Fundamentals of Contemporary Mathematical Sciences
doi:10.54974/fcmathsci.1145287
(**2023**) **4**(2) 56 – 65
Research Article

On Different Definitions of Hyper Pseudo BCC-algebras

Didem Sürgevil Uzay [1]*, **Alev Fırat** [1]
[1] Ege University, Institute of Science, Department of Mathematics, İzmir, Türkiye
alev.firat@ege.edu.tr

Received: 20 July 2022 **Accepted:** 06 June 2023

Abstract: We study hyper pseudo BCC-algebras which are a common generalization of hyper BCC-algebras and hyper BCK-algebras. In particular, we introduce different notion of hyper pseudo BCC-algebras and describe the relationship among them. Then, by choosing one of these definitions, we investigate for its related properties.

Keywords: Hyper pseudo order, hyper operation, hyper pseudo BCC-algebras.

1. Introduction

Hyper structures and pseudo structures have an important place in the field of algebra. These notions help to create new structures in algebraic system and to investigate their properties. The notions of hyper operation and hyper order were first defined by Marty in 1934 [7].

BCK-algebras were first studied by Iseki and Tanaka [4]. BCC-algebras, a generalization of BCK-algebras, were defined in 1990 by Dudek and their related properties were investigated [3]. The concept of Hyper BCK-algebra was introduced in 2000 by Jun, Zahedi, Xin and Borzooei [5]. Borzooei, Dudek and Koohestani in 2006 carried similar definitions and applications of hyper BCK-algebras to hyper BCC-algebras and defined various ideal types [1].

In this study, the notion of hyper pseudo order is defined. Then, different notions of hyper pseudo BCC-algebras are defined and their existences are proven with examples. In addition, the relationship between them is examined and some related properties are obtained. As a result, it is aimed to transfer hyper pseudo structures to BCC-algebras so that new algebraic structures can be built.

2. Preliminaries

Definition 2.1 *[3] Let X be a nonempty set, '$*$' be a operation on X and '0' be a constant*

*Correspondence: didemsurgevil@hotmail.com
2020 *AMS Mathematics Subject Classification*: 06F35, 03G25

element. $(X,*,0)$ *is called to be a BCC-algebra, if it supplies the following conditions for all* $x,y,z\in X$ *:*

(BCC1) $((x*y)*(z*y))*(x*z)=0$,

(BCC2) $x*0=x$,

(BCC3) $x*x=0$,

(BCC4) $0*x=0$,

(BCC5) $x*y$ *and* $y*x=0\Rightarrow x=y$.

Definition 2.2 *[7] Let* H *be a nonempty set*

$$\circ : H\times H\to P(H)-\{\emptyset\}$$

be a hyper operation. If "$x\ll y\Leftrightarrow 0\in x\circ y$ *for all* $x,y\in H$ *and* $S\ll T\Leftrightarrow$ *for every* $S,T\subset H$, $\forall s\in S$, $\exists t\in T$ *such that* $s\ll t$ *", then '*$\ll$*' is named to be a hyper order in* H.

Definition 2.3 *[1] Let* H *be a nonempty set, '*$\circ$*' be a hyper operation on* H*, '*$\ll$*' be a hyper order on* H *and '*0*' be a constant element of* H. $(H,\circ,\ll,0)$ *is called to be a hyper BCC-algebra if it supplies the following conditions, for all* $x,y,z\in H$ *:*

(HBCC1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,

(HBCC2) $0\circ x=0$,

(HBCC3) $x\circ 0=x$,

(HBCC4) $x\ll y$ *and* $y\ll x\Rightarrow x=y$.

