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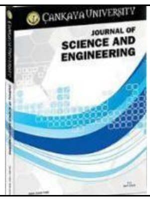

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On The Application of Homotopy Perturbation Method in Simulating the Effect of Double Dose Vaccination on a Mathematical Model of Covid-19 Transmission Dynamics

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Keywords

Covid-19, Vaccination, HPM, Mathematical Modelling.

Abstract

In this research, the impact of first and second dose vaccination in a mathematical model of COVID-19 transmission dynamics with two vaccination classes is investigated. The homotopy perturbation method is applied to obtain the approximate solution of the mathematical model. The impact of the first and second dose vaccinations were analyzed on the susceptible and exposed classes. The results of the analysis reveal that the second dose vaccination strategy is important to control the spread of the COVID-19 virus which is endangering the global existence of human population.

1. Introduction

Mathematical modelling is often applied to track the spread of different diseases. An analysis was carried out by [1] on a mathematical model for an effective management of HIV infection. [2] studied the theoretical and numerical analysis of fractal fractional model of tumor-immune interaction with two different kernels. To solve these mathematical models, researchers often employ the application of different numerical methods. For example, [3] established a new approach for solving the one-dimensional sine-Gordon equation. Also, [4, 5] applied the homotopy perturbation method to obtain the solutions of EIAV infection model and fourth order fractional integro-differential equation respectively. Towards the end of year 2019, the deadly corona virus broke out from the city of Wuhan china. An investigation on the virus by medical agents acknowledged that the causative agent of the sickness is a novel coronavirus known as Severe Acute Respiratory Syndrome Coronavirus 2 (SARS-CoV-2) and the condition produced by the virus was labelled Coronavirus Disease 2019 (COVID-19) by the World Health Organization (WHO,2020b). The spread of this disease was recorded in more than 213 nations as of July 2020, and a total number of 15,969,465 infections and 643,390 deaths were recorded. The incubation period of the disease is 2–14 days and an approximate of 97.5 percent of infected patients were said to develop symptoms by $11\frac{1}{2}$ days [6-9]. In Nigeria, 323 dead victims were recorded from a total of 11,516 confirmed cases as of June 4th, 2020 [10]. Several Mathematicians have proposed different mathematical model to track the spread of this virus, example is [11] who presents a five compartments mathematical model of the transmission dynamics of COVID-19 infection having a particular isolation or quarantine class. In [12], a study of Lyapunov stability and wave analysis of the COVID-19 omicron variant of real data with fractional operator was carried out, it was observed

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that with minimal surveillance and sequencing, the virus could have spread and developed in a population. Also, it might have been conceived in a COVID-19 patient who was persistently infected, and it might have originated in a nonhuman animal before being reintroduced into humans. They utilized the COVID-19 wave model with a fractional operator to examine the dynamical behavior of disease infection in society, and the fractal fractional derivative provided a realistic technique for analyzing illness effects, which proved to be effective. Still researching ways of combating the COVID-19 virus, a study on the modeling of the dynamics of novel coronavirus (COVID-19) via a stochastic epidemic model was published in [13], which proposed a stochastic model to investigate the transmission dynamics of the novel coronavirus using coronavirus disease characteristics. A qualitative study was conducted, which included proving the existence as well as the uniqueness of the global positive solution to demonstrate the well-posedness and feasibility of the problem, and sufficient conditions for the extinction and existence of stationary distributions were established using dynamical system theory and by constructing a suitable stochastic Lyapunov function. Finally, a large-scale numerical simulation was carried out to demonstrate the verification of the analytical results.

As a way of controlling the spread of COVID 19, effective strategy such as vaccination is often introduced. In this research, we apply the homotopy perturbation method proposed by [14] to carry out the analysis and investigation of the effect of two stages vaccination in a modified mathematical model of COVID-19 proposed by [11] by incorporating two additional compartments, which comprises, individuals that are vaccinated with first dose of vaccine $V_1(t)$ and individuals that are vaccinated with second dose of vaccine $V_2(t)$ The force of infection is defined as $\lambda = \alpha SI$, where α is the effective transmission between the susceptible and the infected individuals. The research work with the aid of homotopy perturbation method proved that the method is a powerful tool for simulating the effect of the affected parameters as shown in our simulation results.

