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q-Multiplicative Dirac System

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Abstract

In this study, the classical Dirac equation was investigated on the basis of q-multiplicative calculus. We discuss some spectral properties of the q-multiplicative Dirac system, such as formally self-adjointness, and orthogonality of eigenfunctions. Finally, Green's function for this system has been reconstructed.

Keywords: *q-multiplicave calculus, Dirac equation, self-adjoint operator*
2010 Mathematics Subject Classification: *05A30, 34L40, 39A13, 11D57*

1. Introduction

The Non-Newtonian calculus (or multiplicative calculus) was introduced into the mathematical literature by Grosman and Katz in 1967 as an alternative to Newtonian calculus (see [11, 12]). In [11, 12], Grosman and Katz made a new definition of derivatives and integrals and turned addition and subtraction into multiplication and division. But until recently, multiplicative calculus did not attract much attention from researchers. Recently, studies on this subject have begun to emerge (see [2, 5, 7, 8, 9, 10, 13, 20]). In 2016, Yener and Emiroğlu introduced the concept of multiplicative calculus for quantum calculus ([21]).
On the other hand, Sturm–Liouville and Dirac problems are among the problems that mathematicians are most interested in. There are many studies on these problems ([1, 4, 15, 16, 17, 22]). Especially, the Dirac system is one of the most important equations in physics since this equation predicts the existence of antimatter and gives a description of the electron spin ([15, 18, 19]).
This study aims to investigate the basic properties of the Dirac system in q-multiplicative calculus. According to the authors' knowledge, there is no study on this subject in the literature. Thus, it will contribute to the literature by filling the gap in the literature.

2. Preliminaries

Now, we give some concepts of multiplicative quantum calculus ([3, 6, 14, 21]). Let $0<q<1$ and let $A\subset\mathbb{R}$ is a q-geometric set, i.e., $qx\in A$ for all $x\in A$. The *q-derivative D_q is* defined by

$$D_q y(x)=\frac{1}{qx-x}\left[y(qx)-y(x)\right]$$

for all $x\in A$. A function y which is defined on $A,\ 0\in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} y(xq^n)=y(0),$$

for every $x\in A$. Through the remainder of the paper, we deal only with functions q-regular at zero.

Definition 2.1 ([21])**.** *Let y be a positive function. The q-multiplicative derivative D_q^* is defined by*

$$D_q^* y(x)=\left(\frac{y(qx)}{y(x)}\right)^{\frac{1}{qx-x}}.$$

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From Definition 2.1, one obtains

$$D_q^* y(x) = e^{D_q(\ln y(x))},$$

and

$$\left[D_{q^{-1}}^* y(x)\right]^{1/q} = D_q^* y\left(xq^{-1}\right) \tag{2.1}$$

Theorem 2.2 ([21]). *Let y,z be q^*-differentiable functions. Then we have the following properties.*
i)

$$D_q^*(cy) = D_q^*(y),$$

where c is a positive constant,
ii)

$$D_q^*(yz) = D_q^*(y) D_q^*(z),$$

iii)

$$D_q^*\left(\frac{y}{z}\right) = \frac{D_q^*(y)}{D_q^*(z)},$$

The q-integration is given by

$$\int_a^b y(t)\, d_q t = \int_0^b y(t)\, d_q t - \int_0^a y(t)\, d_q t,$$

where $a,b \in A$ and

$$\int_0^x y(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} q^n y(q^n x), \ (x \in A).$$

Definition 2.3 ([21]). *Let y be a positive bounded function. Then the q-multiplicative integral is defined as*

$$\int y(t)^{d_q t} = e^{\int \ln y(t) d_q t}.$$

Theorem 2.4 ([21]). *Let y,z be q^*-integrable functions. Then we have the following properties.*
i)

$$\int \left(y(t)^k\right)^{d_q t} = \left(\int y(t)^{d_q t}\right)^k, \text{ where } k \in \mathbb{R},$$

ii)

$$\int (y(t) z(t))^{d_q t}) = \int y(t)^{d_q t} \int z(t)^{d_q t},$$

iii)

$$\int (y(t)/z(t))^{d_q t}) = \left(\int y(t)^{d_q t}\right) / \int z(t)^{d_q t}.$$

