Fundamental Journal of Mathematics and Applications, 6 (1) (2023) 12-23
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1147657

On Complete Group Classification of Time Fractional Systems Evolution Differential Equation with a Constant Delay

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Article Info

Keywords: *Delay, Fractional PDEs, Group classification, Lie symmetries*
2010 AMS: *34A08, 35B06, 35R11*
Received: *23 July 2022*
Accepted: *14 November 2022*
Available online: *7 December 2023*

Abstract

A fractional order system of evolution partial differential equations with a constant delay is considered. By exploiting the Lie symmetry method, we give a complete group classification of the system. Furthermore, we establish the corresponding symmetry reductions and construct some analytical solutions to the system.

1. Introduction

Fractional differential equations arise in cases where the extension of differential models to non-integer orders is imperative for more generalized analysis. The theory of fractional differential equations has been considerably developed over the years with some applications in engineering, natural sciences, economic models among others [1]-[5].

Since the extension of Lie symmetry analysis to the theory of differential equations by Ovsiannikov [6], it has remained one of the most powerful technique of studying and constructing analytical solutions to both deterministic and stochastic differential equations, moreover, it has advanced substantially leading to new generalizations and vast applications. For some of the recent work about the classical Lie symmetry theory of differential equations, its applications and extensions, we refer the reader to [7]-[18] and references therein.

Time lags occur naturally in most physical processes because they involve transfer of material or information. Although, time delay effect may improve the system performance [19], oftentimes, it is diagnosed as source of instability [20, 21]. It is therefore important that time delays are included in differential models in order to comprehensively investigate their effect on the systems' performance. Among the most recent applications of Lie symmetry theory is to the functional fractional differential equations [22]-[27].

Analytical solutions to evolution equations play a vital role in mechanics and dynamical systems, because they can naturally represent numerous physical phenomena, for instance, finite speed propagation, perturbations, heat transfer, solitons, among others. Different approaches have been introduced by mathematicians and engineers to construct exact solutions to fractional evolution equations [28]-[31].

In this article, we extend the Lie symmetry theory to the class of time fractional order system of differential equation with a constant delay, and carry out a complete group classification of time fractional system evolution delay differential equation. i.e.,

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Cite as: K. Mpungu, A. Ma'aruf Nass, *On Complete Group Classification of Time Fractional Systems Evolution Differential Equation with a Constant Delay*, Fundam. J. Math. Appl., 6(1) (2023), 12-23.

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x g(w,\overline{w}), \quad g_{\overline{w}} \neq 0, \\ \\ \frac{\partial^\alpha w}{\partial t^\alpha} = f(u,\overline{u})u_x, \quad f_{\overline{u}} \neq 0, \end{cases} \tag{1.1}$$

where $w(t-s,x) = \overline{w}$, $u(t-s,x) = \overline{u}$.
Let us recall that, for a differential equation involving some arbitrary function(s), the group classification problem consists of firstly, finding the Lie symmetries of differential equation with arbitrary function(s) and then determining all possible function(s) for which larger symmetry groups exist.
The motivation for the study in this paper is twofold, symmetry analysis for systems of fractional differential equations carried out in [26, 27]. Secondly, the group classification question for delay differential equations in [32, 33].
We proceed by introducing one of the definition of fractional derivative which will be used throughout this paper, that is, the Riemann-Liouville derivative defined by

$$D_t^\alpha u(t,x) = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(\mu,x)}{(t-\mu)^{\alpha+1-n}} d\mu, & n-1 < \alpha < n, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where Γ is a gamma function.
The rest of the article is organized as follow, a complete group classification of the time evolution fractional system of equations with a constant delay is presented in Section 2 and then followed by symmetry reductions and invariant solutions in Section 3.

2. Admitted Lie group transformation

In this section, we apply the method used in [6, 26, 32], to obtain the admitted Lie groups transformation of equation (1.1). The vector field associated with the one-parameter group of transformation is

$$H = \xi\partial_x + \tau\partial_t + \phi\partial_u + \zeta\partial_w,$$

where the infinitismal with infinitismals ξ, τ, ϕ, ζ depend on the variables t, x, u and v.
The Lie-B$\ddot{a}$cklund generator up to the fractional order corresponding to (1.1) is

$$\overline{H}^\alpha = \phi^u\partial_u + \phi^{u_x}\partial u_x + \zeta^{w_x}\partial w_x + \phi^{u_\alpha}\partial u_\alpha + \zeta^{w_\alpha}\partial w_\alpha + \overline{\phi^u}\partial_{\overline{u}} + \overline{\zeta^u}\partial_{\overline{w}} \tag{2.1}$$

where the coefficient in (2.1) are define as follows;

$$\phi^u = \phi - \xi u_x - \tau u_t, \quad \zeta^w = \zeta - \xi w_x - \tau w_t, \quad \overline{\phi^u} = \overline{\phi} - \overline{\xi}\overline{u_x} - \overline{\tau}\overline{u_t}, \quad \overline{\zeta^w} = \overline{\zeta} - \overline{\xi}\overline{w_x} - \overline{\tau}\overline{w_t} \tag{2.2}$$