1.1. The Modified Model Equation

Here, we present a deterministic mathematical model on transmission dynamics of COVID-19.

$$
\begin{aligned}
\frac{dS}{dt} &= \beta - \alpha SI + \omega V_1 - (\mu + \eta)S \\
\frac{dV_1}{dt} &= \eta S - (\omega + \sigma + \mu)V_1 \\
\frac{dV_2}{dt} &= \sigma V_1 - (\mu + \wedge)V_2 \\
\frac{dE}{dt} &= \alpha SI - (\mu + \rho)E \\
\frac{dI}{dt} &= \rho E - (\mu + \delta + \tau + \phi)I \\
\frac{dQ}{dt} &= \phi I - (\mu + \delta + k)Q \\
\frac{dR}{dt} &= \tau I + kQ - \mu R + \wedge V_2 .
\end{aligned}
\tag{1}
$$

We aim to investigate the effect of second dose vaccination on the prevalence and incidence of COVID-19 pandemic, hence we consider just the following classes' $S - V_1 - V_2 - E - I - Q$ compartment such that we have a six compartment model presented as:

$$\frac{dS}{dt} = \beta - \alpha SI + \omega V_1 - (\mu + \eta)S$$
$$\frac{dV_1}{dt} = \eta S - (\omega + \sigma + \mu)V_1$$
$$\frac{dV_2}{dt} = \sigma V_1 - (\mu + \wedge)V_2 \tag{2}$$
$$\frac{dE}{dt} = \alpha SI - (\mu + \rho)E$$
$$\frac{dI}{dt} = \rho E - (\mu + \delta + \tau + \phi)I$$
$$\frac{dQ}{dt} = \phi I - (\mu + \delta + k)Q$$

Subject to the following initial conditions

$$S(0) = s_0, V_1(0) = v_{10}, V_2(0) = v_{20}, E(0) = e_0, I(0) = i_0, Q(0) = q_0$$

1.2. Homotopy Perturbation Method

The basic idea of homotopy perturbation method in solving differential equations can be illustrated by considering the following differential equation of the form:

$$\varphi(u) = f(r) \qquad r \in \Omega \tag{3}$$

Subject to the boundary condition

$$B(u, u_n) = 0 \qquad r \in \Gamma \tag{4}$$

φ Represent the differential operator, the boundary operator is B, f(r) is an analytic function, Ω is the domain with boundary Γ.The derivative along the normal vector directed externally from Ω is u_n. Let suppose the differential equation consist of both linear and nonlinear terms, then we have

$$\varphi(u) = L(u) + N(u) \tag{5}$$

Where $L(u), N(u)$ represent the linear term, and the nonlinear term of the differential equation respectively. Thus, (3) becomes

$$L(u) + N(u) = f(r) \qquad r \in \Omega \tag{6}$$

Constructing an homotopy for (6),

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[\varphi(v) - f(r)] = 0 \tag{7}$$

Simplifying (8):

$$H(v, p) = L(v) - L(u_0) + p[L(u_0)] + p[N(u_0) - f(r)] = 0 \tag{8}$$

Where the initial approximation is u_0 and occurs on $p \in [0,1]$

As $p \to 0$, we obtain

$$H(v,0) = L(v) - L(u_0) = 0 \tag{9}$$

Also, as $p \to 1$, we obtain.

$$H(v,1) = \varphi(v) - f(r) = 0 \tag{10}$$

The solution of the differential equation can be expressed as a power series such that

$$v(t) = v_0(t) + pv_1(t) + p^2 v_2(t) + \cdots \tag{11}$$

Equation (12) is substituted into (11), coefficient of powers of p are compared and the resulting equation is solved to obtain the value of $v_0(t), v_1(t), v_2(t)$.

And the required approximate solution is obtained as

$$\lim_{p\to 1} v(t) = v_0(t) + v_1(t) + v_2(t) + \cdots \qquad (12)$$

1.3. Application of the Homotopy perturbation method

Here, the homotopy perturbation technique is applied to solve (2). We start by constructing a homotopy for (2):

$$\begin{aligned}
\frac{dS}{dt} &= p(\beta - \alpha SI + \omega V_1 - (\mu+\eta)S) \\
\frac{dV_1}{dt} &= p(\eta S - (\omega+\sigma+\mu)V_1) \\
\frac{dV_2}{dt} &= p(\sigma V_1 - (\mu+\wedge)V_2) \\
\frac{dE}{dt} &= p(\alpha SI - (\mu+\rho)E) \\
\frac{dI}{dt} &= p(\rho E - (\mu+\delta+\tau+\phi)I) \\
\frac{dQ}{dt} &= p(\phi I - (\mu+\delta+k)Q)
\end{aligned} \qquad (13)$$