Definition 2.4 *[1] Let* $(H,\circ,\ll,0)$ *be a hyper BCC-algebra and* I *be a subset of* H *such that* $0\in I$ *is named as follows, for all* $x,y,z\in H$ *:*

(1) a hyper BCC-ideal of type1*, if*
$(x\circ y)\circ z\ll I,\ y\in I\Rightarrow x\circ z\subseteq I$,

(2) a hyper BCC-ideal of type2*, if*
$(x\circ y)\circ z\subseteq I,\ y\in I\Rightarrow x\circ z\subseteq I$,

(3) a hyper BCC-ideal of type3*, if*
$(x\circ y)\circ z\ll I,\ y\in I\Rightarrow x\circ z\ll I$,

(4) a hyper BCC-ideal of type4*, if*
$(x\circ y)\circ z\subseteq I,\ y\in I\Rightarrow x\circ z\ll I$.

Definition 2.5 *[5] Let H be a nonempty set '$\circ$' be a hyper operation on H, '$\ll$' be a hyper order in H and '0' be a constant element of H. $(H,\circ,\ll,0)$ is named to be a hyper BCK-algebra if it supplies the following conditions, for all $x,y,z\in H$:*

(HBCK1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,

(HBCK2) $(x\circ y)\circ z=(x\circ z)\circ y$,

(HBCK3) $x\circ y\ll x$,

(HBCK4) $x\ll y$ *and* $y\ll x\Rightarrow x=y$.

Definition 2.6 *[2] Let H be a nonempty set, '$*$', '$\circ$' be hyper operations on H, '$\ll$' be a hyper order in H and '0' be a constant element of H, $(H,\circ,*,\ll,0)$ is named to be a hyper pseudo BCK-algebra, if it supplies the following conditions, for all $x,y,z\in H$:*

(HPBCK1) $(x\circ z)\circ(y\circ z)\ll x\circ y$, $(x*z)*(y*z)\ll x*y$,

(HPBCK2) $(x\circ y)*z=(x*z)\circ y$,

(HPBCK3) $x\circ y\ll x$, $x*y\ll x$,

(HPBCK4) $x\ll y$ *and* $y\ll x\Rightarrow x=y$.

3. Hyper Pseudo BCC-algebras

In this section, different definitions of Hyper Pseudo BCC-algebras, these definitions relationship between them and some of their related properties are given.

Definition 3.1 *Let H be a nonempty set and*

$$\circ: H\times H\to P(H)-\{\emptyset\}$$

be a hyper operation.
If "$x\ll y\Leftrightarrow 0\in x\circ y\Leftrightarrow 0\in x*y$ for all $x,y\in H$ and $S\ll T\Leftrightarrow$ for every $S,T\subset H$, $\forall s\in S$ $\exists t\in T$ such that $s\ll t$ ", then '$\ll$' is called to be a hyper pseudo order in H.

Definition 3.2 *Let H be a nonempty set, '$\circ$', '$*$' be hyper operations on H, '$\ll$' be a hyper pseudo order in H, '0' be a constant element of H. $(H,\circ,*,\ll,0)$ is named to be hyper pseudo BCC_1-algebra if it supplies the following conditions, for all $x,y,z\in H$:*

$(HPBCC_1 1)$ $(x\circ z)\circ(y\circ z)\ll x\circ y$, $(x*z)*(y*z)\ll x*y$,

$(HPBCC_1 2)$ $0\circ x=\{0\}$, $0*x=\{0\}$,

$(HPBCC_1 3)$ $x\circ 0=\{x\}$, $x*0=\{x\}$,

$(HPBCC_1 4)$ $x \ll y$ *and* $y \ll x \Rightarrow x = y$,

$(HPBCC_1 5)$ $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$.

Example 3.3 *Let* $H = \{0, m, n\}$ *and '*$\circ$*', '*$*$*' be hyper operations on* H *with Cayley table give as in Table 1.*

Table 1: Hyper operations.