$$\phi^{u_x} = D_x(\phi^u), \quad \zeta^{w_x} = D_x(\zeta^w) \quad \phi^{u_\alpha} = D_t^\alpha(\phi^u), \quad \zeta^{w_\alpha} = D_t^\alpha(\zeta^w) \tag{2.3}$$

$$\phi^{u_x} = \phi_x - \xi_x u_x - \tau_x u_t + \phi_u u_x - \xi_u u_x^2 - \tau_u u_x u_t - u_{xt}\tau - u_{xx}\xi + \phi_w w_x - w_t u_x \xi_w - w_x u_t \tau_w \tag{2.4}$$

and

$$\zeta^{w_x} = \zeta_x - \xi_x w_x - \tau_x w_t + \zeta_w w_x - \xi_w w_x^2 - \tau_w w_x w_t - w_{xt}\tau - w_{xx}\xi + \zeta_u u_x - u_t w_x \xi_u - u_x w_t \tau_u. \tag{2.5}$$

The prolongations of the fractional terms above can be expanded as follows;

$$\begin{aligned} \phi^{u_\alpha} = & \frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi_w \frac{\partial^\alpha w}{\partial t^\alpha} + (\phi_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \phi_u}{\partial t^\alpha} - w\frac{\partial^\alpha \phi_w}{\partial t^\alpha} - \tau D_t^{\alpha+1}(u) - \xi D_t^\alpha(u_x) \\ & - \sum_{n=1}^{+\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) + \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n}\frac{\partial^n \phi_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(u) \\ & + \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n}\frac{\partial^n \phi_w}{\partial t^n}\right] D_t^{\alpha-n}(w) + \eta_{\alpha_1} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \zeta^{w_\alpha} = & \frac{\partial^\alpha \zeta}{\partial t^\alpha} + \zeta_u \frac{\partial^\alpha u}{\partial t^\alpha} + (\zeta_w - \alpha D_t(\tau))\frac{\partial^\alpha w}{\partial t^\alpha} - w\frac{\partial^\alpha \zeta_w}{\partial t^\alpha} - u\frac{\partial^\alpha \zeta_u}{\partial t^\alpha} - \tau D_t^{\alpha+1}(w) - \xi D_t^\alpha(w_x) \\ & - \sum_{n=1}^{+\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(w_x) + \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n}\frac{\partial^n \zeta_w}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(w) \\ & + \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n}\frac{\partial^n \zeta_u}{\partial t^n}\right] D_t^{\alpha-n}(u) + \eta_{\alpha_2} \end{aligned} \tag{2.7}$$

with

$$\begin{aligned}\eta_{\alpha_1} &= \sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial u^k}\\ &+\sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-w]^r\frac{\partial^m}{\partial t^m}(w^{k-r})\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial w^k},\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\eta_{\alpha_2} &= \sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-w]^r\frac{\partial^m}{\partial t^m}(w^{k-r})\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial w^k}\\ &+\sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial u^k}.\end{aligned} \tag{2.9}$$

Applying the infinitesimal generator and equation(1.1), we obtain the invariance criterion of the system as;

$$\begin{cases}\left.(\phi^{u_\alpha}-g\zeta^{w_x}-w_xg_w\zeta^{w}-w_xg_{\overline{w}}\zeta^{\overline{w}})\right|_{(1.1)}=0\\[2ex] \left.(\zeta^{w_\alpha}-f\phi^{u_x}-u_xf_u\phi^{u}-u_xf_{\overline{u}}\phi^{\overline{u}})\right|_{(1.1)}=0.\end{cases} \tag{2.10}$$

Substituting (2.2)–(2.7) into the invariance criterion of the system (2.10) , and then equating the coefficients of various derivatives of $u,\overline{u}$ and $w,\overline{w}$ to zero, we have the simplified system of determining equations as follows;

$$\tau_u=\xi_w=\tau_w=\xi_t=\tau_x=\xi_u=0, \tag{2.11}$$

$$\zeta_{uu}=\zeta_{ww}=\zeta_{ut}=\zeta_{wt}=0, \tag{2.12}$$

$$\phi_{uu}=\phi_{ww}=\phi_{ut}=\phi_{wt}=0, \tag{2.13}$$

$$\overline{\tau}=\tau,\quad \overline{\xi}=\xi, \tag{2.14}$$

$$g\zeta_u-f\phi_w=0, \tag{2.15}$$

$$f\zeta_w+f\xi_x-f_{\overline{u}}\overline{\phi}-f_u\phi-f\phi_u=0, \tag{2.16}$$

$$g\phi_u+g\xi_x-g\zeta_w-g_w\zeta-g_{\overline{w}}\overline{\zeta}=0, \tag{2.17}$$

$$\frac{\partial^\alpha\zeta}{\partial t^\alpha}-f\phi_x=0, \tag{2.18}$$

$$\frac{\partial^\alpha\phi}{\partial t^\alpha}-g\zeta_x=0. \tag{2.19}$$

The lower limit in the definition of fractional derivative (1.2) is fixed, the presence of 0 requires that the manifold $t=0$ is invariant i.e.,

$$\left.\tau(t,x,u,\overline{u})\right|_{t=0}=0. \tag{2.20}$$

Differentiating (2.18) and (2.19) with respect to w and u respectively we have

$$\phi_{xw}=0,\quad and \quad \zeta_{xw}=0. \tag{2.21}$$

Solving the system of equation (2.11)–(2.14) using (2.21) and (2.20), we obtain the following infinitesimals

$$\tau=0,\quad \xi=\psi_1(x)\quad \phi=uwc_2+u\psi_2(x)+c_3w+\psi_3(t,x) \tag{2.22}$$

and

$$\zeta = c_4uw + c_5u + w\psi_4(x) + \psi_5(t,x). \tag{2.23}$$

Differentiating (2.18) with respect to u and $u\overline{u}$, we get the following system

$$\begin{cases} f_u\phi_x + f\phi_{ux} = 0 \\ f_{u\overline{u}}\phi_x + f_{\overline{u}}\phi_{ux} = 0. \end{cases} \tag{2.24}$$