Let the approximate solution of each class of (13) be:

$$\begin{aligned}
S(t) &= s_0(t) + ps_1(t) + p^2 s_2(t) + \ldots \\
V_1(t) &= v_{10}(t) + pv_{11}(t) + p^2 v_{12}(t) + \ldots \\
V_2(t) &= v_{20}(t) + pv_{21}(t) + p^2 v_{22}(t) + \ldots \\
E(t) &= e_0(t) + pe_1(t) + p^2 e_2(t) + \ldots \\
I(t) &= i_0(t) + pi_1(t) + p^2 i_2(t) + \ldots \\
Q(t) &= q_0(t) + pq_1(t) + p^2 q_2(t) + \ldots
\end{aligned} \qquad (14)$$

Substituting (14) into (13) and comparing equal powers of p, the following equations are obtained as the coefficient of

$$p^0 : \left.\begin{array}{l} \dfrac{ds_0(t)}{dt} = 0, \dfrac{dv_{10}(t)}{dt} = 0, \dfrac{dv_{20}(t)}{dt} = 0, \\ \dfrac{de_0(t)}{dt} = 0 \dfrac{di_0(t)}{dt} = 0, \dfrac{dq_0(t)}{dt} = 0. \end{array}\right\} \qquad (15)$$

Solving the system of equations obtained in (15) the following initial approximations are obtained for each class.

$$\begin{array}{ll} s_0(t) = s_0, & v_{10}(t) = v_{10}, \\ v_{20}(t) = v_{20}, & e_0(t) = e_0, \\ i_0(t) = i_0, & q_0(t) = q_0, \end{array} \qquad (16)$$

Also, the coefficient of p^1 is obtained as follows,

$$\frac{dS_1(t)}{dt} = \beta - \alpha s_0(t)i_0(t) + \omega v_{10}(t) - (\mu+\eta)s_0(t)$$
$$\frac{dv_{11}(t)}{dt} = \eta s_0(t) - (\omega+\sigma+\mu)v_{10}(t)$$
$$\frac{dv_{21}(t)}{dt} = \sigma v_{10}(t) - (\mu+\wedge)v_{20}(t) \quad (17)$$
$$\frac{de_1(t)}{dt} = \alpha s_0(t)i_0(t) - (\mu+\rho)e_0(t)$$
$$\frac{di_1(t)}{dt} = \rho e_0(t) - (\mu+\delta+\tau+\phi)i_0(t)$$
$$\frac{dq_1(t)}{dt} = \phi i_0(t) - (\mu+\delta+k)q_0(t)$$

Similarly solving the system of equation in (17) gives the first approximation

$$S_1(t) = (\beta - \alpha s_0 i_0 + \omega v_{10} - (\mu+\eta)s_0)t$$
$$v_{11}(t) = (\eta s_0 - (\omega+\sigma+\mu)v_{10})t$$
$$v_{21}(t) = (\sigma v_{10} - (\mu+\wedge)v_{20})t \quad (18)$$
$$e_0(t) = (\alpha s_0(t)i_0(t) - (\mu+\rho)e_0)t$$
$$i_0(t) = (\rho e_0 - (\mu+\delta+\tau+\phi)i_0)t$$
$$q_0(t) = (\phi i_0 - (\mu+\delta+k)q_0)t$$

also the coefficients of p^2 obtained are

$$\frac{dS_2(t)}{dt} = \beta - \alpha s_1(t)i_1(t) + \omega v_{1\backslash 1}(t) - (\mu+\eta)s_1(t)$$
$$\frac{dv_{12}(t)}{dt} = \eta s_1(t) - (\omega+\sigma+\mu)v_{11}(t)$$
$$\frac{dv_{22}(t)}{dt} = \sigma v_{11}(t) - (\mu+\wedge)v_{21}(t) \quad (19)$$
$$\frac{de_2(t)}{dt} = \alpha s_1(t)i_1(t) - (\mu+\rho)e_1(t)$$
$$\frac{di_2(t)}{dt} = \rho e_1(t) - (\mu+\delta+\tau+\phi)i_1(t)$$
$$\frac{dq_2(t)}{dt} = \phi i_1(t) - (\mu+\delta+k)q_1(t)$$