∘	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{0}
n	{n}	{n}	{0,n}

*	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{0}
n	{n}	{n}	{0,m,n}

Then, it is easily controlled that $(H, \circ, *, \ll, 0)$ *is a hyper pseudo* BCC_1*-algebra and hyper pseudo BCK-algebra. Also, '*$\circ$*' and '*$*$*' hyper operations with* $(H, \circ, \ll, 0)$ *and* $(H, *, \ll, 0)$ *be hyper BCC-algebras.*

Remark 3.4 *Let* H *be a nonempty set, '*$\circ$*', '*$*$*' be hyper operations on* H*, '*$\ll$*' be a hyper pseudo order in* H*, '*0*' be a constant element of* H*. According to both hyper operations, the* $(H, \circ, *, 0)$ *system is always a hyper pseudo* BCC_1*-algebra when the system is hyper BCC-algebra.*

Definition 3.5 *Let* H *be a nonempty set, '*$\circ$*', '*$*$*' be hyper operations on* H*, '*$\ll$*' be a hyper pseudo order in* H*, '*0*' be a constant element of* H*.* $(H, \circ, *, \ll, 0)$ *is named to be hyper pseudo* BCC_2*-algebra if it supplies the following conditions, for all* $x, y, z \in H$*:*

$(HPBCC_2 1)$ $(x \circ z) * (y \circ z) \ll x * y$, $(x * z) \circ (y * z) \ll x \circ y$,

$(HPBCC_2 2)$ $0 \circ x = \{0\}$, $0 * x = \{0\}$,

$(HPBCC_2 3)$ $x \circ 0 = \{x\}$, $x * 0 = \{x\}$,

$(HPBCC_2 4)$ $x \ll y$ *and* $y \ll x \Rightarrow x = y$,

$(HPBCC_2 5)$ $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$.

Example 3.6 *Let* $H = \{0, m, n\}$ *and '*$\circ$*', '*$*$*' be hyper operations on* H *with Cayley table give as in Table 2.*

Table 2: Hyper operations.

$\circ$	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{n}
n	{n}	{n}	{0,n}

$*$	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{m}
n	{n}	{n}	{0,m,n}

Then, it is easily controlled that $(H,\circ,*,\ll,0)$ *is a hyper pseudo* BCC_2*-algebra but* $(H,\circ,\ll,0)$ *is not hyper BCC-algebra. Moreover,* $(H,\circ,*,\ll,0)$ *is not hyper pseudo BCK-algebra because it does not satisfy the* $(HPBCK1)$ *condition of hyper pseudo BCK-algebra. For example; it has been* $(m\circ n)\circ(0\circ n)\ll m\circ 0$ *such that* $m,n,0\in H$*. Then, it can be written* $\{n\}\ll\{m\}$ *so that the condition* $(HPBCK1)$ *is satisfied because* 0 *is not an element of this equation* $\{n\}=n\circ m$*.*

Definition 3.7 *Let* H *be a nonempty set, '*$\circ$*', '*$*$*' be hyper operations on* H*, '*$\ll$*' be a hyper pseudo order in* H*, '*0*' be a constant element of* H*.* $(H,\circ,*,\ll,0)$ *is named to be hyper pseudo* BCC_3*-algebra if it supplies the following conditions, for all* $x,y,z\in H$*:*

$(HPBCC_31)$ $(x\circ z)\circ(y\circ z)\ll x\circ y$, $(x*z)*(y*z)\ll x*y$,

$(HPBCC_32)$ $0\circ x=\{0\}$, $0*x=\{0\}$,

$(HPBCC_33)$ $x\circ 0=\{x\}$, $x*0=\{x\}$,

$(HPBCC_34)$ $0\in x\circ y\wedge y*x\Rightarrow x=y$,

$(HPBCC_35)$ $x\ll y\Leftrightarrow 0\in x\circ y\Leftrightarrow 0\in x*y$.

Example 3.8 *Let* $H=\{0,m,n\}$ *and '*$\circ$*', '*$*$*' be hyper operations on* H *with Cayley table give as in Table 3.*

Table 3: Hyper operations.