Theorem 2.5 ([21]). *Let y be q^*-integrable and z be q-differentiable, they are continuous on the interval $0 \le a < b$, then*

$$\left[\int_a^b \left(D_q^* y(t)\right)^{z(t)}\right]^{d_q t} = \frac{y(b)^{z(b)}}{y(a)^{z(a)}} \left(\left[\int_a^b (y(qt))^{D_q z(t)}\right]^{d_q t}\right)^{-1}.$$

Now we will give the notation we will use in our work.

$$y \oplus z = y.z, \ y \ominus z = \frac{y}{z}, \ y \odot z = y^{\ln z} = z^{\ln y},$$

where $y,z \in \mathbb{R}^+$.

Definition 2.6 ([13]). *Let $H \neq \emptyset$ and $\langle .,. \rangle_* : H \times H \to \mathbb{R}^+$ be a function such that the following axioms are satisfied for all $x,y,z \in H$:*
i)

$\langle x \oplus y, z \rangle_* = \langle x,y \rangle_* \oplus \langle y,z \rangle_*,$

ii)

$\langle x,y \rangle_* = \langle y,x \rangle_*,$

iii)

$\langle x,x \rangle_* = 1$ *if and only if* $x = 1,$

iv)

$\langle x,x \rangle_* \geq 1,$

v)

$\langle e^k \odot x, y \rangle_* = e^k \odot \langle x,y \rangle_*,\ k \in \mathbb{R}.$

Then $(H, \langle .,. \rangle_*)$ is called multiplicative inner product space.

Let

$$L^2_{*,q}(0,a) := \left\{ y : \int_0^a |y(x) \odot y(x)|^{d_q x} < \infty \right\}.$$

By Definition 2.6, $L^2_{*,q}(0,a)$ is a multiplicative inner product space with

$$\langle .,. \rangle^{*,q} : L^2_{*,q}(0,a) \times L^2_{*,q}(0,a) \to \mathbb{R}^+, \tag{2.2}$$

$$\langle y,z \rangle^{*,q} = \int_0^a |y(x) \odot z(x)|^{d_q x},$$

where $y,z \in L^2_{*,q}(0,a)$ are positive functions.

3. q-multiplicative Dirac system

In this section, we shall study a q-multiplicative Dirac (q-MD) system.
Let us define a system of equations

$$\begin{cases} D_q^* z_1(x) \oplus \left(e^{-r(x)} \odot z_2(x)\right) = e^{-\lambda} \odot z_2(x) \\ \left(D_{q^{-1}}^*\right)^{1/q} z_2(x) \oplus \left(e^{p(x)} \odot z_1(x)\right) = e^{\lambda} \odot z_1(x) \end{cases},\ x \in [0,a], \tag{3.1}$$

where $r(.)$ and $p(.)$ are real-valued functions on $[0,a]$, and λ is a parameter independent of x; and two supplementary conditions

$$\left(e^{\cos\alpha} \odot z_1(0)\right) \oplus \left(e^{\sin\alpha} \odot z_2(0)\right) = 1, \tag{3.2}$$

$$\left(e^{\cos\beta} \odot z_1(a)\right) \oplus \left(e^{\sin\beta} \odot z_2\left(aq^{-1}\right)\right) = 1, \tag{3.3}$$

where $\alpha, \beta \in \mathbb{R}$. This type of boundary-value problem is called a q-MD system. Denoting in (3.1)

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and

$$\Gamma z := \begin{cases} D_q^* z_1(x) \oplus \left(e^{-r(x)} \odot z_2(x)\right) \\ \left(D_{q^{-1}}^*\right)^{1/q} z_2(x) \oplus \left(e^{p(x)} \odot z_1(x)\right) \end{cases},$$

we can write the system (3.1) in the form

$$\Gamma z = \begin{pmatrix} e^{-\lambda} \odot z_2(x) \\ e^{\lambda} \odot z_1(x) \end{pmatrix}$$

Now let's define a Hilbert space compatible with system (3.1). Let $H = L^2_{*,q}\left[(0,a) : \left(\mathbb{R}^+\right)^2\right]$ be a multiplicative inner product space with

$$\langle y,z \rangle_{*,q} = \int_0^a |(y_1(x) \odot z_1(x)) \oplus (y_2(x) \odot z_2(x))|^{d_q x},$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in H$ and y_1, y_2, z_1, z_2 are positive functions.