Similarly, differentiating (2.19) with respect to w and $w\overline{w}$, we obtain the following system

$$\begin{cases} g_u\zeta_x + g\zeta_{ux} = 0 \\ g_{w\overline{w}}\zeta_x + g_{\overline{w}}\zeta_{ux} = 0. \end{cases} \tag{2.25}$$

Using system (2.24) and (2.25) to eliminate ϕ_{ux} and ζ_{ux} respectively we have

$$\begin{cases} (f_uf_{\overline{u}} - ff_{u\overline{u}})\phi_x = 0 \\ (g_wg_{\overline{w}} - gg_{w\overline{w}})\zeta_x = 0. \end{cases} \tag{2.26}$$

From the system (2.26) and equation (2.18), (2.19) it implies that, ϕ and ζ has no dependency in x i.e.,

$$\phi_x = \zeta_x = 0 \tag{2.27}$$

similarly, using (2.15) and (2.22)

$$\zeta_u = \phi_w = 0. \tag{2.28}$$

This reduces the infinitesimals (2.22) and (2.23) to

$$\tau = 0, \quad \xi = \psi_1(x) \quad \phi = c_6u + \psi_6(t), \quad \zeta = c_7w + \psi_7(t). \tag{2.29}$$

Using (2.27), (2.18), (2.29) and (2.19) we have

$$\frac{\partial^\alpha\psi_7(t)}{\partial t^\alpha} = 0, \quad \frac{\partial^\alpha\psi_6(t)}{\partial t^\alpha} = 0. \tag{2.30}$$

Finally, equations (2.16) and (2.17) are classification equations and are assumed to be satisfied without any restriction on $f(u,\overline{u})$ and $g(w,\overline{w})$. This implies that, to get a minimal symmetry algebra for any choice of functions $f(u,\overline{u}), g(w,\overline{w})$, we have to assume

$$\xi_x = \phi = \zeta = 0.$$

Therefore, for any arbitrary functions $f(u,\overline{u}), g(w,\overline{w})$, the system (1.1) admits a one dimensional symmetry algebra spanned by the infinitesimal generators

$$H_1 = \partial_x.$$

2.1. Classification

To search for a functions $f(u,\overline{u}), g(w,\overline{w})$, that may admit a larger symmetry algebra we have to consider the case when,

$$\xi_x = \phi = \zeta \neq 0. \tag{2.31}$$

Substituting the infinitesimals in (2.29) into equation (2.16) and (2.17), we respectively get

$$c_7f + f\psi_{1_x} - c_6\overline{u}f_{\overline{u}} - f_{\overline{u}}\overline{\psi_6} - c_6f - c_6uf_u - \psi_6f_u = 0 \tag{2.32}$$

and

$$c_6g + g\psi_{1_x} - c_7g - c_7wg_w - g_w\psi_7 - c_7\overline{w}g_{\overline{w}} - g_{\overline{w}}\overline{\psi_7} = 0. \tag{2.33}$$

Differentiating (2.32) with respect to u and $\overline{u}$ we get

$$\begin{cases} -c_6(\overline{u}f_{\overline{u}u} + 2f_u + uf_{uu}) + c_7f_u + f_u\psi_{1_x} = \psi_6f_{uu} + f_{\overline{u}u}\overline{\psi_6} \\ -c_6(\overline{u}f_{\overline{u}\overline{u}} + 2f_{\overline{u}} + uf_{\overline{u}u}) + c_7f_{\overline{u}} + f_{\overline{u}}\psi_{1_x} = \psi_6f_{\overline{u}u} + f_{\overline{u}\overline{u}}\overline{\psi_6}. \end{cases} \tag{2.34}$$

The system (2.34) is algebraic with respect to $\overline{\psi_6}$ and ψ_6, with the determinant of the matrix as;

$$\Delta_1 = f_{u\overline{u}}^2 - f_{uu} f_{\overline{u}\overline{u}}.$$

Similarly, differentiating (2.33) with respect to w and $\overline{w}$, we obtain the system

$$\begin{cases} -c_7(\overline{w} g_{\overline{w}w} + 2g_w + w g_{ww}) + c_6 g_w + g_w \psi_{1_x} = \psi_7 g_{ww} + g_{\overline{w}w} \overline{\psi_7} \\ -c_7(\overline{w} g_{\overline{w}\overline{w}} + 2g_{\overline{w}} + w g_{\overline{w}w}) + c_6 g_{\overline{w}} + g_{\overline{w}} \psi_{1_x} = \psi_7 g_{\overline{w}w} + g_{\overline{w}\overline{w}} \overline{\psi_7}. \end{cases} \tag{2.35}$$

The system (2.35) is algebraic with respect to $\overline{\psi_7}$ and ψ_7, with the determinant of the matrix as

$$\Delta_2 = g_{w\overline{w}}^2 - g_{ww} g_{\overline{w}\overline{w}}.$$

In the subsequent subsections, we discus the cases when the determinants of both matrices are equal to zero and otherwise;

2.2. $\Delta_1 \neq 0$**,** $\Delta_2 \neq 0$.

