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COMPOSITIONS OF INTEGERS AND FIBONACCI NUMBERS

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Abstract. In this paper, we deal with the compositions of the integers. We present the decompositions for both the composition sets and the odd composition sets of the integers. Thus the decompositions provide us to have not only an alternative proof of some well known identies but also many new identities for Fibonacci numbers and Lucas numbers. Thus we investigate the generating functions for the product sum of the odd composition sets of the integers and attain some functional equations.

1. Introduction

Fibonacci numbers and compositions of a positive integer are simply expressed concepts but has many important features with many applications. Since these concepts were defined, these concepts have attracted the attention of many scientists and the results have made incredible contributions to almost all fields of sciences. These discoveries further increased the importance of mathematical analysis and number theory.

The Fibonacci numbers are numbers in which each number is the sum of the two preceding ones, denoted by f_n with the initial conditions, $f_0 = 0$, $f_1 = 1$. That is, $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Moreover, in literature, there are many generalizations of Fibonacci numbers and the other special numbers with many applications.

A composition of an integer n is a way of writing n as a sum of positive integers. The individual summands of a composition called its parts. In the combinatorics, a classical result about the number of compositions of n with an integer k parts is given by the coefficient of x^n of the polynomial or power series $\left(\sum_{i=1}^{\infty} x^i\right)^k$ where

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$|x| < 1$. These coefficients exhibit fascinating mathematical properties, closely resembling Binomial coefficients and have many useful applications ([12], [17], [20], [21], [22]).

By using Binomial properties, Hoggart and Lind ([22]) showed the relationship between a composition of an integer and Fibonacci numbers and proved that

(i) f_n is the number compositions of an integer n into odd parts

(ii) f_{2n} is the sum of the products of the parts over all compositions of an integer n, i.e.

$$f_{2n} = \sum_{a_1+a_2+...+a_k=n} a_1 a_2 ... a_k. \tag{1}$$

Recently, there has been interested n–color compositions of an integer m is defined as composition of m for which a part of size n can take on n colors ([1], [2], [27]). Then by the identity 1, it is clear that the number of n–color compositions of an integer m is f_{2m} the $2m$th Fibonacci number. Therefore, we wonder about the sequence of the sum of the products of the parts over all compositions whose parts are either odd or even. The main purpose of this paper is to investigate what the sum of the products of the parts over all compositions with odd parts is and interpret the relations among the generating functions, the set theory, compositions of an integer, Fibonacci numbers and Lucas numbers.

At first, we decompose the set of compositions of an integer and so give some very useful interpretations of the decompositions. Then we obtain an alternative proof of the above result and well- known identity by using these decompositions and reconstruct the connections between the composition of an integer and the Fibonacci numbers. These decompositions also provide us to derive some new identities and relations including the Fibonacci numbers and Lucas numbers. Next, we investigate some generating functions for the sequence of the sum of the products of the parts over all compositions whose parts are odd, the even term of the sequence and the odd term of the sequence.

Then we acquire the sequence of the sum of the products of the parts over all compositions whose each part is odd. Therefore, we focus on the generating functions for the numbers of n–color compositions with odd parts and so we work out their properties.

2. Decompositions of the Composition Sets of the Integers

In this section, we focus on decomposing the composition sets and the composition sets whose all parts are either odd or even. Then we find out some recurrence relations and also obtain an alternative proof for some well know results by using this decompositions.

We denote the composition set of an integer n as follows

$$P_n = \left\{(a_1, a_2, ..., a_t) : \ a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+\right\}.$$

It is well known that the number of elements of P_n is 2^{n-1}.