Theorem 3.1. *q-MD operator defined by (3.1)-(3.3) is formally self-adjoint on the Hilbert space H.*

Proof. Let $z=\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, t=\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in H$. It follows from (2.2) and (2.1) that

$$\langle \Gamma z,t\rangle_{*,q} = \int_0^a \left|\left(\left[D_q^* z_1(x)\right]\left[z_2(x)^{r(x)}\right]\right)^{-\ln t_2(x)}\right|^{d_q x} \oplus \int_0^a \left|\left(\left[\left(D_{q^{-1}}^*\right)^{1/q} z_2(x)\right]\left[z_1(x)^{p(x)}\right]\right)^{\ln t_1(x)}\right|^{d_q x}$$

$$= \int_0^a \left|\left(\left[D_q^* z_1(x)\right]\right)^{-\ln t_2(x)}\right|^{d_q x} \oplus \int_0^a \left|\left(\left[z_2(x)^{r(x)}\right]\right)^{-\ln t_2(x)}\right|^{d_q x} \oplus \int_0^a \left|\left(\left[D_q^* z_2\left(xq^{-1}\right)\right]\right)^{\ln t_1(x)}\right|^{d_q x} \oplus \int_0^a \left|\left[z_1(x)^{p(x)}\right]^{\ln t_1(x)}\right|^{d_q x}$$

By Theorem 2.5, we get

$$\langle \Gamma z,t\rangle_{*,q} = \frac{(z_1(a))^{-\ln t_2(a)}}{(z_1(0))^{-\ln t_2(0)}} \frac{1}{\int_0^a \left|(z_1(qx))^{-D_q \ln t_2(x)}\right|^{d_q x}} \times \int_0^a \left|\left(\left[z_2(x)^{r(x)}\right]\right)^{-\ln t_2(x)}\right|^{d_q x} \frac{\left(z_2\left(aq^{-1}\right)\right)^{\ln t_1(a)}}{(z_2(0))^{\ln t_1(0)}}$$

$$\times \frac{1}{\int_0^a \left|(z_2(x))^{D_q \ln t_1(x)}\right|^{d_q x}} \int_0^a \left|\left[z_1(x)^{p(x)}\right]^{\ln t_1(x)}\right|^{d_q x}$$

$$= \frac{(z_1(a))^{-\ln t_2(a)}}{(z_1(0))^{-\ln t_2(0)}} \frac{1}{e^{\int_0^a -D_q \ln z_1(qx) D_q \ln t_2(x) d_q x}}$$
$$\times \int_0^a \left|\left(\left[z_2(x)^{r(x)}\right]\right)^{-\ln t_2(x)}\right|^{d_q x} \frac{\left(z_2\left(aq^{-1}\right)\right)^{\ln t_1(a)}}{(z_2(0))^{\ln t_1(0)}} \times \frac{1}{e^{\int_0^a D_q \ln z_2(x) D_q \ln t_1(x) d_q x}} \int_0^a \left|\left[z_1(x)^{p(x)}\right]^{\ln t_1(x)}\right|^{d_q x}. \tag{3.4}$$

Similarly,

$$\langle z,\Gamma t\rangle_{*,q} = \int_0^a \left|\left(\left[D_q^* t_1(x)\right]\left[t_2(x)^{r(x)}\right]\right)^{-\ln z_2(x)}\right|^{d_q x} \oplus \int_0^a \left|\left(\left[\left(D_{q^{-1}}^*\right)^{1/q} t_2(x)\right]\left[t_1(x)^{p(x)}\right]\right)^{\ln z_1(x)}\right|^{d_q x}$$

$$= \int_0^a \left|\left(\left[D_q^* t_1(x)\right]\right)^{-\ln z_2(x)}\right|^{d_q x} \oplus \int_0^a \left|\left(\left[t_2(x)^{r(x)}\right]\right)^{-\ln z_2(x)}\right|^{d_q x}$$

