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# Some generalised extended incomplete beta functions and applications

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*Gamma function, Beta function, Incomplete beta function, Beta distribution*

**Abstract** — This paper introduces generalised incomplete beta functions defined by the generalised beta function. Firstly, we provide some of the generalised beta function's basic properties, such as integral representations, summation formulas, Mellin transform, and beta distribution. We then present several fundamental properties, such as integral representations, summation formulas, and recurrence relations with the help of the generalised incomplete beta functions.

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## 1. Introduction

The classical beta function $B(a,b)$ is defined by [1–4]

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt,\quad \Re(a),\Re(b)>0 \tag{1.1}$$

Moreover, the incomplete beta function $B_\tau(a,b)$ is defined by [1, 2]

$$B_\tau(a,b)=\int_0^\tau t^{a-1}(1-t)^{b-1}dt,\quad \Re(a),\Re(b)>0,\ \text{and } 0<\tau<1 \tag{1.2}$$

In 1997, Chaudhry and Zubair [5] defined and investigated extension of beta function

$$B_p(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\exp\left(\frac{-p}{t(t-1)}\right)dt,\quad \Re(p)>0\,\Re(a),\Re(b)>0 \tag{1.3}$$

Here, if $p=0$, this equation reduces to the classical beta function provided in Equation (1.1). Furthermore, the extension of incomplete beta function $B_\tau(a,b)$ is given by [5]

$$B_p(a,b;\tau)=\int_0^\tau t^{a-1}(1-t)^{b-1}\exp\left(\frac{-p}{t(t-1)}\right)dt,\quad \Re(p)>0,\Re(a),\Re(b)>0 \text{ and } 0<\tau<1 \tag{1.4}$$

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It is easily yielded that setting $\tau = 1$ in Equation (1.4), which provides us the special case of extension of beta function (1.3), and taking $p = 0$ and $\tau = 1$ in Equation (1.4) reduces to Equation (1.1). It will be observed that this expansion is fruitful because it expresses most of the properties of the beta function naturally and simply. They also expressed various integral representations, Mellin transform, a large number of properties and cases from the point of special functions.

Afterwards, Özergin et al. [6] introduced and considered the generalisation of beta function as given:

$$B_p^{\{\alpha,\beta\}}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(t-1)}\right)dt, \quad \Re(p)>0, \Re(\alpha), \Re(\beta)>0, \Re(a), \Re(b)>0 \tag{1.5}$$

In 2011, Parmar and Chopra [7] obtainded the generalisation of incomplete beta function given as follows:

$$B_{p;\tau}^{\{\alpha,\beta\}}(a,b) = \int_0^\tau t^{a-1}(1-t)^{b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(t-1)}\right)dt, \quad \Re(p)>0, \Re(\alpha), \Re(\beta)>0, \Re(a), \Re(b)>0 \text{ and } 0<\tau<1 \tag{1.6}$$

It is observed that putting $\alpha = \beta$ and $p = 0$ in Equation (1.5), which gives us Equation (1.1). Besides, setting $\tau = 1$ in Equation (1.6) reduces to Equation (1.5).

Proceeding from the generalisations of the beta function expressed above, various generalisations of Equation (1.1) have been introduced and investigated by many authors (see [8–23]).

Throughout this paper, let $\mathbb{C}$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ be the sets of complex numbers, non-positive integers, and positive integers, respectively, and assume that $\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0, \Re(\nu) > 0, \Re(x) > 0$. Recently, Şahin et al. [24] proposed a generalisation of the extended beta function as follows:

$$B_{p,q}^{(\kappa,\mu)}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\exp\left(-\frac{p}{t^\kappa}-\frac{q}{(1-t)^\mu}\right)dt \tag{1.7}$$

First, by selecting a known generalisation of Equation (1.7), systematically, we goal to determine further properties and representations for this beta function such as integral representations, Mellin transform. Next, we introduce a new generalisation of the extended incomplete beta function using Equation (1.7). Moreover, we obtain its integral representations and examine its various properties. Finally, we provide the beta distribution for a new generalisation of the extended beta function provided in Equation (1.7).