Now we recall the following operations for the element $a=(a_1,a_2,...,a_t)\in P_n$ and an integer j;

$$\begin{aligned}(j\odot a) &= (j,a_1,a_2,...,a_t)\,,\\ (j\oplus a) &= (a_1+j,a_2,...,a_t)\,.\end{aligned}$$

Then we use the notations $j\oplus P_n$ and $j\odot P_n$ for the following sets,

$$\begin{aligned}j\oplus P_n &= \{j\oplus a : a\in P_n\},\\ j\odot P_n &= \{j\odot a : a\in P_n\}.\end{aligned}$$

Theorem 1. *[6]Let n,r be positive integers $(r\leq n)$. Then the set P_n is disjoint union of the sets $(r\oplus P_{n-r})$ and $(i\odot P_{n-i})$ for all $i\in\{1,\ldots,r\}$,*

$$P_n=(r\oplus P_{n-r})\cup\left(\cup_{i=1}^{r}(i\odot P_{n-i})\right).$$

Proof. It is sufficient to prove the inclusion $P_n\subseteq(r\oplus P_{n-r})\cup(\cup_{i=1}^{r}(i\odot P_{n-i}))$.

Let $x=(a_1,\ldots,a_m)\in P_n$. If $a_1\leq r$ then $x\in\cup_{i=1}^{r}(i\odot P_{n-i})$. Now assume that $r<a_1$. Then $b=a_1-r$ and so define the element $y=(b,a_2,a_3,\ldots,a_m)\in P_{n-r}$. Then it is clear that $x=r\oplus y\in(r\oplus P_{n-r})$.

It is also clear that $(r\oplus P_{n-r})\cap(i\odot P_{n-i})=\varnothing$ for all $i\in\{1,\ldots,r\}$. □

Corollary 1. *[3]For a positive integer n, we have*

$$P_{n+1}=(1\oplus P_n)\cup(1\odot P_n).$$

Let n be a positive integer. It is clear that the number of the elements of both $(1\oplus P_n)$ and $(1\odot P_n)$ are equal, i.e. $|1\oplus P_n|=|1\odot P_n|$ and it follows that $|P_{n+1}|=2\,|1\odot P_n|$ since these sets are disjoint. On the other hand, by $|P_2|=2$, we have that $|P_n|=2^{n-1}$ by induction method. Therefore we have completed an alternative proof by using the set theory for the well-known result as a result of the Corollary 1.

Now we point out our attention to the composition sets whose parts are even or odd. Let us use the notions

$$\begin{aligned}O_n &= \{(a_1,...,a_t):\ a_1+...+a_t=n \text{ and } a_i \text{ is positive odd integer}\}\\ E_{2n} &= \{(2a_1,...2a_t):\ 2a_1+...+2a_t=2n \text{ and } a_i \text{ is positive integer}\}\end{aligned}$$

and we call the set as an odd composition set O_n (even composition set E_n) of an integer n. It is clear that the even composition set of an even integer $2n$ involved to the composition set of an integer n and so the number of elements of the even composition set of $2n$ is 2^{n-1}.

At this moment, we focus on to decompose the odd composition set as union of subset of odd combinations set of integers.

Theorem 2. *For a positive integer n, we decompose the odd composition set of an integer n as a disjoint union of subset of odd combinations set of integers;*

$$O_{2n+1} = \{(2n+1)\}\cup\bigcup_{i=0}^{n-1}\left((2i+1)\odot O_{2(n-i)}\right) \tag{2}$$

$$O_{2n} = \bigcup_{i=0}^{n-1} \left((2i+1) \odot O_{2(n-i)-1} \right). \tag{3}$$

Proof. Let n be a positive integer. It is enough to show one side inclusion for the odd number $2n+1$.

Let $x = (2a_1+1, ..., 2a_t+1)$ and assume that t is different from 1. Then $n - 2a_1 - 1 = 2m$ for an integer even and so the element $b = (2a_2+1, ...2a_t+1)$ is O_{2m}. Therefore $x = (2a_1+1) \odot O_{2n-2a_2}$ and this complete the proof. □

With the decomposition in Theorem 2, we prove again a well-known result using set theory.