and the second approximations obtained by solving (19) are

$$S_1(t) = (2\alpha i_0\eta s_0 - \alpha i_0\beta + \alpha^2 i_0{}^2 s_0 - \alpha i_0\omega v_{10} + 3\alpha i_0\mu s_0 + \alpha i_0 s_0\phi - \alpha e_0\rho s_0 + \alpha i_0 s_0\tau + \alpha i_0 s_0\delta$$
$$-v_{10}\omega^2 + \omega\eta s_0 - 2\mu v_{10}\omega - \sigma v_{10}\omega + \eta^2 s_0 - \beta\eta - \omega v_{10}\eta + 2\eta\mu s_0 - \beta\mu + \mu^2 s_0)\frac{t^2}{2}$$
$$v_{12}(t) = (-v_{10}\omega^2 + \omega\eta s_0 - 2\mu v_{10}\omega - 2\sigma v_{10}\omega + \eta^2 s_0 - \beta\eta + \alpha i_0\eta s_0 - \omega v_{10}\eta + 2\eta\mu s_0 - \mu^2 v_{10}$$
$$-2\mu\sigma v_{10} + \sigma\eta s_0 - \sigma^2 v_{10})\frac{t^2}{2}$$

$$i_2(t)=(2\mu\phi i_0-2\rho e_0\mu+2i_0\mu\tau+2\mu\delta i_0+i_0\mu^2+e_0\rho^2+\alpha\rho i_0 s_0+2\tau\phi i_0-\tau\rho e_0+\tau^2 i_0+2\tau\delta i_0$$
$$+2\delta\phi i_0-\delta\rho e_0+\delta^2 i_0+\phi^2 i_0-\phi\rho e_0)\frac{t^2}{2}$$
$$q_2(t)=(\mu^2 q_0-2\mu\phi i_0+2Kq_0\mu+2\delta q_0\mu-\phi^2 i_0+\phi\rho e_0-\tau\phi i_0-2\delta\phi i_0-K\phi i_0+q_0K^2+2\delta q_0 K$$
$$+q_0\delta^2)\frac{t^2}{2}$$
$$v_{22}(t)=(-\eta v_{10}\omega+\sigma\eta s_0-2\mu\sigma v_{10}-\sigma^2 v_{10}+v_{20}\mu^2+2v_{20}\mu\Lambda-\Lambda\sigma v_{10}+v_{20}\Lambda^2)\frac{t^2}{2}$$
$$e_2(t)=(-2\rho e_0\mu+3\alpha i_0\mu s_0-e_0\mu^2+\alpha i_0\eta s_0-\alpha i_0\beta+\alpha^2 i_0{}^2 s_0-\alpha i_0\omega v_{10}+\alpha s_0\phi i_0$$
$$-\alpha s_0\rho e_0+\alpha s_0\tau i_0+\alpha s_0\delta i_0-e_0\rho^2+\alpha s_0\rho i_0)\frac{t^2}{2}$$

the approximate results of each class is:

$$S(t)=\sum_{n=0}^{2}s_n(t),\;\; v_1(t)=\sum_{n=0}^{2}v_{1n}(t),$$
$$v_2(t)=\sum_{n=0}^{2}v_{2n}(t),\; E(t)=\sum_{n=0}^{2}e_n(t),$$
$$I(t)=\sum_{n=0}^{2}i_n(t),\;\; Q(t)=\sum_{n=0}^{2}q_n(t)$$

2. Numerical Simulation

2.1. Table of Parameters

To conduct the numerical simulation, the results obtained are evaluated using their corresponding values presented in Table 1 such that;

$$S(t)=50+(-50\eta-13.6160250)\text{t}+(14.99897074+107.6320500\eta-2.4\sigma+50\sigma^2)\frac{t^2}{2}$$
$$E(t)=12+(50\eta-2.4038460-12\sigma)\text{t}+(\text{-}0.4815396326+23.6320500\eta-4.8076920\sigma$$
$$+50\eta^2+50\eta\sigma-12\sigma^2)\frac{t^2}{2}$$
$$v_1(t)=18+(50\eta+12\sigma)\text{t}+(0.007432608964-2.6476920\sigma+50\eta\sigma-12\sigma^2)\frac{t^2}{2}$$
$$v_2(t)=25+80.86698750\text{t}-(25.60966730+84\eta)\frac{t^2}{2}$$
$$I(t)=15+1.321347500\text{t}-(4.974744997)\frac{t^2}{2}$$
$$Q(t)=16-1.3858830\text{t}-(0.06254070950)\frac{t^2}{2}$$

Table 1. Parameters, description, value and source.