Solving the system (2.34) and (2.35) for $\psi_6, \overline{\psi_6}$ and $\psi_7, \overline{\psi_7}$ respectively to get

$$\begin{cases} \psi_6 = \dfrac{c_6(u f_{u\overline{u}}^2 + 2 f_{\overline{u}} f_{u\overline{u}} - 2 f_u f_{\overline{u}\overline{u}} - u f_{uu} f_{\overline{u}\overline{u}}) + c_7(f_u f_{\overline{u}\overline{u}} - f_{\overline{u}} f_{\overline{u}u}) + (f_u f_{\overline{u}\overline{u}} - f_{\overline{u}} f_{u\overline{u}})\psi_{1_x}}{\Delta_1}, \\ \\ \overline{\psi_6} = \dfrac{(\overline{u} f_{u\overline{u}}^2 + 2 f_u f_{u\overline{u}} - 2 f_{uu} f_{\overline{u}} - \overline{u} f_{uu} f_{\overline{u}\overline{u}}) c_6 + (f_{uu} f_{\overline{u}} - f_u f_{u\overline{u}}) c_7 + (f_{uu} f_{\overline{u}} - f_u f_{u\overline{u}})\psi_{1_x}}{\Delta_1} \end{cases} \tag{2.36}$$

and

$$\begin{cases} \psi_7 = \dfrac{c_6(w g_{w\overline{w}}^2 + 2 g_{\overline{w}} g_{w\overline{w}} - 2 g_w g_{\overline{w}\overline{w}} - w g_{ww} g_{\overline{w}\overline{w}}) + c_7(g_w g_{\overline{w}\overline{w}} - g_{\overline{w}} g_{\overline{w}w}) + (g_w g_{\overline{w}\overline{w}} - g_{\overline{w}} g_{w\overline{w}})\psi_{1_x}}{\Delta_1}, \\ \\ \overline{\psi_7} = \dfrac{(\overline{w} g_{w\overline{w}}^2 + 2 g_w g_{w\overline{w}} - 2 g_{ww} g_{\overline{w}} - \overline{w} g_{ww} g_{\overline{w}\overline{w}}) c_6 + (g_{ww} g_{\overline{w}} - g_w g_{w\overline{w}}) c_7 + (g_{ww} g_{\overline{w}} - g_w g_{w\overline{w}})\psi_{1_x}}{\Delta_1}. \end{cases} \tag{2.37}$$

Since $\psi_6, \overline{\psi_6}, \psi_7, \overline{\psi_7}$ and ψ_1 are independent of $u, \overline{u}, w, \overline{w}$, we can consider the case when $\psi_6, \overline{\psi_6}, \psi_7, \overline{\psi_7}$ are all constant and ψ_1 is linear in x. However, by the virtue of equation (2.30), i.e., the fractional derivative of a non-zero constant is not zero, consequently it follows that;

$$\psi_6(t) = \psi_7(t) = constant = 0, \quad and \quad \psi_1 = c_9 x + c_8. \tag{2.38}$$

Therefore, equation (2.32) and (2.33) using (2.38) become;

$$c_7 f + f c_9 - c_6 \overline{u} f_{\overline{u}} - c_6 f - c_6 u f_u = 0 \tag{2.39}$$

and

$$c_6 g + g c_9 - c_7 g - c_7 w g_w - c_7 \overline{w} g_{\overline{w}} = 0. \tag{2.40}$$

Any function $f(u, \overline{u})$ and $g(w, \overline{w})$ satisfying (2.39), (2.40) will also satisfy the system (2.36) and (2.37) respectively. The general solutions of (2.39) and (2.40) are;

$$f(u, \overline{u}) = h_1\left(\frac{\overline{u}}{u}\right) u^{\frac{c_7 + c_9 - c_6}{c_6}}, \quad g(w, \overline{w}) = h_2\left(\frac{\overline{w}}{w}\right) w^{\frac{c_6 + c_9 - c_7}{c_7}}.$$

Here, h_1, h_2 are an arbitrary functions.
Thus, the Lie algebra extended by three dimension and is spanned by the following generators

$$H_1, \quad H_2 = x\partial_x, \quad H_3 = u\partial_u, \quad H_4 = w\partial_w.$$

2.3. $\Delta_1 = 0$**,** $\Delta_2 = 0$.

In this subsection, we investigate all the possible functions that are solutions to the determinants of the matrices of the systems (2.34) and (2.35) i.e.

$$\Delta_1 = f_{u\overline{u}}^2 - f_{uu} f_{\overline{u}\overline{u}} = 0, \quad \Delta_2 = g_{w\overline{w}}^2 - g_{ww} g_{\overline{w}\overline{w}} = 0 \tag{2.41}$$

as well as deducing the extra symmetry algebras. The following cases are considered.

