now we denote

$$\Omega_1(x_1, x_2, y_1, y_2) = a[x_1, x_2]a[y_1, y_2] - c[y_1, y_2][x_1, x_2]$$

and

$$\Omega_2(x_1, x_2, y_1, y_2) = \alpha([x_1, x_2])ba[y_1, y_2]$$

obtaining

$$p\Big\{\Omega_1(r_1, r_2, r_3, r_4) + \gamma\Omega_2(r_1, r_2, r_3, r_4)\Big\}^2 = 0$$

for all $r_1, r_2, r_3, r_4 \in Q_r$, with $\gamma = \lambda^{2^h-1} \neq 1$. Thus, as above, for $z_0 = p\Omega_1^2$, $z_1 = p(\Omega_1\Omega_2 + \Omega_2\Omega_1)$ and $z_2 = p\Omega_2^2$, one has

$$z_0 + \gamma z_1 + \gamma^2 z_2 = 0. \tag{34}$$

By the same above Vandermonde determinant argument, we arrive at $z_0 = 0$, that is Q_r satisfies

$$p\Big\{a[x_1, x_2]a[y_1, y_2] - c[y_1, y_2][x_1, x_2]\Big\}^2. \tag{35}$$

Application of Proposition 2.3 to (35) leads to the conclusion $pa = pc = 0$, unless Q_r satisfies s_4.
On the other hand, if we replace x_1 by λx_1 in (32), then Q_r satisfies

$$p\Big\{a[x_1,x_2]\alpha([y_1,y_2])b + \lambda^{2^h-1}\alpha([x_1,x_2])b\alpha([y_1,y_2])b - \alpha([y_1,y_2])u[x_1,x_2]\Big\}^2 \quad (36)$$

Once again, we denote

$$\Psi_1(x_1,x_2,y_1,y_2) = a[x_1,x_2]\alpha([y_1,y_2])b - \alpha([y_1,y_2])u[x_1,x_2]$$

and

$$\Psi_2(x_1,x_2,y_1,y_2) = \alpha([x_1,x_2])b\alpha([y_1,y_2])b$$

obtaining

$$p\Big\{\Psi_1(r_1,r_2,r_3,r_4) + \gamma\Psi_2(r_1,r_2,r_3,r_4)\Big\}^2 = 0$$

for all $r_1,r_2,r_3,r_4 \in Q_r$, with $\gamma = \lambda^{2^h-1} \neq 1$. Therefore, for $w_0 = p\Psi_1^2$, $w_1 = p(\Psi_1\Psi_2 + \Psi_2\Psi_1)$ and $w_2 = p\Psi_2^2$, it follows that

$$w_0 + \gamma w_1 + \gamma^2 w_2 = 0. \quad (37)$$

Similarly to what we saw previously, we get $w_0 = 0$ and $w_2 = 0$, that is both

$$p\Big\{a[x_1,x_2]\alpha([y_1,y_2])b - \alpha([y_1,y_2])u[x_1,x_2]\Big\}^2 \quad (38)$$

and

$$p\Big\{\alpha([x_1,x_2])b\alpha([y_1,y_2])b\Big\}^2 \quad (39)$$

are identities for Q_r. We remark that (39) means that

$$p\Big\{[r_1,r_2]b[s_1,s_2]b\Big\}^2 = 0 \quad \forall r_1,r_2,s_1,s_2 \in Q_r$$

implying $b = 0$ (since $p \neq 0$). Then (38) reduces to

$$p\Big\{\alpha([y_1,y_2])u[x_1,x_2]\Big\}^2$$

that is $u = 0$.
Hence we have proved that either Q_r satisfies s_4, or $F(x) = ax$ and $G(x) = cx$, for any $x \in R$, with $pa = pc = 0$, as required. □

