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HOW THE VAN HIELE THEORY AND THE PIRIE-KIEREN THEORY CAN BE USED TO ASSESS PT'S UNDERSTANDING OF CONCEPT OF REFLECTION?

(Araştırma Makalesi)

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Abstract

This paper seeks to investigate how the van Hiele Theory and the Pirie-Kieren Theory can be used to assess pre-service teachers' understanding of the concept of geometric reflection. These analyses include motivations for ultimately utilizing the van Hiele Theory and Pirie-Kieren Theory to examine how pre-service teachers can develop a mapping view of geometric reflection from a motion view of geometric reflection. Additionally, I contrast previous cases which utilized the van Hiele Theory and Pirie-Kieren Theory separately, noting that there is yet to be work done in which the Pirie-Kieren Theory is utilized in conjunction with dynamic geometry software. While this study is not inherently connected to these existing studies, the utilization of frameworks did play a role in our decisions for deciding on a particular framework, namely the van Hiele Theory. I acknowledge that both the van Viele and Pirie-Kieren frameworks offer insights into pre-service teachers' thinking about geometric reflection (particularly when paired with a dynamic geometry software); however, due to certain characteristics of the van Hiele Theory (namely providing a clear progression in-depth of knowledge), I primarily suggest using the van Hiele Theory in teaching geometric reflection. My findings show that the emphasis on a clear path of progression and requisite knowledge is a critical factor in this change of perspective, as well as the importance of well-designed tasks that illuminate characteristics for a mapping view of geometric reflection.

Keywords: *van Hiele theory, Pirie-Kieren Theory, Geometric reflection, Dynamic geometry software, Motion view, Mapping view.*

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Van Hiele Teorisi ve Pirie-Kieren Teorisi Öğretmen Adaylarının Yansıma Dönüşümü Kavramı Anlayışını Değerlendirmek İçin Nasıl Kullanılabilir?

Öz

Bu çalışma, öğretmen adaylarının yansıma dönüşümü kavramını hareket ve eşleştirme yönünden anlama düzeylerini değerlendirmeyi ve Van Hiele Teorisi ve Pirie- Kieren Teorisinin kullanım şeklini araştırmayı amaçlamaktadır. Ayrıca bu çalışma, Van Hiele Teorisi ve Pirie-Kieren Teorisi'ni ayrı ayrı kullanan önceki çalışmaları karşılaştırarak Pierre Kieren Teorisi'nin, yansıma dönüşümünü anlamada, dinamik geometri yazılımı ile birlikte kullanıldığı çalışmaların henüz yapılmadığına da dikkat çekmektedir. Analizler sonucu, yansıma dönüşümünü anlamada van Hiele Teorisi'nin yansıma dönüşümünü anlamada Pirie-Kieren Teorisi'ne göre daha avantajlı bir teorik çerçeve olduğu söylenebilir. Hem van Viele hem de Pirie-Kieren teorik çerçevelerinin, öğretmen adaylarının geometrik yansımayı anlamaları konusunda (özellikle DGS kullanıldığında) onlara özel bilgiler sunduğu kabul edilebilir; ancak van Hiele Teorisi'nin belirli özelliklerinden dolayı (yani bilgi derinliğinde net bir ilerleme sağlamasi), geometrik yansıma öğretiminde öncelikle van Hiele Teori'sinin kullanılmasını öneriyorum. Bulgular, van Hiele Teorsi'nin hareket perspektifinden eşleştirme perspektifine geçişte hangi alt konseptlerin hangi sırada öğrenilmesi gerektiğini açıkça ortaya koyan ve etkinlik geliştirilmesinde detaylı bir yönerge sunan bir teorik çerçeve olduğunu gösterir.