$$\oplus \int_0^a \left|\left(\left[D_q^* t_2\left(xq^{-1}\right)\right]\right)^{\ln z_1(x)}\right|^{d_q x} \oplus \int_0^a \left|\left[t_1(x)^{p(x)}\right]^{\ln z_1(x)}\right|^{d_q x}$$

$$= \frac{(t_1(a))^{-\ln z_2(a)}}{(t_1(0))^{-\ln z_2(0)}} \frac{1}{e^{\int_0^a -D_q \ln t_1(qx) D_q \ln z_2(x) d_q x}}$$
$$\times \int_0^a \left|\left(\left[t_2(x)^{r(x)}\right]\right)^{-\ln z_2(x)}\right|^{d_q x} \frac{\left(t_2\left(aq^{-1}\right)\right)^{\ln z_1(a)}}{(t_2(0))^{\ln z_1(0)}}$$
$$\times \frac{1}{e^{\int_0^a D_q \ln t_2(x) D_q \ln z_1(x) d_q x}} \int_0^a \left|\left[t_1(x)^{p(x)}\right]^{\ln z_1(x)}\right|^{d_q x} \tag{3.5}$$

From (3.4) and (3.5), simple calculations yield

$$\langle \Gamma z,t\rangle_{*,q} = \frac{\dfrac{\left(z_2\left(aq^{-1}\right)\right)^{\ln t_1(a)}}{(z_2(0))^{\ln t_1(0)}}}{\dfrac{\left(t_2\left(aq^{-1}\right)\right)^{\ln z_1(a)}}{(t_2(0))^{\ln z_1(0)}}} \langle z,\Gamma t\rangle_{*,q}.$$

Then we have

$$\langle \Gamma z,t\rangle_{*,q} = \frac{[z,t](a)}{[z,t](0)} \langle z,\Gamma t\rangle_{*,q}, \tag{3.6}$$

where

$$[z,t](x) := \left(t_1(x) \odot z_2\left(xq^{-1}\right)\right) \ominus \left(z_1(x) \odot t_2\left(xq^{-1}\right)\right).$$

By virtue of (3.2) and (3.3), we conclude that

$$\langle \Gamma z, t\rangle_{*,q} = \langle z, \Gamma t\rangle_{*,q}. \tag{3.7}$$

□

Theorem 3.2. *Eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

Proof. Let ζ, δ be two distinct eigenvalues with corresponding eigenfunctions z, t, respectively. It follows from (3.7) that

$$\langle \Gamma z, t\rangle_{*,q} = \langle z, \Gamma t\rangle_{*,q}$$

$$\langle e^{\zeta} \odot z, t\rangle_{*,q} = \langle z, e^{\delta} \odot t\rangle_{*,q}$$

$$e^{\zeta-\delta} \langle z, t\rangle_{*,q} = 1.$$

Then we obtain

$$\langle z, t\rangle_{*,q} = 1,$$

since $\zeta \neq \delta$. □

The q^*-Wronskian is defined by the formula

$$W_{*,q}(z,t) = \left(t_1(x) \odot z_2\left(xq^{-1}\right)\right) \ominus \left(z_1(x) \odot t_2\left(xq^{-1}\right)\right),$$

where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in H$. Then we have the following theorems.

Theorem 3.3. *The q^*-Wronskian of any two solutions of Eq. (3.1) is independent of x.*

Proof. Let z and t be two solutions of Eq. (3.1). By (3.6), we see that

$$\langle \Upsilon z, t\rangle_{*,q} = \frac{[z,t](a)}{[z,t](0)} \langle z, \Upsilon t\rangle_{*,q}.$$

Then, we obtain

$$\frac{[z,t](a)}{[z,t](0)} = 1,$$

since $\Gamma z = e^{\lambda} \odot z$ and $\Gamma t = e^{\lambda} \odot t$. Thus

$$[z,t](a) = [z,t](0) = W_{*,q}(z,t)(0).$$

□

Theorem 3.4. *Any two solutions of Eq. (3.1) are multiplicative linearly dependent if and only if $W_{*,q} = 1$.*