4. The proof of Theorem 1.1

In this final section we consider the more general situation and write $F(x) = ax + d(x)$, $G(x) = cx + \delta(x)$ for all $x \in R$, where $a, c \in Q_r$ and d, δ are skew derivations of R. Let α be the automorphism associated with d and δ. Thus, for any $x, y \in R$,

$$d(xy) = d(x)y + \alpha(x)d(y)$$

and

$$\delta(xy) = \delta(x)y + \alpha(x)\delta(y).$$

To prove our main result, we always assume that R does not satisfy the standard identity s_4. Under this assumption, and since L is not central, there exists a non-zero ideal I of R such that $0 \neq [I, R] \subseteq L$ ([15, pages 4-5], [12, Lemma 2 and Proposition 1], [17, Theorem 4]). Therefore we have that there exists a non-central ideal I of R such that

$$p\Big\{F(u)F(v) - G(v)u\Big\}^n = 0 \quad \forall u, v \in [I, I].$$

Since R and I satisfy the same generalized differential identities with automorphisms, we may assume that

$$p\Big\{F([x_1, x_2])F([y_1, y_2]) - G([y_1, y_2])[x_1, x_2]\Big\}^n \tag{40}$$

is an identity for R. In other words R satisfies

$$p\Bigg\{\Bigg(a[x_1, x_2]+d([x_1, x_2]\Bigg)\Bigg(a[y_1, y_2])+d([y_1, y_2]\Bigg)-\Bigg(c[y_1, y_2]+\delta([y_1, y_2])\Bigg)[x_1, x_2]\Bigg\}^n. \tag{41}$$

The following results which will be useful in the sequel:

Fact 4.1. ([10, Lemma 3.2]) Let R be a prime ring, $\alpha, \beta \in Aut(Q_r)$ and $d : R \to R$ be a skew derivation, associated with the automorphism α. If there exist $0 \neq \theta \in C$, $0 \neq \eta \in C$ and $u, b \in Q_r$ such that

$$d(x) = \theta\Bigg(ux - \alpha(x)u\Bigg) + \eta\Bigg(bx - \beta(x)b\Bigg), \quad \forall x \in R$$

then d is an inner skew derivation of R. More precisely, either $b = 0$ or $\alpha = \beta$.

Fact 4.2. ([11, Fact 4.2]) Let R be a prime ring, $\alpha, \beta \in Aut(Q_r)$ and $d, \delta : R \to R$ be skew derivations, associated with the automorphism α. If there exist $0 \neq \eta \in C$

and $p \in Q_r$ such that

$$\delta(x) = \eta d(x) + \Big(px - \beta(x)p\Big), \quad \forall x \in R \tag{42}$$

then either $\alpha = \beta$ or $px - \beta(x)p = 0$ and $\delta(x) = \eta d(x)$, for any $x \in R$.

Remark 4.3. If we assume that both F and G are inner generalized skew derivations, then we may write

$$d(x) = bx - \alpha(x)b \text{ and } F(x) = ax + bx - \alpha(x)b \quad \forall x \in R$$

and

$$\delta(x) = ux - \alpha(x)u \text{ and } G(x) = cx + ux - \alpha(x)u \quad \forall x \in R$$

where $a, b, c, u \in Q_r$ and $\alpha \in Aut(R)$.
We would like to point out that, in case R satisfies (41) and by Proposition 3.2, we may conclude that one of the following holds:

(1) $d = \delta = 0$ and $pa = pc = 0$;
(2) R satisfies s_4.

Proof of Theorem 1.1. By Propositions 2.3 and 3.2 we may assume that d, δ are not simultaneously inner skew derivations. In particular d, δ are not simultaneously zero. In all that follows we may also suppose that R does not satisfy s_4.
By (41), R satisfies

$$\begin{aligned} p\Big\{ &\Big(a[x_1,x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)\Big)\cdot \\ &\cdot \Big(a[y_1,y_2] + d(y_1)y_2 + \alpha(y_1)d(y_2) - d(y_2)y_1 - \alpha(y_2)d(y_1)\Big) \\ &- c[y_1,y_2][x_1,x_2] - \Big(\delta(y_1)y_2 + \alpha(y_1)\delta(y_2) - \delta(y_2)y_1 - \alpha(y_2)\delta(y_1)\Big)[x_1,x_2]\Big\}^n. \end{aligned} \tag{43}$$

Let $d \neq 0$ and $\delta \neq 0$ be $C-$linearly independent modulo SD_{int}.