## 2. Integral Representaions of Equation (1.7)

This section presents various integral representations of Equation (1.7), professed in the following theorem.

**Theorem 2.1.** The following integral representation for the function $B_{p,q}^{(\kappa,\mu)}(a,b)$ given by Equation (1.7) holds true:

$$\int_0^\infty\int_0^\infty p^{\zeta-1}q^{\eta-1}B_{p,q}^{(\kappa,\mu)}(a,b)\,dp\,dq = \Gamma(\zeta)\Gamma(\eta)B(a+\kappa\zeta, b+\mu\eta), \quad \Re(a+\kappa\zeta), \Re(b+\mu\eta)>0 \tag{2.1}$$

**Proof.**
Multiplying each side of Equation (1.7) by $p^{\zeta-1}q^{\eta-1}$ and integrating with respect to $0 \le p, q < \infty$, we get

$$\int_0^\infty\int_0^\infty p^{\zeta-1}q^{\eta-1}B_{p,q}^{(\kappa,\mu)}(a,b)\,dp\,dq = \int_0^\infty\int_0^\infty p^{\zeta-1}q^{\eta-1}\left\{\int_0^1 t^{a-1}(1-t)^{b-1}\exp\left(-\frac{p}{t^\kappa}-\frac{q}{(1-t)^\mu}\right)dt\right\}dp\,dq \tag{2.2}$$

Because of the uniform convergence, we can be changed the order of integration in Equation (2.2). Thus, we obtain

$$\int_0^\infty \int_0^\infty p^{\zeta-1} q^{\eta-1} B_{p,q}^{(\kappa,\mu)}(a,b)dpdq = \int_0^1 t^{a-1}(1-t)^{b-1}\left\{\int_0^\infty p^{\zeta-1}\exp\left(-\frac{p}{t^\kappa}\right)dp \int_0^\infty q^{\eta-1}\exp\left(-\frac{q}{(1-t)^\mu}\right)dq\right\}dt \quad (2.3)$$

Further, the above integrals can be reduced in terms of the gamma function to give Theorem 2.1.

$$\begin{aligned}\int_0^\infty \int_0^\infty p^{\zeta-1} q^{\eta-1} B_{p,q}^{(\kappa,\mu)}(a,b)dpdq &= \Gamma(\zeta)\Gamma(\eta)\int_0^1 t^{a+\kappa\zeta-1}(1-t)^{b+\mu\eta-1}dt \\ &= \Gamma(\zeta)\Gamma(\eta)B(a+\kappa\zeta, b+\mu\eta)\end{aligned} \quad (2.4)$$

**Remark 2.2.** Taking $\zeta = 1$ and $\eta = 1$ in (2.1), we have

$$\int_0^\infty \int_0^\infty B_{p,q}^{(\kappa,\mu)}(a,b)dpdq = B(a+\kappa, b+\mu) \quad (2.5)$$

Moreover, taking $\kappa = 1$ and $\mu = 1$ in Equation (2.5), which gives us the special case of integral representation for extended beta function [9].

**Theorem 2.3.** The following integral representations for the function $B_{p,q}^{(\kappa,\mu)}(a,b)$ given by Equation (1.7) hold true, for $\Re(a), \Re(b) > 0$,

$$B_{p,q}^{(\kappa,\mu)}(a,b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}\exp\left(-\frac{p(1+u)^\kappa}{u^\kappa} - q(1+u)^\mu\right)du \quad (2.6)$$

$$B_{p,q}^{(\kappa,\mu)}(a,b) = 2^{1-a-b}\int_{-1}^1 (1+u)^{a-1}(1-u)^{b-1}\exp\left(-\frac{p2^\kappa}{(1+u)^\kappa} - \frac{q2^\mu}{(1-u)^\mu}\right)du \quad (2.7)$$

$$B_{p,q}^{(\kappa,\mu)}(a,b) = 2\int_0^{\frac{\pi}{2}} \cos^{2a-1}\theta \sin^{2b-1}\theta \exp\left(-p\sec^{2\kappa}\theta - q\csc^{2\mu}\theta\right)d\theta \quad (2.8)$$

$$B_{p,q}^{(\kappa,\mu)}(a,b) = (y-x)^{1-a-b}\int_x^y (u-x)^{a-1}(y-u)^{b-1}\exp\left(-\frac{p(y-x)^\kappa}{(u-x)^\kappa} - \frac{q(y-x)^\mu}{(y-u)^\mu}\right)du \quad (2.9)$$

$$B_{p,q}^{(\kappa,\mu)}(a,b) = 2\int_0^\infty \tanh^{2a-1}\theta\, sech^{2b}\theta \exp\left(-p\coth^{2\kappa}\theta - q\cosh^{2\mu}\theta\right)d\theta \quad (2.10)$$