Proof. Let z and t be two multiplicative linearly dependent solutions of Eq. (3.1), i.e, $z = t^{\xi}$, where $\xi \neq 1$ ([20]). Then, we obtain

$$W_{*,q}(z,t)(x) = \left(t_1(x) \odot z_2\left(xq^{-1}\right)\right) \ominus \left(z_1(x) \odot t_2\left(xq^{-1}\right)\right)$$

$$= \left(t_1(x) \odot t_2^{\xi}\left(xq^{-1}\right)\right) \ominus \left(t_1^{\xi}(x) \odot t_2\left(xq^{-1}\right)\right) = 1.$$

Conversely, let

$$W_{*,q}(z,t)(x) = \left(t_1(x) \odot z_2\left(xq^{-1}\right)\right) \ominus \left(z_1(x) \odot t_2\left(xq^{-1}\right)\right) = 1.$$

Then,

$$t_1^{\ln z_2(xq^{-1})} = z_1^{\ln t_2(xq^{-1})}$$

$$\ln t_1 \ln z_2\left(xq^{-1}\right) = \ln z_1 \ln t_2\left(xq^{-1}\right)$$

$$\ln t_1 \ln z_2\left(xq^{-1}\right) - \ln z_1 \ln t_2\left(xq^{-1}\right) = \begin{vmatrix} \ln t_1 & \ln z_1 \\ \ln t_2\left(xq^{-1}\right) & \ln z_2\left(xq^{-1}\right) \end{vmatrix} = 0,$$

i.e., $\ln z = \begin{pmatrix} \ln z_1 \\ \ln z_2 \end{pmatrix}$, and $\ln t = \begin{pmatrix} \ln t_1 \\ \ln t_2 \end{pmatrix}$ are linearly dependent (see [1]). Hence $\ln z = \xi \ln t$, where $\xi \neq 1$. □

Theorem 3.5. *All eigenvalues of (3.1)-(3.3) are simple from the geometric point of view.*

Proof. Let γ be an eigenvalue with eigenfunctions $z(.)$ and $t(.)$. From (3.2), we deduce that

$$W_{*,q}(z,t)(0) = (t_1(0) \odot z_2(0)) \ominus (z_1(0) \odot t_2(0)) = 1,$$

i.e., z and t are multiplicative linearly dependent. □

4. Green's function

In this section, we study Green's function for the following nonhomogeneous system

$$\begin{cases} D_q^* z_1(x) \oplus \left(e^{\lambda - r(x)} \odot z_2(x)\right) = e^{f_2(x)} \\ \left(D_{q^{-1}}^*\right)^{1/q} z_2(x) \oplus \left(e^{p(x)-\lambda} \odot z_1(x)\right) = e^{f_1(x)} \end{cases}, \ x \in [0,a], \tag{4.1}$$

where $r(.)$ and $p(.)$ are real-valued functions on $[0,a]$, and

$$\begin{pmatrix} e^{f_1(.)} \\ e^{f_2(.)} \end{pmatrix} \in H,$$

which satisfy the following conditions

$$\left(e^{\cos\alpha} \odot z_1(0)\right) \oplus \left(e^{\sin\alpha} \odot z_2(0)\right) = 1, \tag{4.2}$$

$$\left(e^{\cos\beta} \odot z_1(a)\right) \oplus \left(e^{\sin\beta} \odot z_2\left(aq^{-1}\right)\right) = 1, \tag{4.3}$$

where $\alpha, \beta \in \mathbb{R}$.
Let

$$\chi(x,\lambda) = \begin{pmatrix} \chi_1(x,\lambda) \\ \chi_2(x,\lambda) \end{pmatrix} \text{ and } \psi(x,\lambda) = \begin{pmatrix} \psi_1(x,\lambda) \\ \psi_2(x,\lambda) \end{pmatrix}$$

be two basic solutions of Eq. (3.1) which satisfy the following initial conditions

$$\chi_1(0,\lambda) = e^{-\sin\alpha},\ \chi_2(0,\lambda) = e^{\cos\alpha},$$
$$\psi_1(a,\lambda) = e^{-\sin\beta},\ \psi_2\left(aq^{-1},\lambda\right) = e^{\cos\beta}.$$