Corollary 2. *The number of element of the odd composition set of an integer n is the n.th Fibonacci number.*

Proof. Let k_n be the number of element of the odd composition set of an integer n. Since the sets in Theorem 2 are disjoint, it is easy to prove that $k_{n+1} = k_n + k_{n-1}$ and $k_1 = 1, k_2 = 1$. □

As a conclusion of Theorem 2, we can reprove the well known identities [25, page 92]

$$f_{2n+1} = 1 + \sum_{i=1}^{n} f_{2i}$$

$$f_{2n} = \sum_{i=0}^{n-1} f_{2i+1}$$

for both the even and odd Fibonacci number.

3. Product Sum Function

By the motivation of the identity 1, we interested in the sequence of the sum of the products the parts over all compositions. In this section, we define function from compositions set to integer to obtain some number sequences and then interpret the relations among the set theory, the compositions of an integer, Fibonacci numbers and Lucas numbers. Thus we attain an alternative proof for the identity 1.

3.1. The composition set of the integers.

Now we establish the function from the composition sets to positive integers defined by

$$T_n := T(P_n) = \sum_{a \in P_n} \bar{a}.$$

We call $T_n = T(P_n)$ as the product sum of the composition set P_n (or the product sum of the integer n). For $n = 0$, we may assume that $T_0 = 1$.

We give an easy numeric example with the new notions;

Example 1. *Let* $n = 4$. *Then it follows that*

$$P_4 = \{(4), (1,1,1,1), (1,1,2), (1,2,1), (2,1,1), (2,2), (3,1), (1,3)\}$$

and $T_4 = T(P_4) = 21$.*Moreover, it follows*

$$1 \odot P_4 = \{(1,4), (1,1,1,1,1), (1,1,1,2), (1,1,2,1), (1,2,1,1), (1,2,2), (1,3,1), (1,1,3)\}$$

$$1 \oplus P_4 = \{(5), (2,1,1,1), (2,1,2), (2,2,1), (2,3), (3,1,1), (3,2), (4,1)\}$$

and so $P_5 = (1 \odot P_4) \cup (1 \oplus P_4)$. *Then* $T_5 = T(P_5) = 55$.

By using Theorem 1, we develop a recurrence for the product sum of the composition sets.

Theorem 3. *For a positive integer* n, *we have*

$$T_{n+1} = T_n + \sum_{i=0}^{n} T_{n-i}. \tag{4}$$

Proof. For an element $a \in P_{n+1}$, there is $b = (b_1, b_2, ..., b_l) \in P_n$ such that either $\overline{a} = \overline{1 \odot b} = \overline{b}$ or $\overline{a} = \overline{1 \oplus b}$ and so $\overline{a} = \overline{1 \odot b} = \overline{b}$ or $\overline{a} = \overline{1 \oplus b} = (b_2....b_l) + \overline{b}$. Hence we have that

$$T(1 \odot P_n) = \sum_{1 \odot b \in 1 \odot P_n} \overline{b} = T_n.$$

Moreover, it follows that

$$\begin{aligned} T(1 \oplus P_n) &= \sum_{a \in P_n} (1 + a_1).a_2.a_3...a_t \\ &= \sum_{a \in P_n} (a_1.a_2.a_3...a_t) + \sum_{i=1}^{n} \sum_{(a_2.a_3...a_t) \in P_{n-i}} (a_2 a_3 ... a_t) \\ &= T_n + \sum_{i=1}^{n} T_{n-i} = \sum_{i=0}^{n} T_{n-i}. \end{aligned}$$

Therefore, we have that

$$T_{n+1} = T(P_{n+1}) = T(1 \odot P_n) + T(1 \oplus P_n) = T_n + \sum_{i=0}^{n} T_{n-i}.$$

Hence we have completed the proof. □

By using the recurrence relation Identity 4, we gain the generating function for the product sum of the positive integers. From [25], we recall the generating function for even Fibonacci numbers is that

$$f(x) = \frac{x}{1 - 3x + x^2} = \sum_{n=1}^{\infty} f_{2n} x^n$$

Thus we give an alternative proof of the result of Hoggart and Lind in [22].