**Proof.**

Putting to use the transformations $t = \frac{u}{u+1}$, $t = \frac{1+u}{2}$, $t = \cos^2\theta$, and $t = \frac{u-x}{y-u}$ in Equation (1.7) and $\sinh^2\theta$ in Equation (2.1). Therefore, we can be obtained Equations (2.6), (2.7), (2.8), (2.9), and (2.10), respectively.

**Corollary 2.4.** If setting $\kappa = \mu$ in Equations (2.6), (2.7), and (2.9), the following integral representations are obtained, respectively.

$$B_{p,q}^{(\kappa,\kappa)}(a,b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}\exp\left((1+u)^\kappa\left[-q - \frac{p}{u^\kappa}\right]\right)du \quad (2.11)$$

$$B_{p,q}^{(\kappa,\kappa)}(a,b) = 2^{1-a-b}\int_{-1}^1 (1+u)^{a-1}(1-u)^{b-1}\exp\left(\left[\frac{2}{1-u^2}\right]^\kappa\left[-p(1-u)^\kappa - q(1+u)^\kappa\right]\right)du \quad (2.12)$$

$$B_{p,q}^{(\kappa,\kappa)}(a,b) = (y-x)^{1-a-b}\int_x^y (u-x)^{a-1}(y-u)^{b-1}\exp\left(\left[\frac{(y-x)}{u^2+(x+y)u-xy}\right]^\kappa\left[-p(y-u)^\kappa - q(u-x)^\kappa\right]\right)du \quad (2.13)$$

**Remark 2.5.** Applying $\kappa = 1$ in Equations (2.11), (2.12), and (2.13), we obtain a special case of integral representation for extended beta function [9].

## 3. Certain Formulas for Equation (1.7)

This part obtains a functional relation and some summation formulas for Equation (1.7).

**Theorem 3.1.** The following functional relation for the function $B_{p,q}^{\kappa,\mu}(a,b)$ given by Equation (1.7) holds true:

$$B_{p,q}^{(\kappa,\mu)}(a,b) = B_{p,q}^{(\kappa,\mu)}(a+1,b) + B_{p,q}^{(\kappa,\mu)}(a,b+1) \tag{3.1}$$

**Proof.**
The right hand side of (3.1) occurs

$$B_{p,q}^{(\kappa,\mu)}(a+1,b) + B_{p,q}^{(\kappa,\mu)}(a,b+1) = \int_0^1 \left[t^a(1-t)^{b-1} + t^{a-1}(1-t)^b\right] \exp\left(-\frac{p}{t^\kappa} - \frac{q}{(1-t)^\mu}\right) dt \tag{3.2}$$

which after a simple arrangement, becomes to the left hand side of (3.1).

**Theorem 3.2.** The following summation formula for the function $B_{p,q}^{\kappa,\mu}(a,b)$ given by Equation (1.7) holds true:

$$B_{p,q}^{(\kappa,\mu)}(a,1-b) = \sum_{m=0}^{\infty} \frac{(b)_m}{m!} B_{p,q}^{(\kappa,\mu)}(a+m,1) \tag{3.3}$$

**Proof.**
Using the following binomial series

$$(1-t)^{-b} = \sum_{m=0}^{\infty} \frac{(b)_m t^m}{m!} \tag{3.4}$$

in Equation (1.7), we have

$$B_{p,q}^{(\kappa,\mu)}(a,1-b) = \int_0^1 \sum_{m=0}^{\infty} \frac{(b)_m t^{a+m-1}}{m!} \exp\left(-\frac{p}{t^\kappa} - \frac{q}{(1-t)^\mu}\right) dt \tag{3.5}$$

Then, shifting integration and summation in Equation (3.5) and taking advantage of Equation (1.7) gives the required result.