In this case, by (43), R satisfies

$$\begin{aligned} p\Big\{ &\Big(a[x_1,x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1\Big)\cdot \\ &\cdot \Big(a[y_1,y_2] + z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1\Big) \\ &- c[y_1,y_2][x_1,x_2] - \Big(w_1y_2 + \alpha(y_1)w_2 - w_2y_1 - \alpha(y_2)w_1\Big)[x_1,x_2]\Big\}^n. \end{aligned} \tag{44}$$

In particular, for $x_1 = t_2 = y_1 = z_2 = 0$, R satisfies

$$p\Big\{\Big(t_1x_2 - \alpha(x_2)t_1\Big) \cdot \Big(z_1y_2 - \alpha(y_2)z_1\Big)\Big\}^n. \tag{45}$$

If α is the identity map, then R satisfies $p[x_1, x_2]^{2n}$, which forces $p = 0$, a contradiction. Thus α is not the identity on R. Since (45) is a non-trivial generalized identity also for Q_r, then Q_r is isomorphic to a dense subring of the ring of linear transformations of a vector space V over a division ring D, containing non-zero linear transformations of finite rank and, as above, there exists a semi-linear automorphism $T \in End(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q_r$.
Hence, Q_r satisfies

$$p\Big\{\Big(t_1x_2 - Tx_2T^{-1}t_1\Big) \cdot \Big(z_1y_2 - Ty_2T^{-1}z_1\Big)\Big\}^n. \tag{46}$$

Let $dim_D V \geq 2$ and suppose that, for any $v \in V$, there exists $\lambda_v \in D$ such that $T^{-1}v = v\lambda_v$. As mentioned above, there exists a unique $\lambda \in D$ such that $T^{-1}v = v\lambda$, for all $v \in V$. In this case α is the identity, a contradiction.
Therefore, there exists $v \in V$ such that $\{v, T^{-1}v\}$ is linearly D-independent. By the density of Q_r, there exist $r_1, r_2, s_1, s_2 \in Q_r$ such that

$$s_1v = 0 \quad s_2v = T^{-1}v \quad s_1T^{-1}v = v \quad r_1v = 0 \quad r_2v = T^{-1}v \quad r_1T^{-1}v = v$$

and, by (46), we get

$$p\Big\{\Big(r_1r_2 - Tr_2T^{-1}r_1\Big) \cdot \Big(s_1s_2 - Ts_2T^{-1}s_1\Big)\Big\}^n v = pv. \tag{47}$$

As above, application of Lemma 2.2 and since $p \neq 0$, it follows $dim_D V = 2$ and Q_r satisfies

$$p\Big\{\Big(t_1x_2 - \alpha(x_2)t_1\Big) \cdot \Big(z_1y_2 - \alpha(y_2)z_1\Big)\Big\}^2. \tag{48}$$

On the other hand, if $dim_D V = 1$, Q_r is a domain satisfying

$$p\Big\{\Big(t_1x_2 - \alpha(x_2)t_1\Big) \cdot \Big(z_1y_2 - \alpha(y_2)z_1\Big)\Big\}.$$

Therefore, more generally we may assume that (48) is an identity for Q_r. In particular, for $t_1 = z_1$ and $x_2 = y_2$, Q_r satisfies $p\Big(z_1y_2 - \alpha(y_2)z_1\Big)^2$. Since $p \neq 0$, this last relation implies $\Big(r_1r_2 - \alpha(r_2)r_1\Big) = 0$, for any $r_1, r_2 \in Q_r$ (see [1, Theorem B and Corollary]). It is easy to see that this case may occur only if R is commutative and α is the identity, a contradiction.

Let $d \neq 0$ and $\delta \neq 0$ be $C-$linearly dependent modulo SD_{int}.

Here we assume that there exist $\lambda, \mu \in C$, $c' \in Q_r$ and $\gamma \in Aut(R)$ such that $\lambda d(x) + \mu\delta(x) = c'x - \gamma(x)c'$ for all $x \in R$.

- We firstly study the case $0 \neq \lambda \in C$ and $0 \neq \mu \in C$.