Theorem 4. *The generating function of the product sum of the positive integer is*

$$\sum_{n=1}^{\infty} T_n x^n = \frac{x}{1-3x+x^2}.$$

i.e. The product sum of the positive integer n is nth even Fibonacci number

Proof. Let $h(x) = \sum_{n=1}^{\infty} T_n x^n$. Then

$$\begin{aligned} h(x) &= x + \sum_{n=1} T_{n+1} x^{n+1} \\ &= x + x \sum_{n=1} \left(T_n + \sum_{i=0}^{n} T_{n-i} \right) x^n \\ &= x + xh(x) - x^2 h(x) + 2xh(x). \end{aligned}$$

Thus we get the function as

$$h(x) = \frac{x}{1-3x+x^2}.$$

□

As a result of Theorem 3 and Theorem 4, we obtain the known identity [25, Page 92- Identity 5.3] for odd Fibonacci numbers and also prove a new identities for Fibonacci numbers in the following;

Theorem 5. *Let n, m be positive integers. Then we have*

$$f_{2n+1} = 1 + \sum_{i=1}^{n} f_{2i} \tag{5}$$

$$f_{2n} = n + \sum_{i=1}^{n-1} (n-i) f_{2i}. \tag{6}$$

Proof. By Theorem 3, we have the recurrence

$$T_{n+1} = T_n + \sum_{i=0}^{n} T_{n-i}. \tag{7}$$

and it follows that $T_{n+1} = T_0 + T_n + \sum_{i=1}^{n} T_i$. Thus we gain that

$$f_{2(n+1)} = 1 + f_{2n} + \sum_{i=0}^{n} f_{2(n-i)}$$

Since $f_{2n+1} = f_{2n+2} - f_{2n}$, we have proved the identity 5.

Now we decompose P_n to get some new equations for the Fibonacci numbers. For an integer i, we define the set

$$(i \odot P_{n-i}) = \{(i, a_1, a_2, ..., a_t) : a_1 + a_2 + ... + a_t = n - i, \quad a_i, t \in \mathbb{Z}^+\}.$$

Then it is easy to check that

$$P_n = \cup_{i=1}^{n} (i \odot P_{n-i})$$

and also for all i, j with $i \neq j$, it follows that $(i \odot P_{n-i}) \cap (j \odot P_{n-j}) = \varnothing$. Therefore it follows that

$$T(i \odot P_{n-i}) = \sum_{(a_1, a_3 \ldots a_t) \in P_{n-i}} i.a_1.a_3 \ldots a_t = iT(P_{n-i}) = iT_{n-i}$$

and so

$$T_n = T(P_n) = \sum_{i=1}^{n} T(i \odot P_{n-i}) = \sum_{i=1}^{n} iT_{n-i} = \sum_{i=0}^{n-1} (n-i)T_i. \tag{8}$$

Thus the we complete the proof. □

Theorem 6. *Let n, m be positive integers $(m \leq n)$. Then we have*

$$f_{2n} - f_{2m} = \sum_{i=1}^{n-m} i f_{2(n-i)} + (n-m) \sum_{i=1}^{m} f_{2(n-i)} \tag{9}$$

$$f_{2n-1} - f_{2m-1} = \sum_{i=1}^{n-m} f_{2(n-i)}. \tag{10}$$

Proof. For any integers n, r we get that

$$f_{2n} - f_{2(n-r)} = \sum_{i=1}^{r} i f_{2(n-i)} + r \sum_{i=1}^{n-r} f_{2(n-i)}$$

and so substituting $m = n - r$, we acquire the identity 9.