It is obvious that

$$\omega(\lambda) = -W_{*,q}(\chi,\psi) \neq 1.$$

Theorem 4.1. *If λ is not an eigenvalue of (3.1)-(3.3), then the function*

$$z(x,\lambda) = \langle G(x,.,\lambda), \begin{pmatrix} e^{f_1(.)} \\ e^{f_2(.)} \end{pmatrix} \rangle_{*,q}, \tag{4.4}$$

where

$$G(x,t,\lambda) = e^{-\frac{1}{\omega(\lambda)}} \odot \begin{cases} \psi(x,\lambda) \odot \chi^T(t,\lambda), & 0 \le t \le x \\ \chi(x,\lambda) \odot \psi^T(t,\lambda), & x < t \le a, \end{cases} \tag{4.5}$$

is the solution of the system (4.1)-(4.3). Conversely, if λ is an eigenvalue of (3.1)-(3.3), then the system (4.1)-(4.3) is generally unsolvable.

Proof. Assume that λ is not an eigenvalue of (3.1)-(3.3). From (4.5), we have

$$G(x,t,\lambda) \odot \begin{pmatrix} e^{f_1(t)} \\ e^{f_2(t)} \end{pmatrix} =$$

$$\begin{cases} \begin{pmatrix} \chi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)} f_1(t) \ln \psi_1(t,\lambda)} \chi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)} f_2(t) \ln \psi_2(t,\lambda)} \\ \chi_2(x,\lambda)^{-\frac{1}{\omega(\lambda)} f_1(t) \ln \psi_1(t,\lambda)} \chi_2(x,\lambda)^{-\frac{1}{\omega(\lambda)} f_2(t) \ln \psi_2(t,\lambda)} \end{pmatrix}, & 0 \le t \le x \\ \begin{pmatrix} \psi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)} f_1(t) \ln \chi_1(t,\lambda)} \psi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)} f_2(t) \ln \chi_2(t,\lambda)} \\ \psi_2(x,\lambda)^{\frac{1}{\omega(\lambda)} f_1(t) \ln \chi_1(t,\lambda)} \psi_2(x,\lambda)^{-\frac{1}{\omega(\lambda)} f_2(t) \ln \chi_2(t,\lambda)} \end{pmatrix}, & x < t \le a. \end{cases} \tag{4.6}$$

By (4.6), we get

$$z_1(x,\lambda)=\int_0^x \left|\psi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)}f_1(t)\ln\chi_1(t,\lambda)}\psi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)}f_2(t)\ln\chi_2(t,\lambda)}\right|^{d_qt}\times\int_x^a \left|\chi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)}f_1(t)\ln\psi_1(t,\lambda)}\chi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)}f_2(t)\ln\psi_2(t,\lambda)}\right|^{d_qt}$$

$$\begin{aligned}&=e^{-\frac{q}{\omega(\lambda)}\ln\psi_1(x,\lambda)\int_0^x f_1(qt)\ln\chi_1(qt,\lambda)d_qt}e^{-\frac{q}{\omega(\lambda)}\ln\psi_1(x,\lambda)\int_0^x f_2(qt)\ln\chi_2(qt,\lambda)d_qt}\\&\times e^{-\frac{q}{\omega(\lambda)}\ln\chi_1(x,\lambda)\int_x^a f_1(qt)\ln\psi_1(qt,\lambda)d_qt}\\&\times e^{-\frac{q}{\omega(\lambda)}\ln\chi_1(x,\lambda)\int_x^a f_2(qt)\ln\psi_2(qt,\lambda)d_qt},\end{aligned}\tag{4.7}$$

and

$$z_2(x,\lambda)=\int_0^x \left|\psi_2(x,\lambda)^{-\frac{1}{\omega(\lambda)}f_1(t)\ln\chi_1(t,\lambda)}\psi_2(x,\lambda)^{-\frac{1}{\omega(\lambda)}f_2(t)\ln\chi_2(t,\lambda)}\right|^{d_qt}\times\int_x^a \left|\chi_2(x,\lambda)^{-\frac{1}{\omega(\lambda)}f_1(t)\ln\psi_1(t,\lambda)}\chi_2(x,\lambda)^{-\frac{1}{\omega(\lambda)}f_2(t)\ln\psi_2(t,\lambda)}\right|^{d_qt}$$