2.3.1. $f_{\overline{u}\overline{u}} \neq 0, \quad g_{\overline{w}\overline{w}} \neq 0.$

The general solutions of equation (2.41) are;

$$f_u = \delta_1(f_{\overline{u}}), \quad g_w = \delta_1(g_{\overline{w}}). \tag{2.42}$$

where $\delta_1(u)$ and $\delta_2(w)$ are an arbitrary functions of integrations.
Substituting equation (2.42) into the systems of equations (2.34) and (2.35)

$$\begin{cases} (2c_6 - c_7 - \psi_{1_x})(\delta_1' f_{\overline{u}} - f_u) = 0 \\ (2c_7 - c_6 - \psi_{1_x})(\delta_2' g_{\overline{w}} - g_w) = 0 \end{cases} \tag{2.43}$$

This leads to two cases;
Case I: $\delta_1' f_{\overline{u}} = f_u, \ \delta_2' g_{\overline{w}} = g_w$ This implies

$$\delta_1' f_{\overline{u}} = f_u, \quad \delta_2' g_{\overline{w}} = g_w. \tag{2.44}$$

Equation (2.44) has a general solutions

$$f(u,\overline{u}) = f_1(c_{16}u + \overline{u}), \quad g(w,\overline{w}) = g_1(c_{17}w + \overline{w}). \tag{2.45}$$

Substituting equation (2.45) into (2.32) and (2.33), we have

$$\frac{c_7 - c_6 + \psi_{1x}}{c_6\overline{u} + c_6c_{16}u + c_{16}\psi_6 + \overline{\psi}_6} = \frac{f_1{}'}{f_1} \tag{2.46}$$

and

$$\frac{c_6 - c_7 + \psi_{1x}}{c_7\overline{w} + c_7c_{17}w + c_{17}\psi_7 + \overline{\psi}_7} = \frac{g_1{}'}{g_1}. \tag{2.47}$$

Now considering the fact that f_1, g_1 are not functions of x,t, it is clear that, ψ_1, ψ_6, ψ_7 must be constants and from equation (2.30), it follows that ψ_6, ψ_7 are not constant except zero, since fractional derivatives of non-zero constant are not zero. Therefore, equation (2.46) and (2.47) reduces to

$$\frac{f_1'}{f_1} = \frac{c_7 - c_6 + c_{18}}{c_6(c_{16}u + \overline{u})} \tag{2.48}$$

and

$$\frac{g_1'}{g_1} = \frac{c_6 - c_7 + c_{18}}{c_7(c_{17}w + \overline{w})} \tag{2.49}$$

with

$$\psi_1 = c_{18}x + c_{19}. \tag{2.50}$$

Solving equation (2.48) and (2.49) we obtain another set of functions

$$f(u,\overline{u}) = (c_{16}u + \overline{u})^{\frac{c_7 - c_6 + c_{18}}{c_6}}, \quad g(w,\overline{w}) = (c_{17}w + \overline{w})^{\frac{c_6 - c_7 + c_{18}}{c_7}}, \tag{2.51}$$

that give an extra algebra. The Lie algebra is extended by three dimension and is spanned by the following generators;

$$H_1, \quad H_2, \quad H_3, \quad H_4.$$

Case II: $\delta_1' f_{\overline{u}} \neq f_u, \ \delta_2' g_{\overline{w}} \neq g_w$ Under this case, it is clear from equation (2.43) that;

$$\psi_1 = c_6x + c_{16}, \qquad c_6 = c_7. \tag{2.52}$$

Substituting (2.52) into the systems (2.34) and (2.35), we have

$$\begin{cases} (c_6u + \psi_6)\delta_1' = -(\overline{\psi_6} + c_6\overline{u}) \\ (c_7w + \psi_7)\delta_2' = -(\overline{\psi_7} + c_7\overline{w}). \end{cases} \tag{2.53}$$

If $\delta_1' = \delta_2' = 0$, no extra symmetry algebra is possible, so we proceed by considering the case when $\delta_1' \neq 0$ and $\delta_2' \neq 0$.

Differentiating the system (2.53) with respect to $\overline{u},\overline{w}$ we get

$$\begin{cases} (c_6u+\psi_6)\delta_1^{''}f_{\overline{uu}} = -c_6 \\ (c_7w+\psi_7)\delta_2^{''}g_{\overline{ww}} = -c_7 \end{cases} \tag{2.54}$$

and differentiating (2.54) with respect to temporal variable we obtain

$$\begin{cases} \psi_{6_t}\delta_1^{''} = 0 \\ \psi_{7_t}\delta_2^{''} = 0. \end{cases}$$