**Theorem 3.3.** The following summation formula for the function $B_{p,q}^{\kappa,\mu}(a,b)$ given by Equation (1.7) holds true:

$$B_{p,q}^{(\kappa,\mu)}(a,b) = \sum_{m=0}^{\infty} B_{p,q}^{(\kappa,\mu)}(a+m,b+1) \tag{3.6}$$

**Proof.**
Setting the following binomial series

$$(1-t)^{b-1} = (1-t)^b \sum_{m=0}^{\infty} t^m \tag{3.7}$$

in Equation (1.7), we have

$$B_{p,q}^{(\kappa,\mu)}(a,b) = \int_0^1 (1-t)^b \sum_{m=0}^{\infty} t^{a+m-1} \exp\left(-\frac{p}{t^\kappa} - \frac{q}{(1-t)^\mu}\right) dt \tag{3.8}$$

Then, changing integration and summation in Equation (3.8) and taking into consideration of Equation (1.7) obtains the required result.

**Definition 3.4.** The extended Gamma function is defined by

$$\Gamma_p^{(\nu)}(x) = \int_0^\infty t^{x-1} \exp\big(-t - \frac{p}{t^\nu}\big) dt \tag{3.9}$$

**Theorem 3.5.** The following product formula for the function $B_{p,q}^{\kappa,\mu}(a,b)$ given by Equation (1.7) holds true:

$$\Gamma_p^{(\kappa)}(a)\Gamma_q^{(\mu)}(b) = 2\int_0^\infty r^{2(a+b)-1} \exp(-r^2) B^{(\kappa,\mu)}_{\frac{p}{r^{2\kappa}},\frac{q}{r^{2\mu}}}(a,b) dr \tag{3.10}$$

**Proof.**
Putting $t = \zeta^2$ and $t = \eta^2$ in Equation (3.9), we obtain

$$\Gamma_p^{(\kappa)}(x) = 2\int_0^\infty \zeta^{2a-1} \exp\left(-\zeta^2 - \frac{p}{\zeta^{2\kappa}}\right) d\zeta \tag{3.11}$$

and

$$\Gamma_q^{(\mu)}(b) = 2\int_0^\infty \eta^{2b-1} \exp\left(-\eta^2 - \frac{q}{\eta^{2\mu}}\right) d\eta \tag{3.12}$$

Therefore,

$$\Gamma_p^{(\kappa)}(a)\Gamma_q^{(\mu)}(b) = 4\int_0^\infty \int_0^\infty \zeta^{2a-1}\eta^{2b-1} \exp(-\zeta^2 - \eta^2) \exp\left(-\frac{p}{\zeta^{2\kappa}} - \frac{q}{\eta^{2\mu}}\right) d\zeta\, d\eta \tag{3.13}$$

Replacing $\zeta = r\cos\theta$ and $\eta = r\sin\theta$ in Equation (3.13), we get

$$\Gamma_p^{(\kappa)}(a)\Gamma_q^{(\mu)}(b) = 2\int_0^\infty r^{2(a+b)-1} \exp(-r^2) \left[2\int_0^\infty \cos^{2a-1}\theta \sin^{2b-1}\theta \exp\left(-\frac{p}{r^{2\kappa}\cos^{2\kappa}\theta} - \frac{q}{r^{2\mu}\sin^{2\mu}\theta}\right) d\theta\right] dr \tag{3.14}$$

Taking advantage of Equation (2.8) in Equation (3.14) obtains the required result.