$$\begin{aligned}&=e^{-\frac{q}{\omega(\lambda)}\ln\psi_2(x,\lambda)\int_0^x f_1(qt)\ln\chi_1(qt,\lambda)d_qt}\\&\times e^{-\frac{q}{\omega(\lambda)}\ln\psi_2(x,\lambda)\int_0^x f_2(qt)\ln\chi_2(qt,\lambda)d_qt}\\&\times e^{-\frac{q}{\omega(\lambda)}\ln\chi_2(x,\lambda)\int_x^a f_1(qt)\ln\psi_1(qt,\lambda)d_qt}\\&\times e^{-\frac{q}{\omega(\lambda)}\ln\chi_2(x,\lambda)\int_x^a f_2(qt)\ln\psi_2(qt,\lambda)d_qt}.\end{aligned}$$

It follows from (4.7) that

$$\begin{aligned}D_q^*z_1(x)&=e^{D_q(\ln z_1(x))}=e^{D_q\left(-\frac{q}{\omega(\lambda)}\ln\psi_1(x,\lambda)\int_0^x f_1(qt)\ln\chi_1(qt,\lambda)d_qt\right)}\times e^{D_q\left(-\frac{q}{\omega(\lambda)}\ln\psi_1(x,\lambda)\int_0^x f_2(qt)\ln\chi_2(qt,\lambda)d_qt\right)}\\&\times e^{D_q\left(-\frac{q}{\omega(\lambda)}\ln\chi_1(x,\lambda)\int_x^a f_1(qt)\ln\psi_1(qt,\lambda)d_qt\right)}\times e^{D_q\left(-\frac{q}{\omega(\lambda)}\ln\chi_1(x,\lambda)\int_x^a f_2(qt)\ln\psi_2(qt,\lambda)d_qt\right)}\\&=e^{-\frac{q}{\omega(\lambda)}D_q\ln\psi_1(x,\lambda)\int_0^x f_1(qt)\ln\chi_1(qt,\lambda)d_qt}\times e^{-\frac{q}{\omega(\lambda)}D_q\ln\psi_1(x,\lambda)\int_0^x f_2(qt)\ln\chi_2(qt,\lambda)d_qt}\\&\times e^{-\frac{q}{\omega(\lambda)}D_q\ln\chi_1(x,\lambda)\int_x^a f_1(qt)\ln\psi_1(qt,\lambda)d_qt}\times e^{-\frac{q}{\omega(\lambda)}D_q\ln\chi_1(x,\lambda)\int_x^a f_2(qt)\ln\psi_2(qt,\lambda)d_qt}\times e^{-\frac{f_2(x)W_{*,q}(\chi,\psi)}{\omega(\lambda)}}.\end{aligned}$$

Hence, we see that

$$\begin{aligned}D_q^*z_1(x)&=e^{-\frac{q}{\omega(\lambda)}(r(x)-\lambda)\ln\psi_2(x,\lambda)\int_0^x f_1(qt)\ln\chi_1(qt,\lambda)d_qt}\\&\times e^{-\frac{q}{\omega(\lambda)}(r(x)-\lambda)\ln\psi_2(x,\lambda)\int_0^x f_2(qt)\ln\chi_2(qt,\lambda)d_qt}\\&\times e^{-\frac{q}{\omega(\lambda)}(r(x)-\lambda)\ln\chi_2(x,\lambda)\int_x^a f_1(qt)\ln\psi_1(qt,\lambda)d_qt}\\&\times e^{-\frac{q}{\omega(\lambda)}(r(x)-\lambda)\ln\chi_2(x,\lambda)\int_x^a f_2(qt)\ln\psi_2(qt,\lambda)d_qt}\\&\times e^{f_2(x)}=z_2^{(r(x)-\lambda)}\oplus e^{f_2(x)}.\end{aligned}$$