Denote $\eta = -\mu^{-1}\lambda$ and $p' = \mu^{-1}c'$. So $\delta(x) = \eta d(x) + p'x - \gamma(x)p'$ for all $x \in R$. By Fact 4.2, we know that either $\delta(x) = \eta d(x)$ for all $x \in R$, or $\gamma = \alpha$.
In case $\gamma = \alpha$, one has $\delta(x) = \eta d(x) + p'x - \alpha(x)p'$ for all $x \in R$. Therefore by (43), Q_r satisfies

$$p\Bigg\{\Bigg(a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)\Bigg)\cdot \\ \cdot\Bigg(a[y_1, y_2] + d(y_1)y_2 + \alpha(y_1)d(y_2) - d(y_2)y_1 - \alpha(y_2)d(y_1)\Bigg) \\ - c[y_1, y_2][x_1, x_2] - \Bigg(\eta d(y_1)y_2 + \alpha(y_1)\eta d(y_2) - \eta d(y_2)y_1 - \alpha(y_2)\eta d(y_1)\Bigg)[x_1, x_2] \\ - \Bigg(p'[y_1, y_2] - [\alpha(y_1), \alpha(y_2)]p'\Bigg)[x_1, x_2]\Bigg\}^n. \tag{49}$$

Applying Fact 4.1 we may assume that d is not inner. By (49) Q_r satisfies

$$p\Bigg\{\Bigg(a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1\Bigg)\cdot \\ \cdot\Bigg(a[y_1, y_2] + z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1\Bigg) \\ - c[y_1, y_2][x_1, x_2] - \Bigg(\eta z_1y_2 + \alpha(y_1)\eta z_2 - \eta z_2y_1 - \alpha(y_2)\eta z_1\Bigg)[x_1, x_2] \\ - \Bigg(p'[y_1, y_2] - [\alpha(y_1), \alpha(y_2)]p'\Bigg)[x_1, x_2]\Bigg\}^n. \tag{50}$$

In particular, for $x_1 = t_2 = y_1 = z_2 = 0$ in (50), it follows that Q_r satisfies again relation (45), so that a contradiction follows as above.

Analogously, for $\delta = \eta d$, the relation (49) reduces to

$$p\Bigg\{\Bigg(a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)\Bigg)\cdot \\ \cdot\Bigg(a[y_1, y_2] + d(y_1)y_2 + \alpha(y_1)d(y_2) - d(y_2)y_1 - \alpha(y_2)d(y_1)\Bigg) \\ - c[y_1, y_2][x_1, x_2] - \Bigg(\eta d(y_1)y_2 + \alpha(y_1)\eta d(y_2) - \eta d(y_2)y_1 - \alpha(y_2)\eta d(y_1)\Bigg)[x_1, x_2]\Bigg\}^n. \tag{51}$$

It is easy to see that Q_r satisfies again (45) and we conclude as above.

- Assume now $\lambda = 0$.

Hence $\delta(x) = p'x - \gamma(x)p'$ for all $x \in R$, where $p' = \mu^{-1}c'$ and d is not inner. Then, by relation (43), Q_r satisfies

$$p\Big\{\Big(a[x_1,x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1\Big)\cdot \cdot\Big(a[y_1,y_2] + z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1\Big) - c[y_1,y_2][x_1,x_2] - \Big(p'[y_1,y_2] - [\gamma(y_1),\gamma(y_2)]p'\Big)[x_1,x_2]\Big\}^n. \tag{52}$$

Also in this case, for $x_1 = t_2 = y_1 = z_2 = 0$ in (52), Q_r satisfies (45) and we are done.

- The case $\mu = 0$

In this case, $d(x) = p'x - \gamma(x)p'$ for all $x \in R$, where $p' = \lambda^{-1}c'$ and δ is not inner. Moreover $\alpha = \gamma$ (as a reduction of Fact 4.2). Relation (43) implies that Q_r satisfies

$$p\Big\{\Big(a[x_1,x_2] + p'[x_1,x_2] - [\alpha(x_1),\alpha(x_2)]p'\Big)\Big(a[y_1,y_2] + p'[y_1,y_2] - [\alpha(y_1),\alpha(y_2)]p'\Big) - c[y_1,y_2][x_1,x_2] - \Big(\delta(y_1)y_2 + \alpha(y_1)\delta(y_2) - \delta(y_2)y_1 - \alpha(y_2)\delta(y_1)\Big)[x_1,x_2]\Big\}^n. \tag{53}$$