Parameters	**Description**	**Values**	**Source.**
s_0	Initial susceptible population	50	Assumed
e_0	Initial exposed population	25	Assumed
i_0	Initial infected population	15	Assumed
q_0	Initial quarantined population	16	Assumed
β	Recruitment rate	68	Estimated
α	Effective contact rate	0.112 per day	Estimated
ρ	Progression rate from Exposed to Infected class	$1/8$ per day	(Lauer, et al 2020)
ϕ	Progression rate from Infected to Quarantine Class	1.923×10^{-3} per day	(Matthew et al 2021)
τ	Recovery rate	$1/10$ per day	(Tang et al 2020)
k	Treatment rate	0.0701	(Garba et al 2020)
μ	Natural death rate	0.0003205	Estimated
δ	Covid-19 Induced death rate	0.018	(Garba et al 2020)
η	First dosage vaccination rate	$0 \le \eta < 1$	
ω	Rate at which vaccinated population move to susceptible class	0.2	Assumed
σ	Second dose of vaccine rate	$0 \le \eta < 1$	Assumed
$\wedge$	Rate at which population move from second vaccination to Recovery	0.02	Assumed

2.2. Figures

In this section we present the graphs (Figs. 1-6) representing the impact of implementing the second dose vaccination σ on the susceptible and exposed population who embraced the first dosage vaccination at different level on interval $0 \le \eta < 1$. The second dose vaccination σ is examined on the course $0 \le \sigma < 1$.

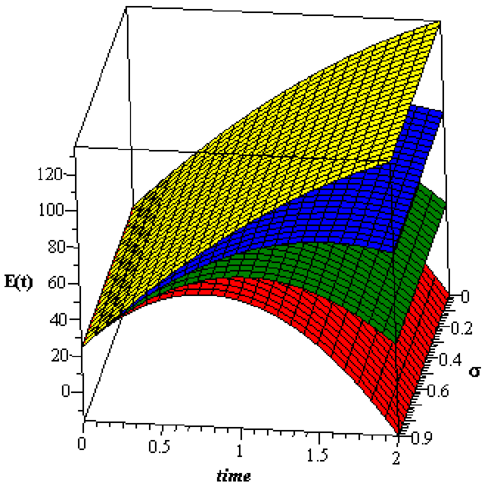

Figure 1. Effect of second dose vaccination on susceptible class.