**Theorem 3.6.** The following summation formula for the function $B_{p,q}^{\kappa,\mu}(a,b)$ given by Equation (1.7) holds true:

$$B_{p,q}^{(\kappa,\mu)}(-\zeta, -\zeta-n) = \sum_{m=0}^{n} \binom{n}{m} B_{p,q}^{(\kappa,\mu)}(\zeta+m, -\zeta-m) \tag{3.15}$$

**Proof.**
Applying $a = \zeta$ and $b = -\zeta - n$ in Equation (3.1), we have

$$B_{p,q}^{(\kappa,\mu)}(\zeta, -\zeta-n) = B_{p,q}^{(\kappa,\mu)}(\zeta+1, -\zeta-n) + B_{p,q}^{(\kappa,\mu)}(\zeta, -\zeta-n+1) \tag{3.16}$$

Begin with $n = 1$, we can write this formula recursively to have

$$\begin{aligned}
B_{p,q}^{(\kappa,\mu)}(\zeta,-\zeta-1) &= B_{p,q}^{(\kappa,\mu)}(\zeta+1,-\zeta-1) + B_{p,q}^{(\kappa,\mu)}(\zeta,-\zeta)\\
B_{p,q}^{(\kappa,\mu)}(\zeta,-\zeta-2) &= B_{p,q}^{(\kappa,\mu)}(\zeta+2,-\zeta-2) + 2B_{p,q}^{(\kappa,\mu)}(\zeta+1,-\zeta-1) + B_{p,q}^{(\kappa,\mu)}(\zeta,-\zeta)\\
B_{p,q}^{(\kappa,\mu)}(\zeta,-\zeta-3) &= B_{p,q}^{(\kappa,\mu)}(\zeta+3,-\zeta-3) + 3B_{p,q}^{(\kappa,\mu)}(\zeta+2,-\zeta-2) + 3B_{p,q}^{(\kappa,\mu)}(\zeta+1,-\zeta-1) + B_{p,q}^{(\kappa,\mu)}(\zeta,-\zeta)
\end{aligned}$$

and so on. Then, it can be seen from the above equation that the coefficients of the expression come from the finite binomial expansion. Thus, we can get the required result.

## 4. Mellin Transform of Equation (1.7)

This section provides the Mellin transform and recurrence relation for Equation (1.7).

**Theorem 4.1.** The following Mellin transform for the function $B_{p,q}^{\kappa,\mu}(a,b)$ given by Equation (1.7) holds true:

$$B_{p,q}^{(\kappa,\mu)}(a,b) = \frac{1}{(2\pi i)^2}\int_{\nu_1-i\infty}^{\nu_1+i\infty}\int_{\nu_2-i\infty}^{\nu_2+i\infty}\frac{\Gamma(\zeta)\Gamma(\eta)\Gamma(a+\kappa\zeta)\Gamma(b+\mu\eta)}{\Gamma(a+b+\kappa\zeta+\mu\eta)}p^{-\zeta}q^{-\eta}d\zeta d\eta \tag{4.1}$$

**Proof.**
Using the Mellin transform in [1–3] for Equation (1.7), we have

$$M\left\{B_{p,q}^{(\kappa,\mu)}(a,b); p\to\zeta, q\to\eta\right\} = \frac{\Gamma(\zeta)\Gamma(\eta)\Gamma(a+\kappa\zeta)\Gamma(b+\mu\eta)}{\Gamma(a+b+\kappa\zeta+\mu\eta)} \tag{4.2}$$

Then, taking the inverse Mellin transform in [1–3] for Equation (4.2), we can obtain the required result.

**Theorem 4.2.** The following recurrence relation for the function $B_{p,q}^{\kappa,\mu}(a,b)$ given by Equation (1.7) holds true:

$$aB_{p,q}^{(\kappa,\mu)}(a,b+1) - bB_{p,q}^{(\kappa,\mu)}(a+1,b) = \mu q B_{p,q}^{(\kappa,\mu)}(a+1,b-1) - \kappa p B_{p,q}^{(\kappa,\mu)}(a-1,b+1) \tag{4.3}$$