By Theorem 3, we have the recurrence

$$T_{n+1} = T_n + \sum_{i=0}^{n} T_{n-i}. \tag{11}$$

and it follows that

$$\begin{aligned} \sum_{i=1}^{n-m} T_{n-i} &= \sum_{i=m}^{n-1} T_n + \sum_{i=1}^{m-1} T_n - \sum_{i=1}^{m-1} T_n \\ &= \sum_{i=1}^{n-1} T_n - \sum_{i=1}^{m-1} T_n = f_{2n-1} - f_{2m-1} \end{aligned}$$

Thus we achieve the identity 10. □

By by Theorem 5 we have the following equation

$$f_{2n+2} = 1 + n + f_{2n} + \sum_{i=1}^{n-1} (n-i+1) f_{2i}$$

and we also obtain

$$f_{2n+3} = n + 2 + 2f_{2n} + \sum_{i=1}^{n-1}(n-i+2)f_{2i}.$$

For an integer r, we have

$$\begin{aligned} f_{2n+r} &= (f_{2n}+1)f_r + nf_{r-1} + \left[f_{r-1}\sum_{i=1}^{n-1}(n-i)f_{2i} + f_r\sum_{i=1}^{n-1}f_{2i}\right] \\ &= f_r f_{2n+1} + f_{r-1}f_{2n}. \end{aligned}$$

Therefore we just gain the combinatorial proof of the Honsberger's formula by using the compositions of an integer.

Corollary 3. *For positive integers n, m, we have*

$$\begin{aligned} f_{2n+2m} &= f_{2m}f_{2n+1} + f_{2m-1}f_{2n} \\ f_{2n+2m+1} &= f_{2m+1}f_{2n+1} + f_{2m}f_{2n}. \end{aligned}$$

Corollary 4. *Let n be positive integer. Then we have*

$$f_{4n} = f_{2n}f_{2n-1} + f_{2n}f_{2n+1} \quad (12)$$

$$f_{4n+1} = f_{2n}^2 + f_{2n+1}^2 \quad (13)$$

$$\begin{aligned} f_{4n+2} &= f_{2n}f_{2n+1} + f_{2n+1}f_{2n+2} \\ f_{4n+3} &= f_{2n}f_{2n+2} + f_{2n+1}f_{2n+3} \end{aligned}$$

Proof. It is clear from Corollary 3. □

Let l_n be the n *th* term of Lucas sequence, defined by $l_0 = 2$, $l_1 = 1$, and $l_n = l_{n-1} + l_{n-2}$, $n > 3$. Also, one of the well-known relation between Fibonacci numbers and Lucas numbers is

$$l_n = f_{n-1} + f_{n+1}. \quad (14)$$

Thus by using the identity 13 and Cassini's formula, we obtain

$$\begin{aligned} f_{4n+1} &= f_{2n}^2 + 1 + f_{2n}f_{2n+2} \\ &= f_{2n}l_{2n+1} + 1 \end{aligned}$$

and it follows that

$$\begin{aligned} f_{4n+2} &= f_{4n} + f_{4n+1} = f_{2n}(l_{2n} + l_{2n+1}) + 1 \\ &= f_{2n}l_{2n+1} + f_2. \end{aligned}$$

Therefore, we just gain the following identity which is the general form of the well known result ([25, page 90]).

Corollary 5. *For positive integers r, n, we have the equality*

$$f_{4n+r} = f_{2n}l_{2n+r} + f_r.$$

Theorem 7. *For a positive integer n, we have the identities for Lucas numbers*

$$l_{2n+1} = 2n+1+\sum_{i=1}^{n-1} l_{2i+1}(n-i), \tag{15}$$

$$l_{2n} = 3n+1+f_{2n-2}+\sum_{i=1}^{n-2}\left(2f_{2i}+(n-1-i)l_{2i+1}\right). \tag{16}$$

Proof. By Theorem 5, we get the following result

$$f_{2n} = n+\sum_{i=1}^{n-1}(n-i)f_{2i}.$$

Thus,

$$\begin{aligned} l_{2n+1} &= f_{2n}+f_{2n+2} \\ &= \left[n+\sum_{i=1}^{n-1}(n-i)f_{2i}\right]+\left[n+1+\sum_{i=0}^{n-1}(n-i)f_{2(i+1)}\right] \end{aligned}$$

and so we have proved the identity 15.