$\circ$	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{0}
n	{n}	{0}	{0,n}

$*$	0	m	n
0	{0}	{0}	{0}
m	{m}	{0}	{m}
n	{n}	{n}	{0,m,n}

Then, it is easily controlled that $(H, \circ, *, \ll, 0)$ *is a hyper pseudo* BCC_3*-algebra but according to operation '*$\circ$*',* $(H, \circ, \ll, 0)$ *is not hyper BCC-algebra because it does not satisfy the* $(HBCC4)$ *condition of hyper BCC-algebra. Also, this structure isn't hyper pseudo BCK-algebra because the system does not satisfy the condition* $(HPBCK4)$*.*

Definition 3.9 *Let* H *be a nonempty set, '*$\circ$*', '*$*$*' be hyper operations on* H*, '*$\ll$*' be a hyper pseudo order in* H*, '*0*' be a constant element of* H*.* $(H, \circ, *, \ll, 0)$ *is named to be hyper pseudo* BCC_4*-algebra if it supplies the following conditions, for all* $x, y, z \in H$*:*

$(HPBCC_41)$ $(x \circ z) * (y \circ z) \ll x * y$, $(x * z) \circ (y * z) \ll x \circ y$,

$(HPBCC_42)$ $0 \circ x = \{0\}$, $0 * x = \{0\}$,

$(HPBCC_43)$ $x \circ 0 = \{x\}$, $x * 0 = \{x\}$,

$(HPBCC_44)$ $0 \in x \circ y$, $0 \in y * x \Rightarrow x = y$,

$(HPBCC_45)$ $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$.

Example 3.10 *Let* $H = \{0, m, n, k\}$ *and '*$\circ$*', '*$*$*' be hyper operations on* H *with Cayley table give as in Table 4.*

Table 4: Hyper operations.

$\circ$	0	m	n	k
0	{0}	{0}	{0}	{0}
m	{m}	{0}	{0}	{n}
n	{n}	{0}	{0,n}	{n}
k	{k}	{0}	{0}	{0,k}

$*$	0	m	n	k
0	{0}	{0}	{0}	{0}
m	{m}	{0}	{k}	{n}
n	{n}	{n}	{0,m,n}	{m}
k	{k}	{k}	{0}	{0,k}

Then, it is easily controlled that $(H, \circ, *, \ll, 0)$ *is a hyper pseudo* BCC_4*-algebra. Also,* $(H, \circ, \ll, 0)$ *and* $(H, *, \ll, 0)$ *systems built with* H *and hyper operations '*$\circ$*', '*$*$*' are not hyper BCC-algebra as they do not satisfy* $(HBCC4)$ *and* $(HBCC1)$*, respectively. Finally, it is not hyper pseudo BCK-algebra because the system does not satisfy the conditions* $(HPBCK1)$ *and* $(HPBCK4)$*.*

Definition 3.11 *Let H be a nonempty set, '$\circ$', '$*$' be hyper operations on H, '$\ll$' be a hyper pseudo order in H, '0' be a constant element of H. $(H, \circ, *, \ll, 0)$ is named to be hyper pseudo BCC_5-algebra if it supplies the following conditions, for all $x, y, z \in H$:*

$(HPBCC_5 1)$ $(x \circ z) * (y \circ z) \ll x * y$, $(x * z) \circ (y * z) \ll x \circ y$,

$(HPBCC_5 2)$ $x * (0 \circ y) = \{x\}$, $x \circ (0 * y) = \{x\}$,

$(HPBCC_5 3)$ $x \ll y$ *and* $y \ll x \Rightarrow x = y$,

$(HPBCC_5 4)$ $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$.

Example 3.12 *Let $H = \{0, m, n, k\}$ and '$\circ$', '$*$' be hyper operations on H with Cayley table give as in Table 5.*

Table 5: Hyper operations.

$\circ$	0	m	n	k
0	{0}	{0}	{0}	{0}
m	{m}	{0}	{k}	{m}
n	{n}	{0}	{0,n}	{k}
k	{k}	{0}	{0}	{0,k}

$*$	0	m	n	k
0	{0}	{0}	{0}	{0}
m	{m}	{0}	{n}	{k}
n	{n}	{0}	{0,m,n}	{n}
k	{k}	{0}	{k}	{0,k}

Then, it is easily controlled that $(H, \circ, *, \ll, 0)$ is a hyper pseudo BCC_4-algebra. Also, $(H, \circ, \ll, 0)$ and $(H, *, \ll, 0)$ systems built with H and hyper operations '$\circ$', '$*$' are not hyper BCC-algebra as they do not satisfy $(HBCC1)$. Finally, it is not hyper pseudo BCK-algebra because the system does not satisfy the condition $(HPBCK1)$.