**Proof.**
The Mellin transform of Equation (1.7) is

$$B_{p,q}^{(\kappa,\mu)}(a,b) = M\left\{f_{p,q}^{(\kappa,\mu)}(t,b); a\right\}$$

and

$$H(1-t) = \begin{cases} 0, & t>1 \\ 1, & t<1 \end{cases}$$

Differentiating with respect to t, we have

$$\frac{d}{dt}\left\{f_{p,q}^{(\kappa,\mu)}(t,b)\right\} = \left[-\delta(1-t)(1-t)^{b-1} - (b-1)H(1-t)(1-t)^{b-2} + H(1-t)(1-t)^{b-1}\left(-\frac{\kappa p}{t^{\kappa+1}} - \frac{\mu q}{(1-t)^{\mu+1}}\right)\right]\exp\left(-\frac{p}{t^{\kappa}} - \frac{q}{(1-t)^{\mu}}\right)$$

where $\frac{d}{dt}H(1-t) = -\delta(1-t)$ and $\delta$ symbolizes the Dirac delta function such that $\delta(1-t) = \delta(t-1)$, for $t \neq 0$ [1–3]. Taking advantage of the relationship between the Mellin transform of a function and its derivative:

$$M\left\{f(t); a\right\} = F(a) \implies M[f'(t)] = -(a-1)F(a-1)$$

Then, making a simple arrangment, we have that

$$(a-1)B_{p,q}^{(\kappa,\mu)}(a-1,b) - (b-1)B_{p,q}^{(\kappa,\mu)}(a,b-1) = \mu q B_{p,q}^{(\kappa,\mu)}(a,b-2) - \kappa p B_{p,q}^{(\kappa,\mu)}(a-2,b) \tag{4.4}$$

Setting $a$ and $b$ by $a+1$ and $b+1$ in Equation (4.4), respectively, yields the required result.

## 5. Generalisation of Extended Incomplete Beta Function

This section defines the generalisation of the extended incomplete beta function using Equation (1.7) as follows:

$$B_{p,q}^{(\kappa,\mu)}(x,y;\tau)=\int_0^\tau t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p}{t^\kappa}-\frac{q}{(1-t)^\mu}\right)dt,\quad -\infty<x,y<\infty,\text{ and } 0<\tau<1 \tag{5.1}$$

Here, the special cases of Equation (5.1) are $B_{0,0}^{(\kappa,\mu)}(x,y;\tau)=B_\tau(x,y)$, $B_{p,p}^{(1,1)}(x,y;\tau)=B_\tau(x,y;p)$, $B_{p,p}^{(m,m)}(x,y;\tau)=B_\tau(x,y;\ p:m)$, and $B_{p,q}^{(1,1)}(x,y;\tau)=B_\tau(x,y;p,q)$ where $B_\tau(x,y)$ is the incomplete beta function provided in Equation (1.2). $B_\tau(x,y;p)$ is extended incomplete beta function given in Equation (1.4), $B_\tau(x,y;p:m)$ and $B_\tau(x,y;p,q)$ are generalised incomplete beta functions defined in [8, 9], respectively.

## 6. Integral Representations of Equation (5.1)

This section presents various integral representations for the generalisation of the extended incomplete beta function provided in Equation (5.1), professed in the following theorem.

**Theorem 6.1.** The following integral representation for the function $B_{p,q}^{(\kappa,\mu)}(a,b;\tau)$ given by Equation (5.1) holds true:

$$\int_0^\infty\int_0^\infty p^{\zeta-1}q^{\eta-1}B_{p,q}^{(\kappa,\mu)}(a,b;\tau)\,dp\,dq=\Gamma(\zeta)\Gamma(\eta)B_\tau(a+\kappa\zeta,b+\mu\eta),\quad \Re(a+\kappa\zeta),\Re(b+\mu\eta)>0 \tag{6.1}$$

where $B_\tau(a,b)$ is the incomplete beta function provided in Equation (1.2).

**Proof.**
The proof of Equation (6.1) is similar to that of Equation (2.1).

**Theorem 6.2.** The following integral representations for the function $B_{p,q}^{\kappa,\mu}(a,b;\tau)$ given by Equation (5.1) hold true:

$$B_{p,q}^{(\kappa,\mu)}(a,b;\tau)=\int_0^\sigma \frac{u^{a-1}}{(1+u)^{a+b}}\exp\left(-\frac{p(1+u)^\kappa}{u^\kappa}-q(1+u)^\mu\right)du,\quad 0<\sigma=\frac{\tau}{1-\tau}<\infty \tag{6.2}$$