1. If $\delta_1^{''} \neq 0, \delta_2^{''} \neq 0$
   It implies ψ_6, ψ_7 are constant, and by using equation (2.30) we obtain specifically;
   $$\psi_6 = \psi_7 = 0.$$
   From (2.53), we can assumes the following functions
   $$\begin{cases} f(u,\overline{u}) = uF\left(\frac{\overline{u}}{u}\right)+c_{21}, \quad F_{u\overline{u}} \neq 0 \\ g(w,\overline{w}) = wG\left(\frac{\overline{w}}{w}\right)+c_{22}, \quad G_{u\overline{u}} \neq 0. \end{cases} \tag{2.55}$$
   Substituting equation (2.55) into (2.32) and (2.33) using (2.52) we note that an extra symmetry algebra can obtained if $c_{21} = c_{22} = 0$. Thus, the symmetry algebra is spanned by the generators;
   $$H_1, \quad H_5 = x\partial_x + u\partial_u + w\partial_w.$$
2. If $\delta_1^{''} = 0, \delta_2^{''} = 0$ then,
   $$f_u = c_{23}f_{\overline{u}} + c_{24}, \quad g_w = c_{25}g_{\overline{w}} + c_{26}.$$
   This can be solved to obtain
   $$f(u,\overline{u}) = c_{24}u + F(\overline{u}+c_{23}u), \quad g(w,\overline{w}) = c_{26}w + G(\overline{w}+c_{25}w),$$
   with $F_{u\overline{u}} \neq 0$ and $F_{w\overline{w}} \neq 0$.
   From the system (2.54) and (2.53), we have $c_6 = 0$ and
   $$c_{23}\psi_6 = -\overline{\psi_6}, \quad c_{25}\psi_7 = -\overline{\psi_7}. \tag{2.56}$$
   However, ψ_6 and ψ_7 have to satisfy equation (2.30), which in turn pushes them to zero, so in this case there is no any extra symmetry algebra possible.

2.3.2. $f_{\overline{uu}} = 0, \quad g_{\overline{ww}} = 0.$

Since $f_{\overline{u}} \neq 0, \quad g_{\overline{w}} \neq 0$, equations (2.41) have the following solutions;

$$f(u,\overline{u}) = c_9\overline{u} + h_1(u), \quad g(w,\overline{w}) = c_{10}\overline{w} + h_2(w) \tag{2.57}$$

where $h_1(u)$, $h_2(w)$ are an arbitrary functions of integrations.
Substituting equation (2.57) into the systems of equations (2.34) and (2.35) gives;

$$\begin{cases} (c_6u+\psi_6)h_1^{''} = 0 \\ (c_7u+\psi_7)h_2^{''} = 0 \end{cases}$$

as well as;

$$\psi_1 = c_6x + c_{11}, \qquad c_6 = c_7. \tag{2.58}$$

For $h_i^{''} \neq 0$, we have no extra symmetry i.e., we get the minimal algebra, therefore we consider the case $h_i^{''} = 0$, from which equation (2.57) becomes;

$$f(u,\overline{u}) = c_9\overline{u} + c_{12}u + c_{13}, \quad g(w,\overline{w}) = c_{10}\overline{w} + c_{14}w + c_{15}. \tag{2.59}$$

Substituting equation (2.58) and (2.59) into (2.31) and (2.32), it follows that extra symmetry algebra are possible if

$$\begin{cases} c_{13} = c_{15} = 0 \\ c_9\overline{\psi_6(t)} = -c_{12}\psi_6(t), \quad c_{10}\overline{\psi_7(t)} = -c_{13}\psi_7(t), \end{cases}$$

but ψ_6 and ψ_7 have to satisfy equation (2.30), which implies that $\psi_6 = \psi_7 = 0$. Therefore, the equation admits one additional symmetry algebra which is the linear combination of H_2, H_3, H_4 and it is spanned by;

$$H_1, \quad H_5 = x\partial_x + u\partial_u + w\partial_w.$$

2.4. Summary of the classification

In the previous sections, we have carried out a complete group classification of the fractional evolution systems of partial differential equations with a constant delay i.e.,

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x g(w,\overline{w}), \quad g_{\overline{w}} \neq 0 \\ \\ \frac{\partial^\alpha w}{\partial t^\alpha} = f(u,\overline{u})u_x, \quad f_{\overline{u}} \neq 0, \end{cases}$$

where $w(t-s,x) = \overline{w}$, $u(t-s,x) = \overline{u}$. We have proved that for any arbitrary functions f,g, the system admits one dimensional symmetry Lie algebra, which is a shift in the temporal and spatial variables i.e.,

$$H_1 = \partial_x.$$

We have also shown that, the Lie symmetry algebra can be extended up to dimension four in the following cases;

1. For functions;

$$f(u,\overline{u}) = h_1\left(\frac{\overline{u}}{u}\right)u^{\frac{c_7+c_9-c_6}{c_6}}, \quad g(w,\overline{w}) = h_2\left(\frac{\overline{w}}{w}\right)w^{\frac{c_6+c_9-c_7}{c_7}},$$

   and

$$f(u,\overline{u}) = (c_{16}u+\overline{u})^{\frac{c_7-c_6+c_{18}}{c_6}}, \quad g(w,\overline{w}) = (c_{17}w+\overline{w})^{\frac{c_6-c_7+c_{18}}{c_7}}$$

   The Lie algebra is extended by three dimension and is spanned by the following infinitesimal generators.

$$H_1, \quad H_2 = x\partial_x, \quad H_3 = u\partial_u, \quad H_4 = w\partial_w.$$

2. For functions

$$\begin{cases} f(u,\overline{u}) = uF\left(\frac{\overline{u}}{u}\right), \quad F_{u\overline{u}} \neq 0 \\ g(w,\overline{w}) = wG\left(\frac{\overline{w}}{w}\right), \quad G_{u\overline{u}} \neq 0. \end{cases}$$

   and

$$f(u,\overline{u}) = c_9\overline{u}+c_{12}u, \quad g(w,\overline{w}) = c_{10}\overline{w}+c_{14}w.$$

   The Lie symmetry algebra was extended by one dimension and it is spanned by;

$$H_1, \quad H_5 = x\partial_x + u\partial_u + w\partial_w.$$

It is easy to check that all the generators obtained form a Lie algebra and this takes us to the next section, were we implement one of the applications of group classification to determine symmetry reductions and invariant solutions of the system.