Theorem 3.13 *Let $(H, \circ, *, \ll, 0)$ be a hyper pseudo BCC_1-algebra or hyper pseudo BCC_3-algebra. If $x * y = x \circ y$ for all $x, y \in H$, then H is a hyper BCC-algebra.*

Proof Let H be a hyper pseudo BCC_1-algebra. If $x * y = x \circ y$ for all $x, y \in H$, then proof follows from conditions of hyper pseudo BCC_1-algebra. Let H be a hyper pseudo BCC_3-algebra. If $x * y = x \circ y$ for all $x, y \in H$, then proof follows from conditions of hyper pseudo BCC_3-algebra. □

Proposition 3.14 *Let $(H, \circ, *, \ll, 0)$ be any of the hyper pseudo BCC_1-algebra, hyper pseudo BCC_2-algebra, hyper pseudo BCC_3-algebra, hyper pseudo BCC_4-algebra. Then, the following conditions are satisfied for every nonempty subset S, T of H and for all $x, y, z \in H$:*

(i) $0 \circ 0 = \{0\}\,,\ 0 * 0 = \{0\}\,,$

(ii) $0 \ll x\,,$

(iii) $x \ll x\,,$

(iv) $x \circ y \ll \{x\}\,,\ x * y \ll \{x\}\,,$

(v) $S \circ 0 = S\,,\ S * 0 = T\,,$

(vi) $0 \circ S = \{0\}\,,\ 0 * S = \{0\}\,,$

(vii) $x * y = \{0\} \Rightarrow x \circ z \ll y \circ z\,,\ x \circ y = \{0\} \Rightarrow x * z \ll y * z\,,$

(viii) $S \ll S\,,$

(ix) $S \subseteq T \Rightarrow S \ll T\,,$

(x) $S \ll \{0\} \Rightarrow S = \{0\}\,,$

(xi) $x \circ 0 \ll \{y\} \Rightarrow x \ll y\,,\ x * 0 \ll \{y\} \Rightarrow x \ll y\,.$

Proof Let $(H, \circ, *, \ll, 0)$ be a hyper pseudo BCC_4-algebra.

(i) In $(HPBCC_4 2)$, let $x = 0$. Then

$$0 \circ 0 = \{0\},\ 0 * 0 = \{0\}.$$

(ii) Using $(HPBCC_4 2)$ condition,

$$0 \in 0 \circ x\,,\ 0 \in 0 * x$$

and so $0 \ll x$.

(iii) Using $(HPBCC_4 1)$ condition, let $y = z = 0$. Then, by (i) and $(HPBCC3)$ condition, we get that $x \ll x$.

(iv) By $(HPBCC_4 1)$ condition, we conclude that

$$(x \circ y) * (z \circ y) \ll (x * z)\,,\ (x \circ y) * (z \circ y) \ll (x * z).$$

Therefore let $z = 0$. Then, by $(HPBCC_4 2)$ and $(HPBCC_4 3)$ we can write,

$$x \circ y \ll \{x\},\ x * y \ll \{x\}.$$

(v) Using $(HPBCC_4 3)$ condition,

$$S \circ 0 = S\,,\ S * 0 = S$$

is shown.

(vi) Using $(HPBCC_4 2)$ condition,

$$0 \circ S = \{0\},\ 0 \circ S = \{0\}$$

is shown.