Anahtar Kelimeler: *van Hiele Teori, Pirie-Kieren Teori, geometrik yansıma, dinamik geometri programı, hareket perspektivi, eşleştirme perspektifi.*

1. Introduction

Many students have difficulties understanding basic concepts in the area of geometry (Adolphus, 2011; Clements, Sarama, Yelland, & Glass, 2008; Luneta, 2014; Strutchens & Blume, 1997). The National Council of Teachers of Mathematics (NCTM, 2000) emphasizes that instruction should guide students to investigate properties of geometric figures and understand relationships among these properties. Specifically, perpendicularity and equidistance properties of geometric reflection are important for mapping the points of pre-image to image points which is a good mastery of reflection. Besides reflections through coordinates, geometric reflection has a significant role to understand other mathematical topics such as functions, symmetry. All the above testifies that reflections are in a particularly important stance in geometry and measurement learning and teaching. An inference drawn from the literature is that there are two perspectives to understand the concept of geometric reflection: motion and mapping perspectives (Akarsu, 2022; Hollebrands, 2003; Yanik, 2006). The motion perspective in understanding geometric reflection is not mathematically correct because students with motion perspective consider the plane to be empty and apply the geometric reflection only to the given shape. However, the plane consists of infinite points, and when applying the geometric reflection, the students need to apply the geometric reflection not only to the given shape but also to all the points in the plane, which is the mapping perspective. From this viewpoint, the number of studies examining the transition of pre-service teachers from the motion perspective to the mapping perspective in understanding geometric reflection and the number of theoretical frameworks explaining the mental structures in this transition are limited. Therefore, this research inquiry is aimed to explore how the van Hiele Theory and the Pirie-Kieren Theory can be used to assess PTs' understanding of the concept of reflection in terms of motion and mapping view. I work to summarize these two frameworks and their particular applications to geometric reflection. I also seek to examine how these theoretical frameworks can work in conjunction with Dynamic Geometry Software (DGS) as a means to analyze PTs' thinking about geometric reflection. This decision is explained along with an analysis of why I believe the Pirie-Kieren theory is not compatible with the progression of moving from a motion view to a mapping view of geometric reflection. PTs' initial conceptual understanding of reflection and discover common successes and failures in this understanding to come up with useful suggestions for reflection teaching and learning.

2. The van Hiele Theory

The van Hiele levels of thinking model (1986) is one of the most well-known frameworks guiding research on students' difficulties and their level of understanding. The van Hiele framework provides a lens through which to examine and understand students' geometry thinking. In the van Hiele model, there are five levels of geometric thinking: Level 1 (Visualization), Level 2 (Analysis), Level 3 (Informal Deduction), Level 4 (Deduction), and Level 5 (Rigor). These levels provide a general framework that can inform the design of instructional activities and through which student activity can be interpreted. Van Hiele proposed that there are four crucial characteristics of the levels. First, the levels are sequential and hierarchical. Put simply, if one level is not completed successfully, a student may perform only algorithmically at higher levels. NCTM (1989) supports the idea that there should be a hierarchy of levels because this provides a sequence for guiding students' learning; first to learn to identify whole shapes and then to explore the properties of shapes. From that stage, they can perceive relationships between properties and make basic deductions. Due to the progression of developing depth knowledge, "Curriculum development and instruction must consider this hierarchy" (NCTM, p. 48).

Second, students' progression from one level to the next level depends on the quality of instruction rather than age, maturation, environment, or parental support(s). Third, geometric experience is one of the most important features of the van Hiele model (1986) for helping students to progress through the levels. Fourth, language has a significant role in learning. All levels of the van Hiele theory uses the same terms but with augmented meanings as students progress through the levels. Thus, teachers and students may use the same terms while referring to different levels of meaning. For instance, if a student uses the word "geometric reflection" at level 1, s/he means that the geometric reflection is moving a pre-image figure to image figure over the reflection line without considering the properties of equidistance and perpendicularity, but at level 2 s/he is reflecting the pre-image figure considering the properties of equidistance and perpendicularity to determine where to place the figure. Hence, the van Hiele theory provided guidelines to show which levels students should reach to succeed in a high school geometry class. For example, students should achieve at least level 3 or level 4 to demonstrate a high level of thinking. To have a high level of thinking about geometric reflection, students or PTs need to have a mapping view of the reflection line, domain, and plane (Akarsu, 2022; Yanik, 2006). PTs with a mapping view of the reflection line know the role of the reflection line using the properties of perpendicularity and equidistance to position the pre-image figure correctly. PTs with a mapping view of the domain consider the domain as all points in the plane rather than as a single figure. PTs with a mapping view plane consider the points or figures as a subset of the plane rather than separated from the plane.