3. Applications

3.1. $\Delta_1 \neq 0, \Delta_2 \neq 0$

In the case above, the system (1.1) becomes

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x h_2\left(\frac{\overline{w}}{w}\right)w^{k_1}, \\ \frac{\partial^\alpha w}{\partial t^\alpha} = u_x h_1\left(\frac{\overline{u}}{u}\right)u^{k_1}, \end{cases} \tag{3.1}$$

We carry out the symmetry reduction using the sub-algebras X_1 and X_2 as presented below.

3.1.1. Sub-algebra $X_1 = H_2 + H_3 + H_4$,

$$\frac{dx}{x} = \frac{du}{u} = \frac{dw}{w}.$$

The similarity transformation obtained are

$$z = t, \quad u(t,x) = V_1(z)x, \quad w(t,x) = V_2(z)x.$$

which reduces the system (3.1) to

$$\begin{cases} x\frac{\partial^\alpha V_1}{\partial z^\alpha} = h_2\left(\frac{\overline{V_2}}{V_2}\right)V_2^{(1+k_1)}, \\ x\frac{\partial^\alpha V_2}{\partial z^\alpha} = h_1\left(\frac{\overline{V_1}}{V_1}\right)V_1^{(1+k_1)}. \end{cases}$$

3.2. $\Delta_1 = 0, \Delta_2 = 0.$

In this section, we discuss case by case resulting from the different functions obtained in Section 2.3 above. Symmetry reductions are obtained and some invariant solutions of the system are constructed.

3.2.1. $f_{\overline{u}\overline{u}} \neq 0, \quad g_{\overline{w}\overline{w}} \neq 0$ **and** $\delta_1' f_{\overline{u}} = f_u,\ \delta_2' g_{\overline{w}} = g_w.$

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x(c_{17}w + \overline{w})^{k_1} \\ \frac{\partial^\alpha w}{\partial t^\alpha} = u_x(c_{16}u + \overline{u})^{k_1}, \end{cases}$$

Sub-algebra $X_1 = H_2 + H_3 + H_4,$

$$\frac{dx}{x} = \frac{du}{u} = \frac{dw}{w}.$$

The similarity variables obtained are

$$z = t, \quad u(t,x) = V_1(z)x, \quad w(t,x) = V_2(z)x,$$

which are used to transform the system as below;

$$\begin{cases} \frac{\partial^\alpha V_1}{\partial z^\alpha} = V_2\left(c_{17}V_2 + \overline{V_2}\right)^{k_1} x^{(k_1-1)}, \\ \frac{\partial^\alpha V_2}{\partial z^\alpha} = V_1\left(c_{16}V_1 + \overline{V_1}\right)^{k_1} x^{(k_1-1)}. \end{cases}$$

3.2.2. $f_{\overline{u}\overline{u}} \neq 0, \quad g_{\overline{w}\overline{w}} \neq 0$ **and** $\delta_1' f_{\overline{u}} \neq f_u,\ \delta_2' g_{\overline{w}} \neq g_w.$

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = w_x w G\left(\frac{\overline{w}}{w}\right), \quad G_{u\overline{u}} \neq 0, \\ \frac{\partial^\alpha w}{\partial t^\alpha} = u_x u F\left(\frac{\overline{u}}{u}\right), \quad F_{u\overline{u}} \neq 0, \end{cases}$$

Sub-algebra $X_2 = H_6$

$$\frac{dx}{x} = \frac{du}{u} = \frac{dw}{w}.$$

The similarity transformations obtained are

$$z = t, \quad u(t,x) = V_1(z)x, \quad w(t,x) = V_2(z)x,$$

which are utilized to reduce the system to single variable fractional delay differential equations;

$$\begin{cases} \frac{\partial^\alpha V_1}{\partial t^\alpha} = V_2^2 G\left(\frac{\overline{V_2}}{V_2}\right), \\ \frac{\partial^\alpha V_2}{\partial t^\alpha} = V_1^1 F\left(\frac{\overline{V_1}}{V_1}\right). \end{cases} \tag{3.2}$$

3.2.3. $f_{\overline{u}\overline{u}} = 0, \quad g_{\overline{w}\overline{w}} = 0$

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = (c_{10}\overline{w} + c_{14}w)w_x, \\ \frac{\partial^\alpha w}{\partial t^\alpha} = (c_9\overline{u} + c_{12}u)u_x, \end{cases} \tag{3.3}$$