For the second the identity, it is known that

$$f_{2n-1} = 1+\sum_{i=1}^{n-1} f_{2i}. \tag{17}$$

and

$$l_{2n-2} = f_{2n-3}+f_{2n-1} \ . \tag{18}$$

Then we gain the equation

$$l_{2n-2} = 2\left[1+\sum_{i=1}^{n-2} f_{2i}\right]+f_{2n-2}.$$

On the other hand, by the identity 15, we get

$$\begin{aligned} l_{2n} &= 2n+1+f_{2n-2}+\sum_{i=1}^{n-2}\left(2f_{2i}+(n-1-i)l_{2i+1}\right) \\ &= 3n+1+2f_{2n-2}+\left(\sum_{i=2}^{n-2}(2n+1-2i)f_{2i}\right). \end{aligned}$$

□

Corollary 6. *Let n, r be positive integers. Then we have*

$$l_{4n+r} = f_{2n}l_{2n+(r-1)}+f_{2n+1}l_{2n+r}$$

3.2. **The odd composition set of the integers.** Now we focus on the combinations of an integer whose each part is either odd nor even and we reach to the main goal of the paper which is to investigate the product sum of both an odd and even composition of an integer n.

Let us define the number sequence such as

$$o_n \quad : \quad = \sum_{a\in O_n} \bar{a} \tag{19}$$

$$e_n \quad : \quad = \sum_{a\in E_n} \bar{a}. \tag{20}$$

One may compute the sequence as

$$\begin{aligned} o_1 &= 1, o_2 = 1, o_3 = 4, o_4 = 7,\ o_5 = 15, o_6 = 32, o_7 = 65, o_8 = 137 \\ e_2 &= 2, e_4 = 16, e_6 = 48. \end{aligned}$$

By using the decomposition of an odd composition of an integer n, we figure out a recurrence relations for the product sum of an odd composition of an integer n.

Theorem 8. *For a positive integer $n \geq 1$, we have the recurrence relations for both an even and an odd term of the product sum of an odd composition of an integer*

$$o_{2n+2} = o_{2n+1} + 2o_{2n} + o_{2n-1} - o_{2n-2} \tag{21}$$

$$o_{2n+3} = 3o_{2n} + 3o_{2n+1} - o_{2n-2}. \tag{22}$$

Proof. Let n be an positive integer. Then we apply the the definition of the product sum function to the decomposition in Theorem 2 and so we get

$$\begin{aligned} o_{2n+1} &= 2n+1+\sum_{i=0}^{n-1}\sum_{b\in O_{2(n-i)}}(2i+1)\bar{b} \\ &= 2n+1+\sum_{i=0}^{n-1}(2i+1)o_{2(n-i)} \end{aligned}$$

and it also follows that

$$\begin{aligned} o_{2n+3} &= 2+o_{2n+2}+\left(2n+1+\sum_{i=0}^{n-1}(2i+1)o_{2(n-i)}\right)+2\sum_{i=0}^{n-1}o_{2(n-i)} \\ &= 2+o_{2n+2}+o_{2n+1}+2\sum_{i=1}^{n}o_{2i}. \end{aligned}$$

When we compute the difference between o_{2n+3} and o_{2n+1}, we get the recurrence for the odd term of the product sum of an odd composition of an integer n

$$o_{2n+3} = o_{2n+2} + 2o_{2n+1} + o_{2n} - o_{2n-1}. \tag{23}$$

On the other hand, by the decomposition in Theorem 2, we point out the recurrence for the even term of the product sum of an odd composition of an integer n as

$$o_{2n} = \sum_{i=0}^{n-1}(2i+1)o_{2(n-i)-1}.$$

Then we compute

$$\begin{aligned} o_{2n+2} &= o_{2n+1} + \sum_{i=0}^{n-1}(2i+1+2)o_{2(n-i)-1} \\ o_{2n+2} &= o_{2n+1} + o_{2n} + 2\sum_{i=1}^{n} o_{2i-1}. \end{aligned}$$

By the difference between o_{2n+2} and o_{2n+2}, we obtain the recurrence for the even terms

$$o_{2n+2} = o_{2n+1} + 2o_{2n} + o_{2n-1} - o_{2n-2}.$$

By substituting o_{2n+2} in the identity 23, we figure out

$$o_{2n+3} = 3o_{2n} + 3o_{2n+1} - o_{2n-2}$$

This completes the proof. □

Theorem 9. *The generating function for the product sum of an odd composition sets is*

$$U(x) = 1 + x^2(x+1)\frac{-2x + x^2 - 1}{x + 2x^2 + x^3 - x^4 - 1},$$

where $|x| < 1$.