Initial support for our argument comes from a study conducted by Akarsu (2022) in which he used the Action, Process, Object, Schema (APOS) framework (Dubinsky, 1992) to PTs' levels of understanding in the geometric reflection in terms of motion and mapping view. Akarsu explained each characteristic of the APOS levels based on the geometric reflection in his study. His work provides a good example and starting point for research and for the development of an assessment tool to assess PTs' levels of understanding of geometric reflection in terms of motion and mapping view. Therefore, I want to identify each characteristic of the van Hiele levels (1986) based on the geometric reflection in terms of motion and mapping view (see Table 1).

Table 1. The van Hiele Levels of Motion and Mapping Views in Geometric Reflection

van Hiele Levels:	Motion View	Mapping view
Level 1 (Visualization)	PTs reflect the figure as a whole rather than as a collection of points	PTs reflect the figure as a collection of points (every pre-image point of the figure has a corresponding image point of the figure)
Level 2 (Analysis)	PTs do not use the properties of equidistance and perpendicular to determine the position of the points of the figure	PTs use the properties of equidistance and perpendicularity to determine the position of the points of the figure
Level 3 (Informal Deduction)	-PTs consider the domain as a single figure in the plane -PTs conceptualize definitions of plane metaphorically (i.e., know definitions verbally without being able to operate with them to perform geometric reflection)	-PTs consider the domain as all points in the plane. -PTs conceptualize definitions of plane mathematically (i.e., know definitions verbally and being able to operate with them to perform geometric reflection)
Level 4 (Deduction)	PTs consider geometric points or figures as moveable on the plane	PTs consider geometric points or figures as a subset of the plane.
Level 5 (Rigor)	PTs do not make connection with other geometric transformations	PTs know that geometric reflections produce other geometric transformations (e.g., translation, rotation)

In sum, based on table 1, I can hypothesize that level 4 is sufficient to have a mapping view of geometric reflection. Therefore, I can argue that the van Hiele theory (1986) is useful to determine students' levels of understanding of a specific topic such as geometric reflection. Determining students' levels of understanding enables teachers to plan and design their learning and teaching activities for their classrooms effectively. In line with these considerations, the van Hiele theory and the reconstructive approach are useful frameworks for designing tasks to expand teachers' understandings of the geometric reflection.

3. The Pirie-Kieren Theory

Another leading framework, the Pirie-Kieren theory (1989) has been adopted in many studies to analyze the growth of students' mathematical understanding. Some studies have specifically focused on analyzing the effects of processes such as folding back (Martin, 2008; Pirie & Martin, 2000), while several others have investigated their growth of understanding of specific mathematical topics, such as combinatorics (Warner, 2008), right triangle trigonometry (Cavey & Berenson, 2005), and geometric transformations (Gülkılık, Uğurlu & Yürük, 2015).

Over the last few decades, there has been growing interest in developing frameworks of mathematical understanding. Some researchers attempted to explore understanding by classifying it as either various types or levels (Pirie & Kieren, 1992; Skemp, 1976; Sierpinska, 1990). Skemp proposed that there are three different kinds of understanding, namely, instrumental, relational, and formal (logical) understanding. Skemp's categories of understanding left some questions unanswered for mathematics educators who sought to make sense of the fact that students can have some relational understanding that is not as yet useful to solve certain categories of problems. Drawing from this observation, there seemed to be an implication of either levels or, degrees of understanding, rather than just categories. Likewise, Sierpinska (1990) pursued this idea, asking: "Are there levels, degrees, or rather kinds of understanding? … Is understanding an act, an emotional experience, an intellectual process, or a way of knowing?… What are the conditions for understanding as an act to occur? ... How do we come to understand? … Can understanding be measured and how?" (p. 24). To address Sierpinska's thought-provoking questions, Pirie and Kieren (1992) sought to center their work around the growth of mathematical understanding, which they considered to be a complex process that cannot be characterized in terms of two or three categories.