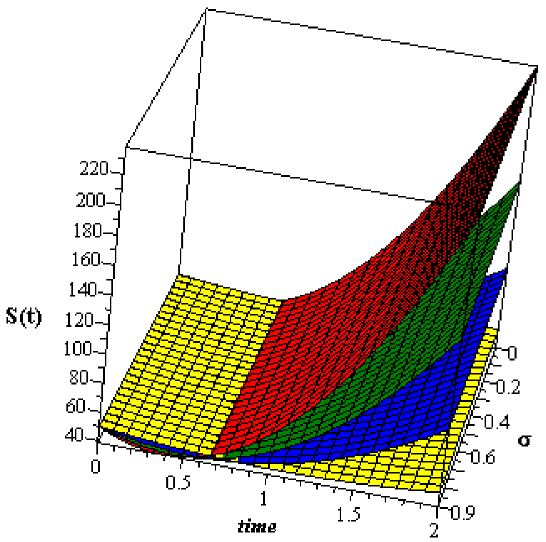

Figure 2. Effect of second dose vaccination on exposed class

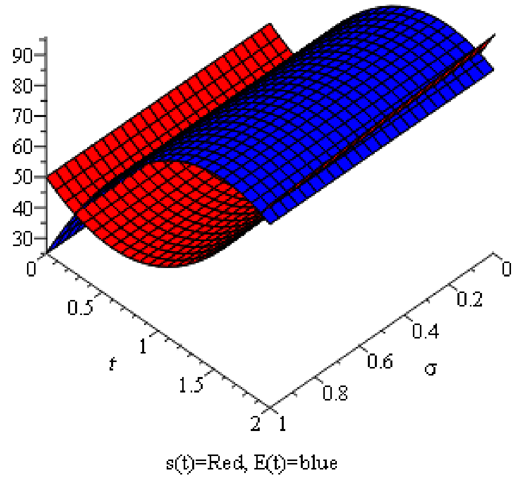

Figure 3. Graph of susceptible and exposed class when $\sigma = 0$

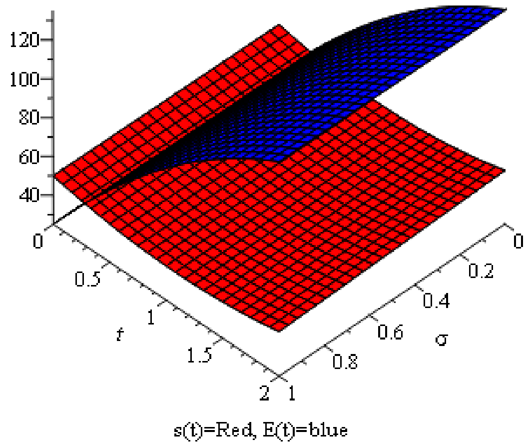

Figure 4. Graph of susceptible and exposed class when $\sigma = 0.3$

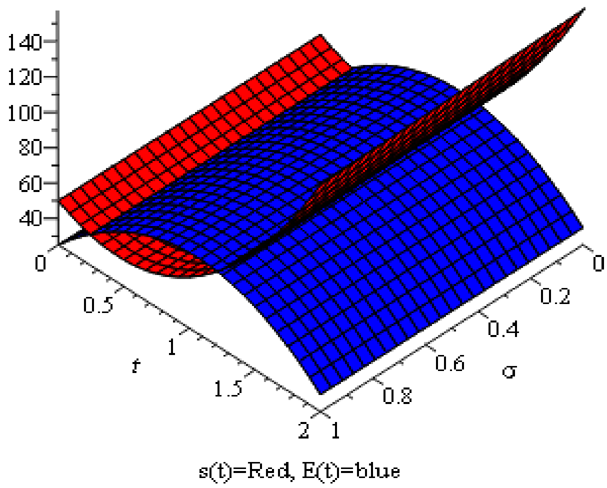

Figure 5. Graph of susceptible and exposed class when $\sigma = 0.6$

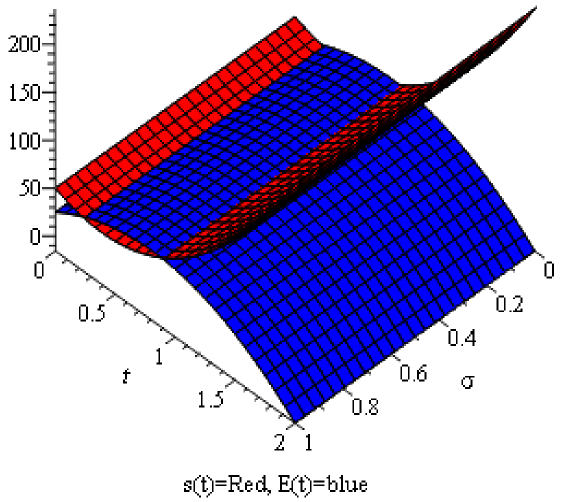

Figure 6. Graph of susceptible and exposed class when $\sigma = 0.9$

2.3. Discussion

Figure 1 reveals the impact of second dose vaccination on a susceptible class initially vaccinated at different levels of first dose η .It is observed that second vaccination tends to be more effective in eradicating the disease if the first vaccination stage is well embraced.

Figure 2 shows that fewer people will be exposed to contacting the COVID-19 virus by implementing the second stage vaccination on the basis that first stage vaccination is well implemented.

In Figure 3-5, the effect of second dose vaccination was simultaneously examined in the two classes it could be observed that even though the exposed population drastically reduces during second stage vaccination the decrement rate depends on the level of first vaccination.

3. Conclusions

In this paper, a modified mathematical model of COVID-19 transmission dynamics with two dose vaccination classes is solved numerically with the aid of homotopy perturbation method. The simulation of the obtained results shows the importance of two stage vaccination in controlling the spread of COVID-19 virus. It was discovered that the second dose vaccination is more effective if the level at which the population embrace the first vaccination is high. It is therefore recommended that masses should be enlightened to know the importance of been vaccinated and it should be implemented in high coverage.