$$B_{p,q}^{(\kappa,\mu)}(a,b;\tau)=2\int_0^\sigma \cos^{2a-1}\theta\sin^{2b-1}\theta\exp\left(-p\sec^{2\kappa}\theta-q\csc^{2\mu}\theta\right)d\theta,\quad 0<\sigma=\arcsin(\sqrt{\tau})\leq\frac{\pi}{2} \tag{6.3}$$

$$B_{p,q}^{(\kappa,\mu)}(a,b;\tau)=2\int_0^\sigma \tanh^{2a-1}\theta\, sech^{2b}\theta\exp\left(-p\coth^{2\kappa}\theta-q\cosh^{2\mu}\theta\right)d\theta,\quad 0<\sigma=\sinh^{-1}\left(\sqrt{\frac{\tau}{1-\tau}}\right)<\infty \tag{6.4}$$

**Proof.**
Putting to use the transformations $t=\frac{u}{u+1}$ and $t=\cos^2\theta$ in Equation (5.1) and $t=\sinh^2\theta$ in Equation (6.2), we can be obtained Equations (6.2), (6.3), and (6.4), respectively.

## 7. Certain Formulas for Equation (5.1)

This part obtains functional relation and summation formulas for the generalisation of the extended incomplete function (5.1). Besides, we present the relationship between Equation (1.7) and Equation (5.1).

**Theorem 7.1.** The following functional relation for the function $B_{p,q}^{\kappa,\mu}(a,b;\tau)$ given by Equation (5.1) holds true:

$$B_{p,q}^{(\kappa,\mu)}(a,b;\tau)=B_{p,q}^{(\kappa,\mu)}(a+1,b;\tau)+B_{p,q}^{(\kappa,\mu)}(a,b+1;\tau) \tag{7.1}$$

**Proof.**
The proof of Equation (7.1) is similar to that of Equation (3.1).

**Theorem 7.2.** The following summation formula for the function $B_{p,q}^{\kappa,\mu}(a,b;\tau)$ given by Equation (5.1) holds true:

$$B_{p,q}^{(\kappa,\mu)}(a,1-b;\tau)=\sum_{m=0}^{\infty}\frac{(b)_m}{m!}B_{p,q}^{(\kappa,\mu)}(a+m,1;\tau) \tag{7.2}$$

**Proof.**
The proof of Equation (7.2) is similar to that of Equation (3.3).

**Theorem 7.3.** The following summation formula for the function $B_{p,q}^{\kappa,\mu}(a,b;\tau)$ given by Equation (5.1) holds true:

$$B_{p,q}^{(\kappa,\mu)}(a,b;\tau)=\sum_{m=0}^{\infty}B_{p,q}^{(\kappa,\mu)}(a+m,b+1;\tau) \tag{7.3}$$

**Proof.**
The proof of Equation (7.3) is similar to that of Equation (3.6).

**Theorem 7.4.** The following identity for the function $B_{p,q}^{\kappa,\mu}(a,b;\tau)$ given by Equation (5.1) holds true:

$$B_{q,p}^{(\kappa,\mu)}(b,a;\tau)=B_{p,q}^{(\kappa,\mu)}(a,b)-B_{p,q}^{(\kappa,\mu)}(a,b;1-\tau) \tag{7.4}$$

**Proof.**
The right hand side of Equation (7.4) gives

$$B_{p,q}^{(\kappa,\mu)}(a,b)-B_{p,q}^{(\kappa,\mu)}(a,b;1-\tau)=\int_{1-\tau}^{1}x^{b-1}(1-x)^{a-1}\exp\left(-\frac{p}{x^{\kappa}}-\frac{q}{(1-x)^{\mu}}\right)dx \tag{7.5}$$

Setting $x=1-t$ in Equation (7.4), we obtain

$$\int_{0}^{\tau}t^{b-1}(1-t)^{a-1}\exp\left(-\frac{q}{t^{\kappa}}-\frac{p}{(1-t)^{\mu}}\right)dt \tag{7.6}$$

which gives the left hand side of (7.4).