To define mathematical understanding, Pirie and Kieren (1989) used von Glasersfeld's (1987) constructivist view of understanding. Accordingly, von Glaserfeld states, "The experiencing organism now turns into a builder of cognitive structures, intended to solve such problems as the organism perceives or conceives…among which is the never-ending problem of consistent organizations [of cognitive structures] that we call understanding" (p. 7). From this perspective, Pirie and Kieren (1992) used this characterization as a basis for defining understanding as "a whole dynamic, leveled but nonlinear, recursive process" (p. 243). In brief, they proposed that the growth of mathematical understanding is a dynamic and progressive process, but it is not linear.

Extending the existing view of Glasersfeld, Pirie, and Kieren (1994a) described eight levels of understanding: Primitive Knowing, Image-Making, Image-Having, Property Noticing, Formalizing, Observing, Structuring and Inventising (see Figure 1).

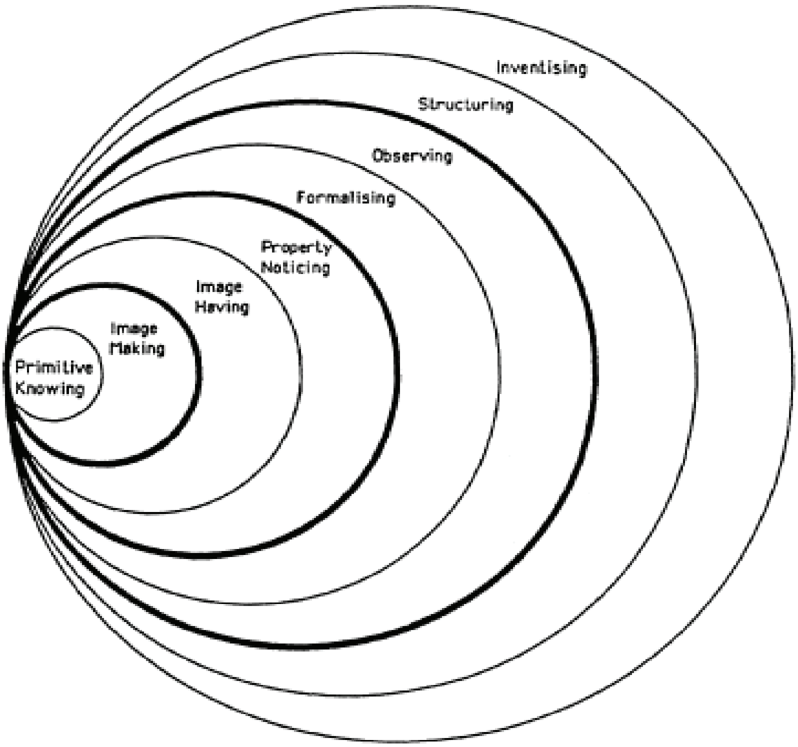

Figure 1. The model for the growth of mathematical understanding by (Martin and Pirie-Kieren, 2003, p. 174).

Pirie and Kieren (1992) define the first four levels as inner (informal) levels and the last four levels as outer (formal) levels. The Pirie-Kieren theory also proposes two crucial characteristics of their levels of understanding: *folding back* and *don't-need boundaries* (Gülkılık et al., 2015; Pirie & Kieren, 1992). When identifying processes learners use to switch between different levels of understanding, Pirie and Kieren (1994) suggested that students might *fold back* to an earlier level of understanding before progressing to a more advanced level. In other words, when a student encounters a difficult problem that cannot be resolved at the student's current level and requires a more advanced outer level of understanding, s/he may first need to switch back to an inner level of understanding to reconstruct new and appropriate images about the topic. Furthermore, as illustrated in the figure above, some boundaries between levels have thicker lines (see Figure 1) that are classified as "don't need boundaries" by Pirie and Kieren (1992). These lines represent that students have progressed in their abstract understanding. For instance, a student may have an image but does not need to give any examples of image-making, or s/he may formalize the concept without needing to show an image, or at the structuring level, s/he does not need to insert concrete meanings or apply formal algorithms. In other words, "don't-need" boundaries indicate that students do not need to focus on specific actions that they bring inside the boundary. They have formulated their understanding in a way that allows them to work with an abstract level of understanding outside the boundary (Pirie & Kieren, 1994).