Since δ is not inner, Q_r satisfies

$$p\Big\{\Big(a[x_1,x_2] + p'[x_1,x_2] - [\alpha(x_1),\alpha(x_2)]p'\Big)\Big(a[y_1,y_2] + p'[y_1,y_2] - [\alpha(y_1),\alpha(y_2)]p'\Big) - c[y_1,y_2][x_1,x_2] - \Big(z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1\Big)[x_1,x_2]\Big\}^n. \tag{54}$$

For $z_1 = z_2 = 0$ in (54), it follows that

$$p\Big\{\Big(a[x_1,x_2] + p'[x_1,x_2] - [\alpha(x_1),\alpha(x_2)]p'\Big)\Big(a[y_1,y_2] + p'[y_1,y_2] - [\alpha(y_1),\alpha(y_2)]p'\Big) - c[y_1,y_2][x_1,x_2]\Big\}^n. \tag{55}$$

is an identity for Q_r. Application of Proposition 3.2 implies $p'x - \alpha(x)p' = 0$, for any $x \in Q_r$, that is $d = 0$, which is a contradiction.

The case $\delta = 0$

Here we have to consider the only case when $0 \neq d$ is an outer skew derivation. By (43), R satisfies

$$p\Big\{\Big(a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)\Big)\cdot \\ \cdot\Big(a[y_1, y_2] + d(y_1)y_2 + \alpha(y_1)d(y_2) - d(y_2)y_1 - \alpha(y_2)d(y_1)\Big) \\ - c[y_1, y_2][x_1, x_2]\Big\}^n. \tag{56}$$

Then, since $0 \neq d$ is outer, R satisfies

$$p\Big\{\Big(a[x_1, x_2] + t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1\Big)\cdot \\ \cdot\Big(a[y_1, y_2] + z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1\Big) \\ - c[y_1, y_2][x_1, x_2]\Big\}^n. \tag{57}$$

As above, for $x_1 = t_2 = y_1 = z_2 = 0$ in (57), (45) is an identity for R and we are done again.

The case $d = 0$

In this final case, relation (43) reduces to

$$p\Big\{a[x_1, x_2]a[y_1, y_2] - c[y_1, y_2][x_1, x_2] \\ - \Big(\delta(y_1)y_2 + \alpha(y_1)\delta(y_2) - \delta(y_2)y_1 - \alpha(y_2)\delta(y_1)\Big)[x_1, x_2]\Big\}^n. \tag{58}$$

Moreover, we may assume that $0 \neq \delta$ is not inner. Therefore (58) implies that R satisfies

$$p\Big\{a[x_1, x_2]a[y_1, y_2] - c[y_1, y_2][x_1, x_2] \\ - \Big(z_1y_2 + \alpha(y_1)z_2 - z_2y_1 - \alpha(y_2)z_1\Big)[x_1, x_2]\Big\}^n \tag{59}$$

and in particular, for $y_1 = z_2 = 0$ in (59), it follows that

$$p\Big\{\Big(z_1y_2 - \alpha(y_2)z_1\Big)[x_1, x_2]\Big\}^n \tag{60}$$

is satisfied by R, as well as by Q_r.

Now let's fix any two elements $r_1, r_2 \in Q_r$ and denote $w = r_1r_2 - \alpha(r_2)r_1$. By (60) we have that

$$p\Big\{w[x_1,x_2]\Big\}^n$$

is an identity for Q_r. This last implies $pw=0$ (see for instance [8, Theorem]). By the arbitrariness of $r_1,r_2\in Q_r$, it follows that Q_r satisfies the generalized identity

$$p\Big\{z_1y_2-\alpha(y_2)z_1\Big\}.$$

Since $p\neq 0$, as above we get $\Big(r_1r_2-\alpha(r_2)r_1\Big)=0$, for any $r_1,r_2\in Q_r$ (see [1, Theorem B and Corollary]). Once again, since R is not commutative, a contradiction follows. □

Availability of data and material. No datasets were generated or analysed during the current study.

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