**Theorem 7.5.** The following summation formula for the function $B_{p,q}^{\kappa,\mu}(a,b;\tau)$ given by Equation (5.1) holds true:

$$B_{p,q}^{(\kappa,\mu)}(-\zeta,-\zeta-n;\tau)=\sum_{m=0}^{n}\binom{n}{m}B_{p,q}^{(\kappa,\mu)}(\zeta+m,-\zeta-m;\tau) \tag{7.7}$$

**Proof.**
The proof of Equation (7.7) is similar to that of Equation (3.15).

## 8. Beta Distribution of Equation (1.7)

This section anticipated that $B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y})$ would have several applications in generalising Equation (1.7). One of these that comes to mind is applications in statistics. For example, the conventional beta distribution can be expanded, by considering $B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y})$, to variables $\mathfrak{x}$ and $\mathfrak{y}$ with an infinite range. Such an extension seems desirable for the project consideration and review technique used in certain special cases.

We define the beta distribution of Equation (1.7) by

$$f(t)=\begin{cases}\frac{1}{B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y})}t^{\mathfrak{x}-1}(1-t)^{\mathfrak{y}-1}\exp\left(-\frac{p}{t^{\kappa}}-\frac{q}{(1-t)^{\mu}}\right), & 0<t<1\\ 0, & \text{otherwise}\end{cases}\tag{8.1}$$

A random variable $\mathfrak{X}$ with probability density function (pdf) given by Equation (8.1) will be said to have the generalisation of the extended beta distribution with parameters $\mathfrak{x}$ and $\mathfrak{y}$ such that $-\infty<\mathfrak{x},\mathfrak{y}<\infty$. If $\delta$ is any real number

$$E(\mathfrak{X}^{\delta})=\frac{B_{p,q}^{(\kappa,\mu)}(\mathfrak{x}+\delta,\mathfrak{y})}{B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y})}\tag{8.2}$$

In particular, $\delta=1$

$$\nu=E(\mathfrak{X})=\frac{B_{p,q}^{(\kappa,\mu)}(\mathfrak{x}+1,\mathfrak{y})}{B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y})}\tag{8.3}$$

shows the mean of the distribution, and

$$\sigma^2=E(\mathfrak{X}^2)-(E(\mathfrak{X}))^2=\frac{B_{p,q}^{(\kappa,\mu)}(\mathfrak{x}+2,\mathfrak{y})B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y})-[B_{p,q}^{(\kappa,\mu)}(\mathfrak{x}+1,\mathfrak{y})]^2}{[B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y})]^2}\tag{8.4}$$

is the variance of the distribution. The moment generating function of the distribution is

$$M(t)=\sum_{m=0}^{\infty}\frac{t^m}{m!}E(\mathfrak{X}^m)=\frac{1}{B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y})}\sum_{m=0}^{\infty}B_{p,q}^{(\kappa,\mu)}(\mathfrak{x}+m,\mathfrak{y})\frac{t^m}{m!}\tag{8.5}$$

The cumulative distribution of Equation (8.1) can be written as

$$F(\tau)=\frac{B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y};\tau)}{B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y})}\tag{8.6}$$

where $B_{p,q}^{(\kappa,\mu)}(\mathfrak{x},\mathfrak{y};\tau)$ is the generalisation of the extended incomplete beta function provided in Equation (5.1). Probably, this distribution should be useful in expanding the statical results for quite simply positive variables to deal with variables that can take arbitrarily large negative values as well.

## 9. Conclusion

Recently, Şahin et al. [24] have defined and investigated certain properties of Equation (1.7). This paper obtains several new formulas for generalising the beta function provided in Equation (1.7), for example, several integral representations, functional relations, summation formulas, Mellin transform and recurrence relation. Furthermore, we defined and studied a generalisation of the extended incomplete beta function provided in Equation (5.1) with the help of Equation (1.7). Then, we present some essential properties, for instance, integral representations, functional relations, summation formulas and recurrence relations. Finally, the conventional beta distribution can be extended using Equation (1.7).

This paper produces results with a general character and encourages further interesting studies involving integral representations. Moreover, opening up creative horizons in applied mathematics, this paper inspires the researchers to define and study various new fractional derivatives and integral operators.

## Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

## Conflicts of Interest

The authors declare no conflict of interest.

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