As another example of application of the theory, Gülkılık et al., (2015) investigated four 10th-grade students' mathematical understanding of geometric transformations such as translation, reflection, rotations, and dilation. In this study, students were observed during their classes on geometric transformations, and then semi-structured interviews were conducted to analyze the growth of their mathematical understanding of geometric transformations. These authors used the levels of understanding of the Pirie-Kieren theory (1989) to analyze the complex and dynamic nature of the process of mathematical understanding. The findings of the study indicated that the Pirie-Kieren theory provides a useful foundation to examine the process of students' mathematical understanding of geometric transformations. They claimed that primitive knowing is an important level for understanding geometric transformations. According to these authors, students should first have primitive knowledge about vector, reflection line, and plane before working with geometric transformations, because these components are foundations for moving from informal definitions to formal definitions of geometric transformations. For instance, a student faced a challenge in understanding translations because s/he had not developed an understanding of the concept of the vector. To my knowledge, this is the first study that explains the importance of primitive knowing with a specific topic such as geometric transformations. Also, acting and expressing activities in image-making and property noticing provided a good basis for analyzing students' processes of mathematical understanding because students are more active at these levels than others (Pirie & Kieren, 1989). For instance, a student working with geometric reflections at the formalizing level noticed that his/her understanding was not adequate for this level, and s/he needed to go back to inner levels (fold back) to develop some of the properties of geometric reflections (e.g., geometric reflections preserve the equidistance between pre-image to image points). Pirie and Kieren (1994) point out that the observing level of understanding enables students to see patterns and connections in their reasoning to construct theories.

In conclusion, based on empirical studies, I observe the Pirie-Kieren theory (1989) is used for both mathematics in general and with a focus on geometry. Concerning the growth of mathematical understanding, both studies emphasized that the property noticing level of understanding provides an important foundation for students to be able to make generalizations, use formal definitions and create theorems. In addition, Gülkılık et al., (2015) showed the importance of primitive knowing for achieving higher levels of understanding of geometric transformations.

Ethical Procedures

This study was conducted based on research and publication ethics. The articles used in the research are fully expressed by the rules determined in the text and the references.

4. How the van Hiele Theory and the Pirie-Kieren Theory Can be used to Assess PTs' Understanding of Concept of Reflection

To answer the question "How do the van Hiele Theory and the Pirie-Kieren Theory can be used to assess PTs' understanding of the concept of reflection". I should first identify how mathematical understanding occurs. Then, I will present the van Hiele theory (1986) and Pirie-Kieren theory (1989) as two alternative perspectives that researchers can potentially use to assess PTs' level of mathematical understanding. In the following, I will discuss how each perspective can influence how we pursue our research question, and then I will make an argument to justify our choice of the van Hiele theory as more useful than Pirie-Kieren theory for assessing PTs' level of mathematical understanding in geometric reflections.

First, as suggested by Hiebert and Carpenter (1992) understanding can be defined as, "making connections, or establishing relationships, either between knowledge already internally represented or between existing networks and new information" (p. 80). This entails that understood mathematical concepts are part of an internal network of representations and constructing relations between these mental objects produces networks of knowledge.

Focusing on students' mental structures then is useful to analyze how they learn, and what they understand (Hiebert & Carpenter, 1992; Steffe & Kieren 1994). These processes are a key factor for understanding, and to aid in the analysis of students' mental structures many researchers have discussed the development and refinement of related skills (e.g. conceptualization, reasoning) in terms of levels (Battista & Clements, 1996; Battista, 2004; Pirie & Kieren, 1989; van Hiele, 1986). Models based around a levels framework can not only describe cognitive plateaus, but also aid in determining what students can and cannot do (Battista, 2004). Further, these frameworks provide a means for students to demonstrate primary milestones, as well as learning trajectories for a topic (Battista, 2004). Hence, to identify mental processes in which students understand a mathematical topic, I infer that defined sub-concepts in the learning process are necessary, and they provide characteristics of levels, as well as milestones of the particular topic.