(vii) Let $x * y = \{0\}$. From the $(HPBCC_4 1)$ condition, since

$$(x \circ z) * (y \circ z) \ll (x * y),\ (x * z) \circ (y * z) \ll (x \circ y),$$

then for all

$$a \in (x \circ z) * (y \circ z),$$

$a \ll 0$ and then for all

$$b \in (x * z) \circ (y * z),$$

$b \ll 0$ and so, by the help of conditions $(HPBCC_4 3)$ and $(HPBCC_4 4)$, we can find $a = 0$ and $b = 0$. Hence

$$(x \circ z) * (y \circ z) = \{0\},\ (x * z) \circ (y * z) = \{0\}.$$

Then, we can write this,

$$x \circ z \ll y \circ z,\ x * z \ll y * z.$$

(viii) By (iii), $S \ll S$ can be proved.

(ix) Let $S \subseteq T$ and $m \in S$. For $n = m$ we can find $n \in T$. Hence, by (iii), we get $m \ll n$. Therefore we have $S \ll T$.

(x) Let $s \in S$ and $S \ll \{0\}$. Then using $s \ll 0$ and (i) we can find $s = 0$. Hence $S = \{0\}$ is satisfied.

(xi) From $(HPBCC_4 3)$ condition,

$$0 \in (x \circ 0) \circ \{y\} = 0 \in \{x\} \circ \{y\},$$

we can get $x \ll y$. Similarly, using $(HPBCC_4 3)$, since

$$0 \in (x * 0) * \{y\} = 0 \in \{x\} * \{y\},$$

then we can find $x \ll y$.

□

Theorem 3.15 *Let $(H, \circ, *, \ll, 0)$ be a hyper pseudo BCK-algebra. Then, $(H, \circ, *, \ll, 0)$ is a hyper pseudo BCC_1-algebra and hyper pseudo BCC_3-algebra.*

Proof Using the $(HPBCK1)$, $(HPBCK4)$ conditions hyper pseudo BCC_1-algebra and hyper pseudo BCC_3-algebra are obtained. □

Theorem 3.16 *Let $(H, \circ, *, \ll, 0)$ be a hyper pseudo BCC_1-algebra. Then, H is a hyper pseudo BCK-algebra if and only if $(x \circ y) * z = (x * z) \circ y$, for all $x, y, z \in H$.*

Proof Every hyper pseudo BCC_1-algebra supplies this identity. Conversely, using $(HPBCC_11)$, we have $(HPBCK1)$ and using $(HPBCC_14)$, we get $(HPBCK4)$. Next in a hyper pseudo BCC_1-algebra satisfying this identity, for all $x, y \in H$, we get using Proposition 3.14 (iv);

$$x \circ y \ll \{x\} \Leftrightarrow x * y \ll \{x\}.$$

Then, we have the $(HPBCK_13)$ condition. Hence, H is a hyper pseudo BCK-algebra. □

Example 3.17 *Let $(H, \circ, *, \ll, 0)$ given in Example 3.3 be a hyper pseudo BCC_1-algebra. We can find*

$$(n \circ m) * n \neq (n * n) \circ m$$

for $m, n \in H$. Hence, H is not hyper pseudo BCK-algebra.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Didem Sürgevil Uzay]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%60).

Author [Alev Fırat]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%40).

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Borzooei R.A., Dudek W.A., Koohestani N., *On hyper BCC-algebras*, International Journal of Mathematics and Mathematical Sciences, 1-18, 2006.

[2] Borzooei R.A., Rezazadeh A., Ameri R., *On hyper pseudo BCK-algebras*, Iranian Journal of Mathematical Sciences and Informatics, 9(1), 13-29, 2014.

[3] Dudek W.A., *On BCC-algebras*, Logique & et Analyse Nouvelle Série, 33(129-130), 103-111, 1990.

[4] Iseki K., Tanaka S., *An introduction to the theory of BCK-algebras*, Mathematica Japonica, 23, 1-26, 1978.

[5] Jun Y.B., Zahedi M.M., Xin X.L., Borzooei R.A., *On hyper BCK-algebras*, Italian Journal of Pure and Applied Mathematics, 8, 127-136, 2000.

[6] Komori Y., *The class of BCC-algebras is not a variety*, Mathematica Japonica, 29(3), 391-394, 1984.

[7] Marty F., *Sur une generalization de la notion de groupe*, In Eighth Congress Math. Scandinaves, 45-49, 1934.