Sub-algebra $X_2 = H_6$

$$\frac{dx}{x} = \frac{du}{u} = \frac{dw}{w}.$$

The similarity variables obtained are

$$z = t, \quad u(t,x) = V_1(z)x, \quad w(t,x) = V_2(z)x.$$

which leads to

$$\begin{cases} \frac{\partial^\alpha V_1}{\partial z^\alpha} = (c_{10} + c_{14})V_2^2, \\ \frac{\partial^\alpha V_2}{\partial z^\alpha} = (c_9 + c_{12})V_1^2. \end{cases} \tag{3.4}$$

The system (3.4) has a solution of the form $V_1(z) = k_2 z^{\lambda_1}$, and $V_2(z) = k_3 z^{\lambda_2}$. Substituting this back to the system, we have

$$\begin{cases} k_2 \frac{\Gamma(\lambda_1+1)}{\Gamma(\lambda_1+1-\alpha)} z^{\lambda_1-\alpha} = (c_{10}+c_{14}) k_3^2 z^{2\lambda_2}, \\ k_3 \frac{\Gamma(\lambda_2+1)}{\Gamma(\lambda_2+1-\alpha)} z^{\lambda_2-\alpha} = (c_9+c_{12}) k_2^2 z^{2\lambda_1}. \end{cases}$$

To obtain the values of the constants k_2 and k_3, we assume the powers of z in the system to be the same leading to, $\lambda_1 = \lambda_2 = -\alpha$. As a result, we have

$$\begin{cases} k_2 = \left(\frac{a_2^2}{a_1^5}\right)^{\frac{1}{3}} \frac{\Gamma(-\alpha+1)}{\Gamma(1-2\alpha)}, \\ k_3 = \left(\frac{a_2}{a_1^4}\right)^{\frac{1}{3}} \frac{\Gamma(-\alpha+1)}{\Gamma(1-2\alpha)}. \end{cases}$$

Therefore, the exact solution of the system (3.3) is

$$\begin{cases} u(t,x) = \left(\frac{a_2^2}{a_1^5}\right)^{\frac{1}{3}} \frac{\Gamma(-\alpha+1)}{\Gamma(1-2\alpha)} x t^{-\alpha}, \\ w(t,x) = \left(\frac{a_2}{a_1^4}\right)^{\frac{1}{3}} \frac{\Gamma(-\alpha+1)}{\Gamma(1-2\alpha)} x t^{-\alpha}, \end{cases} \tag{3.5}$$

where $a_1 = (c_{10}+c_{14})$ and $a_2 = (c_9+c_{12})$.

Figures 3.1 and 3.2 illustrate the solution of the system (3.5).

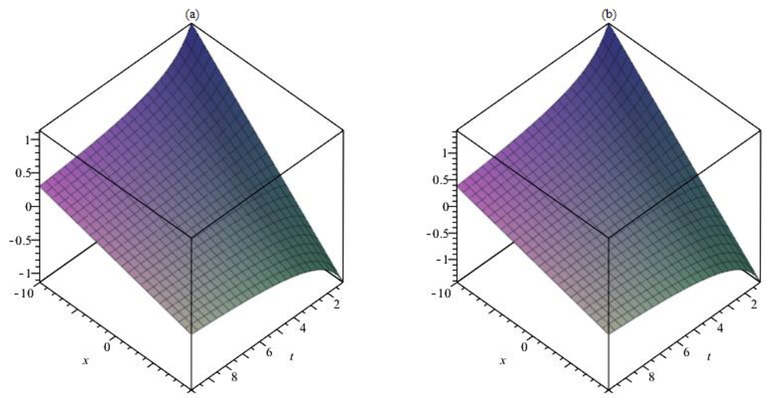

Figure 3.1: u, w when $a_1 = 2,\ a_2 = 1,\ \alpha = 0.6$ with $x = -10\ldots10,\ t = 1.1\ldots10$.

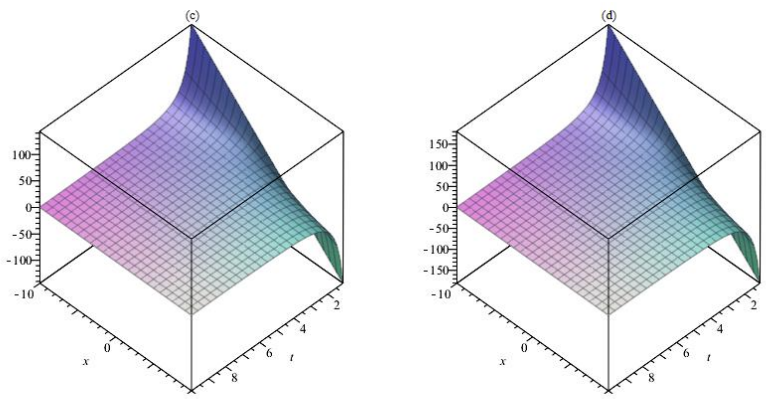

Figure 3.2: u, w when $a_1 = 2,\ a_2 = 1,\ \alpha = 1.6$ with $x = -10\ldots10,\ t = 1.1\ldots10$.

4. Conclusion

Lie symmetry analysis of fractional order evolution system equations with a constant delay was investigated. A one dimensional minimal symmetry algebra corresponding to an arbitrary function was obtained

$$H_1 = \partial_x.$$

Functions that lead to larger symmetry algebra were also found as well as their extended symmetry algebras. Further more, invariant solutions were obtained in addition to one new exact solution.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

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