A levels-model offers a suitable conceptual framework to understand and reason about the paths students take to complete the procedure of learning a topic. Hence, I intend to suggest the van Hiele Theory (1986), which represents a levels-model perspective, as a framework to investigate PTs' understanding of geometric reflection. I reason that the van Hiele theory's five levels of understanding (visualization, analysis, informal deduction, deduction, and rigor) can help researchers to determine PTs' learning paths, levels of thinking, language difficulties, thinking processes, and the primary milestones of their learning trajectories for geometric reflection. Therefore, I consider carefully defined levels to be indispensable in an investigation of the process of students' mathematical understanding. Additionally, teachers can use the five levels of the van Hiele model to design the geometric reflection tasks for their instruction. The NCTM (2000) has pointed out that mathematical tasks play a crucial role in the learning of mathematics. Likewise, Krainer (1993) stated, "Powerful tasks are important points of contact between the actions of the teacher and those of the student" (p. 68). Therefore, the van Hiele levels can provide teachers with guidelines to design appropriate tasks to deeply analyze PTs' understanding of geometric reflection.

Several studies have examined the effects of the use of dynamic geometry software (DGS) programs on students' van Hiele levels (Breen, 1999; Clements & Battista, 1990; Kutluca, 2013) and provided support for its positive effect on the development of students' van Hiele levels (1986) in geometry. These studies prompted us to think about the impact of technology on van Hiele levels and the transitions between the levels. In particular, DGS provides opportunities for students to draw, construct, and measure (Hollebrands, 2007) and to recognize patterns, make conjectures, and formulate conclusions (Tikoo, 1998) as a result of constructing objects and acting upon them. Consistently, Hollebrands (2007) found that when students interact with DGS, they observe and experience geometric transformations that serve as the basis for making connections between representations. Therefore, I infer from these findings that one of the major goals of using DGS is to facilitate students' transitions from one van Hiele level to the next. Furthermore, unique to DGS, dragging supports students' ability to explore objects' invariant properties and to make conjectures (Arzarello, Olivero, Paola, & Robutti, 2002; Hollebrands, 2007; İbili, 2019). For instance, students can use DGS to draw a triangle and then drag a vertex of the triangle and change the properties of the shape. Interacting with shapes in this way helps students to observe whether or not the properties of the object remain invariant or not (Tikoo, 1998). In other words, students can switch from the visualization level to the analysis level by using the dragging feature of DGS. I believe that all these studies can guide teachers as they design their tasks for each of the van Hiele levels to prepare their instruction.

Researchers can also draw from the Pirie-Kieren theory (1989) to explore PTs' growth of understanding of geometric reflections. Pirie and Kieren defined growth as a "whole, dynamic, leveled but a non-linear, transcendently recursive process" (Pirie & Kieren, 1991a, p. 1). In keeping with this view, they focused on an action to characterize understanding rather than a product resulting from such action. Adopting the Pirie-Kieren theory will allow researchers to explore PTs' understanding of geometric reflections as an ongoing process in action. However, the Pirie-Kieren theory does not provide learning trajectories to determine which level of a specific topic a student currently demonstrates because it focuses on the process of growth of mathematical understanding as dynamic and non-linear.

Folding back, which refers to moving back and forth between levels of understanding to promote understanding, is an important part of the Pirie-Kieren theory (1989). When a student has difficulties solving a problem, s/he might need to go back to an inner level of understanding to create new images or reorganize his/her prior understanding to improve his/her current understanding. Martin (2000) stated that a student needs to be “self-aware of the nature of his or her existing understanding of the folding back is to be effective” (p.145). This process is effective when moving back to an inner level enables the student to extend his/her understanding to solve the problem. The use of the Pirie-Kieren theory in geometric reflections could help determine the extent to which PTs are aware of the limitations of their understanding. By using the folding back strategy, they might examine what they learned about a topic and notice where gaps exist in their mathematical understanding. In the following paragraph, I explain the effects of DGS on Pirie-Kieren levels.