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Declaration of Competing Interest

The authors declare that there is no conflict of interest.

Authorship Contribution Statement

Kayode M. Kolawole: Model Formulation, and Analysis.
Tawakalt A. Ayoola: Results Computation.
Adedapo A. Alaje: Typesetting, Reviewing.
Amos O. Popoola: Supervision, Editing.
Kehinde A. Bashiru: Data preparation, Simulation.

References

[1] O. M. Ogunlaran, and S. C. Oukouomi Noutchie, “Mathematical model for an effective management of HIV infection,” *Hindawi Publishing Corporation*, vol. 2016, no. 4217548, pp. 6, 2016. http://dx.doi.org/10.1155/2016/4217548.

[2] S. Ahmad, A. Ullah, A. Akgül, and D. Baleanu, “Theoretical and numerical analysis of fractal fractional model of tumor-immune interaction with two different kernels,” *Alexandria Engineering Journal*, vol. 61, no. 7, pp. 5735-5752, 2022.

[3] A. Akgül, M. Inc, A. Kilicman, and D. Baleanu, “A new approach for one-dimensional sine-Gordon equation,” *Advances in Difference Equations*, 2016. http://dx.doi.org/10.1186/s13662-015-0734-x.

[4] S. Balamuralitharan, and S. Geethamalini, “Solutions of epidemic of EIAV infection by HPM,” *Journal of Physics:* conf.series 1000, 012023, 2018.

[5] A. I. Alaje, M. O. Olayiwola, M. O. Ogunniran, J.A. Adedeji, and K. A. Adedokun, “Approximate Analytical Methods for the Solution of Fractional Order Integro-differential Equations,” *Nigerian Journal of Mathematics and Applications*, vol. 31, pp.175 – 190, 2021.

[6] S. A. Lauer, K. H. Grantz, Q Bi, F. K. Jones, Q. Zheng, H. R. Meredith, A.S. Azman, N.G. Reich, and J. Lessler “The incubation period of coronavirus disease 2019 (COVID-19) from publicly reported confirmed cases: estimation and application,” *Ann. Internal Med.*, vol. 172, no. 9, pp. 577–582, 2020.

[7] Q. Li, X. Guan, and P. Wu, “Early transmission dynamics in Wuhan, China, of novel coronavirus–infected pneumonia,” *New England Journal of Medicine*, vol. 382, no. 13, pp. 1199–1207, 2020.

[8] C. Lai, T. Shih, W. Ko, H. Tang, and P. Hsueh, “Severe acute respiratory syndrome coronavirus the challenges,” *International Journal of Antimicrobial Agents*, vol. 55, no. 3, 2020.

[9] D. Rio, and P. N. Malani, “Covid-19. New insights on a rapidly changing epidemic,” *JAMA*, vol. 323, no. 14, pp. 1339-1340, 2020.

[10] O. A. Adegboye, A. I. Adekunle, and E. Gayawan, “Early transmission dynamics of novel coronavirus (COVID-19) in Nigeria,” *Int. J. Environ. Res. Public Health*, vol. 17, no. 3054, pp. 1–10, 2020.

[11] D. Kalyan, G. K. Ranjith, R. Madhusudhan, and K. Lakshminarayan, “Sensitivity and elasticity analysis of novel corona virus transmission model: A mathematical approach,” *Sensors International*, vol. 2, 1999.

[12] C. Xu, M. Farman, A. Hasan, A. Akgül, M. Zakarya, W. Albalawi, and C. Park, “Lyapunov stability and wave analysis of Covid-19 omicron variant of real data with fractional operator,” *Alexandria Engineering Journal*, vol. 61, no. 12, pp. 11787–802, 2022.

[13] G. Hussain, T. Khan, A. Khan, M. Inc, G. Zaman, K. Sooppy Nisar, and A. Akgül, “Modeling the dynamics of novel coronavirus (COVID-19) via stochastic epidemic model,” *Alexandria Engineering Journal*, vol. 60, no. 4, pp. 4121-4130, 2021.

[14] J. H. He, “Homotopy perturbation technique,” *Comput. Methods. Appl. Mech. Eng.*, vol. 178, pp. 257, 1999.