It is proved similarly that the validity of the second equation in (4.1). Further, it is easy to check that (4.4) satisfies (4.2)-(4.3). □

Theorem 4.2. *Green's function $G(x,t,\lambda)$ defined by (4.5) is unique.*

Proof. Suppose that there is another Green's function $\widetilde{G}(x,t,\lambda)$ for the system (4.1)-(4.3). Then, we have

$$z(x,\lambda)=\langle\widetilde{G}(x,.,\lambda),\begin{pmatrix}e^{f_1(.)}\\e^{f_2(.)}\end{pmatrix}\rangle_{*,q}.$$

Thus,

$$\langle G(x,t,\lambda)\ominus\widetilde{G}(x,.,\lambda),\begin{pmatrix}e^{f_1(.)}\\e^{f_2(.)}\end{pmatrix}\rangle_{*,q}=0.\tag{4.8}$$

Putting $f(x)=\ln\left[G(x,t,\lambda)\ominus\widetilde{G}(x,t,\lambda)\right]$ in (4.8), we conclude that

$$G(x,t,\lambda)=\widetilde{G}(x,t,\lambda).$$

□

Theorem 4.3. *Green's function $G(x,t,\lambda)$ is defined by (4.5) satisfies the following properties.*
i) $G(x,t,\lambda)$ *is continuous at* $(0,0)$.
ii) $G(x,t,\lambda)=G(t,x,\lambda)$.
iii) For each fixed $t\in(0,qa]$, *as a function of* x, $G(x,t,\lambda)$ *satisfies Eq. (4.1) in the intervals* $[0,t),(t,qa]$ *and it satisfies (4.2)-(4.3).*

Proof. *i)* Since $\psi(.,\lambda)$ and $\chi(.,\lambda)$ are continuous at 0, we conclude that $G(x,t,\lambda)$ is continuous at $(0,0)$.
ii) It is easy to be checked.
iii) Let $t\in(0,qa]$ be fixed and $x\in[0,t]$. Then, we get

$$G(x,t,\lambda)=\begin{pmatrix} G_1(x,t,\lambda)\\ G_2(x,t,\lambda)\end{pmatrix}$$

$$=\begin{pmatrix} \chi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)}\ln\psi_1(t,\lambda)}\chi_1(x,\lambda)^{-\frac{1}{\omega(\lambda)}\ln\psi_2(t,\lambda)}\\ \chi_2(x,\lambda)^{-\frac{1}{\omega(\lambda)}\ln\psi_1(t,\lambda)}\chi_2(x,\lambda)^{-\frac{1}{\omega(\lambda)}\ln\psi_2(t,\lambda)}\end{pmatrix}.$$

Thus,

$$\Gamma G(x,t,\lambda)=e^{\lambda}\odot G(x,t,\lambda).$$

Similarly for $x\in(t,qa]$. Furthermore, we see that

$$\left(e^{\cos\alpha}\odot G_1(0,t,\lambda)\right)\oplus\left(e^{\sin\alpha}\odot G_2(0,t,\lambda)\right)$$

$$=\left(\chi_1(0,\lambda)^{\cos\alpha}\chi_2(0,\lambda)^{\sin\alpha}\right)^{-\frac{1}{\omega(\lambda)}\ln\psi_1(t,\lambda)-\frac{1}{\omega(\lambda)}\ln\psi_2(t,\lambda)}$$

$$=1,$$

and

$$\left(e^{\cos\beta}\odot G_1(a,t,\lambda)\right)\oplus\left(e^{\sin\beta}\odot G_2\left(aq^{-1},t,\lambda\right)\right)$$

$$=\left(\psi_1(a,\lambda)^{\cos\beta}\psi_2\left(aq^{-1},\lambda\right)^{\sin\beta}\right)^{-\frac{1}{\omega(\lambda)}\ln\chi_1(t,\lambda)-\frac{1}{\omega(\lambda)}\ln\chi_2(t,\lambda)}$$

$$=1.$$

□

5. Conclusion

In this study, the q-multiplicative Dirac system is defined. Then, some spectral properties of this problem were examined. Green's function is created for this system. Some properties of this function have been given.
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