There is no research on the effects of the use of DGS programs on students’ Pirie-Kieren levels (1989) in geometric reflections. I infer that since the Pirie-Kieren focus is on action to characterize the level of understanding, use of DGS with the Pirie-Kieren theory may not be helpful for geometric reflections in terms of the motion and mapping views because understanding geometric transformations has been conceptualized in terms of the broad notions “motion” and “mapping” (Edwards, 1997; Flanagan, 2001; Yanik, 2013). A motion-oriented view entails seeing the plane as a background, separate from geometric objects (Yanik, 2013). This view is “erroneous” since the plane is a set of infinite points, and geometric objects are not separate from the plane; but rather a subset of the points in it. Now, a mapping view acknowledges, “all points in the plane [are mapped] to other points in the plane rather than removing images/points from their original locations to different locations.” (Yanik & Flores, 2009, p. 42).

Therefore, Yanik (2013) points out that when students focus on the action of geometric transformations by using DGS, they may promote their motion view rather than mapping view. To improve the mapping view, I need to focus on the result of the geometric reflections rather than action. Hence, the use of DGS with Pirie-Kieren's theory may not be helpful for research on geometric reflections.

To recap, I suggest using the van Hiele theory (1986) in geometric reflections research specifically for geometric transformations research for several reasons. First, the van Hiele model was written specifically for geometry as opposed to the Pirie-Kieren theory (1989), which applies across a variety of subject areas (e.g., algebra, computation, geometry). Second, while the van Hiele levels can help to determine PTs’ learning paths, levels of thinking, language difficulties, thinking processes, and the primary milestones of their learning trajectories for geometric reflections, the Pirie-Kieren theory cannot support the exploration of their learning trajectories in geometric reflections. Third, I used van Hiele Levels to determine the characteristics of the van Hiele levels in geometric transformations in terms of motion and mapping views (see Table 1). These characteristics of levels might be helpful for the design of tasks that investigate PT's understanding of geometric reflections. However, there is no research to determine the characteristics of the Pirie-Kieren theory in geometric transformations. Keeping that in mind, I believe that there is merit in exploring how the Pirie-Kieren theory might be applied to a similar context particularly because of the lack of research in this aspect. Looking more carefully into potential extensions of the Pirie-Kieren theory to fill an existing gap in the current literature might be an important goal to set up for future studies. Fourth, van Hiele explained five interim phases of learning in progressing from one level to the next level. I suggest using the van Hiele theory to determine students' particular levels and provide appropriate tasks to observe whether and how the movement from one level to the next level is achieved. The Pirie-Kieren theory, however, does not provide a learning path to explain how to move from one level to the next.

5. Conclusion

To discuss and investigate how PTs (or students) understand the concept of geometric reflections using DGS, the researchers, educators, and teachers use the van Hiele framework (1986). This is primarily due to the fact that the van Hiele framework provides clear descriptors and pathways for developing a deeper understanding of a topic (see Table 1). While this feature is applicable in many mathematical settings, it plays a fairly significant role in monitoring the progression from a motion perspective of geometric reflection to the mapping perspective. This study has shown that Pirie-Kieren's theory offers a unique perspective for understanding how PTs (or students) develop an understanding of a mathematical topic. However, this contrasts with the depths of knowledge provided by the van Hiele levels, although it does allow for different types of knowledge. The lack of a well-defined progression of knowledge types is the primary source of the difficulty in monitoring PT (or student) development. While the "pull-back" feature of Pirie-Kieren's theory acknowledges and, in some respects, implies a depth of knowledge, without a pathway to track the development/changes to a PTs understanding it does not lend itself to our study. The evidence this study has provided throughout suffices to show that the van Hiele levels meshes in a more effective manner with our work of understanding (and monitoring) PTs' understandings of geometric reflections utilizing DGS. This is a result of the descriptors and criteria laid out in the van Hiele levels, along with the linear progression of the development of understandings. These levels to some extent also correspond nicely with the descriptors for the types of understanding for geometric reflection. As a result, it can be seen similarities of the van Hiele levels with the motion and mapping perspectives and hence can track PTs (or student) progression of understanding of geometric reflection. Therefore, the studies mentioned above can be used as a guideline to design observations, interview protocols, and the setting of studies related to levels and types of understanding of geometric reflections.

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