Proof. For an integer n, we have the recurrence relations for either an even or an odd term of the product sum of an odd composition of an integer

$$\begin{aligned} o_{2n+3} &= 3o_{2n} + 3o_{2n+1} - o_{2n-2} \\ o_{2n+2} &= o_{2n+1} + 2o_{2n} + o_{2n-1} - o_{2n-2}. \end{aligned}$$

Let $U(x) = \sum_{n=1}^{\infty} o_n x^n = 1 + \sum_{n=1}^{\infty} o_{2n}x^{2n} + \sum_{n=1}^{\infty} o_{2n+1}x^{2n+1}$ be the generating function for the product sum of an odd composition of integers and so it is enough to investigate

$$\begin{aligned} A(x) &= \sum_{n=1} o_{2n}x^{2n} \\ B(x) &= \sum_{n=1}^{\infty} o_{2n+1}x^{2n+1}. \end{aligned}$$

By using the recurrence identity 22, it is easy to compute that

$$(1 - 3x^2)B(x) = x^3(3 - x^2)A(x) + 4x^3. \tag{24}$$

Similarly it is also easy to compute

$$A(x)=\frac{x\left(x^2+1\right)}{\left(x^2-1\right)^2}B(x)+\frac{x^2\left(x^2+1\right)}{\left(x^2-1\right)^2}, \tag{25}$$

due to the recurrence identity 21. Then combining the equations 24 and 25, we figure out both A and B and so it follows that

$$\begin{aligned} B(x) &= -x^3\frac{5x^2-6x^4+x^6-4}{\left(x+2x^2+x^3-x^4-1\right)\left(x-2x^2+x^3+x^4+1\right)} \\ A(x) &= x^2\frac{\left(x^2+1\right)^2}{\left(x-2x^2+x^3+x^4+1\right)\left(-x-2x^2-x^3+x^4+1\right)}. \end{aligned}$$

Therefore we investigate the generating function

$$U(x)=1+x^2\left(x+1\right)\frac{-2x+x^2-1}{x+2x^2+x^3-x^4-1}.$$

□

Moreover, we study out the generating function for either an odd or even term of product sum of an odd composition.

Theorem 10. *The generating function for the either an odd or an even term of product sum of an odd composition sets are*

$$\begin{aligned} O(x) &= -x\frac{5x-6x^2+x^3-4}{x^4-5x^3+4x^2-5x+1}, \\ E(x) &= x\frac{\left(x+1\right)^2}{x^4-5x^3+4x^2-5x+1}, \end{aligned}$$

where $|x|<1$.

Proof. Let

$$\begin{aligned} E &= E(x)=\sum_{n=1}^{\infty} o_{2n}x^n \\ O &= O(x)=\sum_{n=1}^{\infty} o_{2n+1}x^n. \end{aligned}$$

be the generating function for the either an odd or an even term of product sum of an odd composition sets. Then by using the recurrence identity 21 and 22, we compute

$$(1-3x)O=x(3-x)E+4x$$

and due to the recurrences, we compute

$$E=\frac{x\left(x+1\right)\left(O+1\right)}{\left(x-1\right)^2}=\frac{\left(x^2+x\right)}{\left(x-1\right)^2}O+\frac{\left(x^2+x\right)}{\left(x-1\right)^2}.$$

Therefore we figure out the generating function for the either an odd or even term of product sum of an odd composition and this completes the proof. □

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